Modern Optics
Prof. Partha Roy Chaudhuri
Department of Physics
Indian Institute of Technology, Kharagpur

Lecture – 10
Wave propagation in anisotropic media (Contd.)

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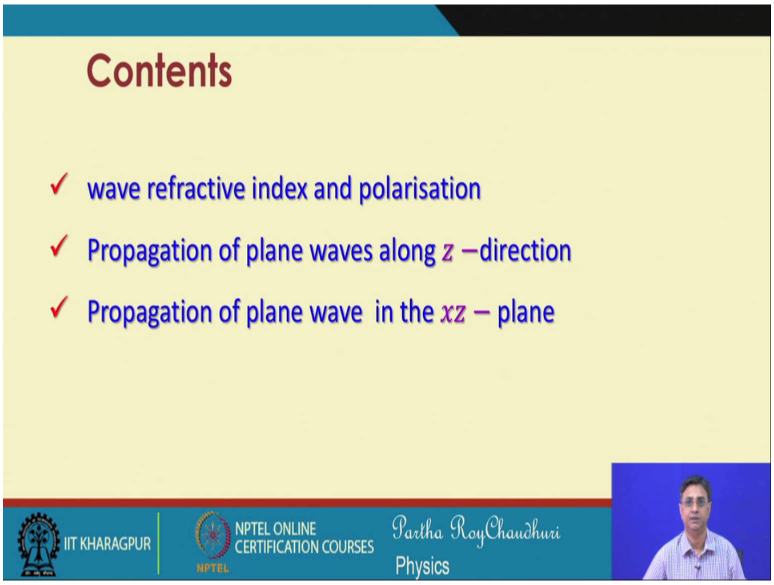

We are discussing the propagation of electromagnetic waves in anisotropic media. And in the last section, we have discussed how we can represent the direction of propagation of the electromagnetic waves, when it is given then to determine the refractive wave refractive indices and the direction of the d vectors. So, we will make use of that characteristic equation to solve the direction of the d vectors as well as the wave refractive indices.

So, the direction of the d vectors which are the direction of the polarisation that we will discuss in this connection; then we will consider two example cases: one is along the direction of propagation of the electromagnetic waves in the z direction which is a simple case and then slightly different case when the electromagnetic wave is propagating in the xz plane.

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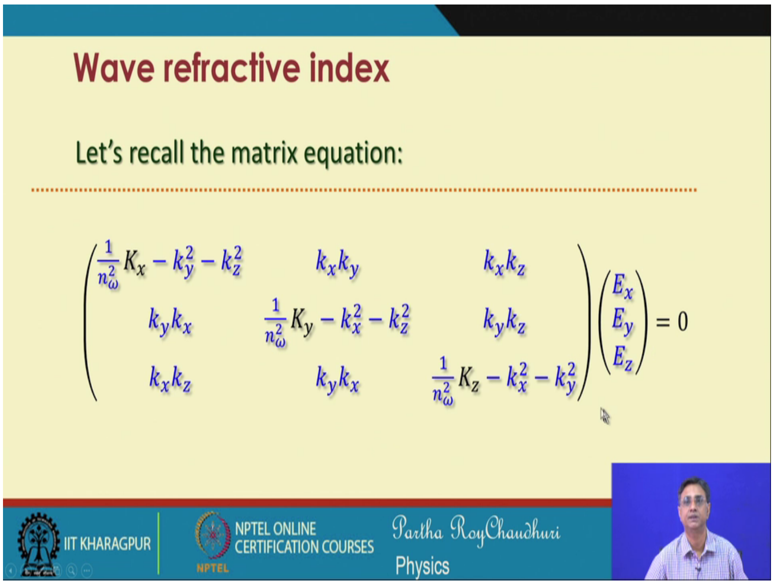

So, we have seen this characteristic equation that is an equation which has similar diagonal elements and similar half diagonal elements connected with this E x E y and E z equal to 0. So, this matrix the determinant value of this matrix will be equal to 0 for the non-trivial solution for this n omega square, the refractive index indices of associated with the wave.

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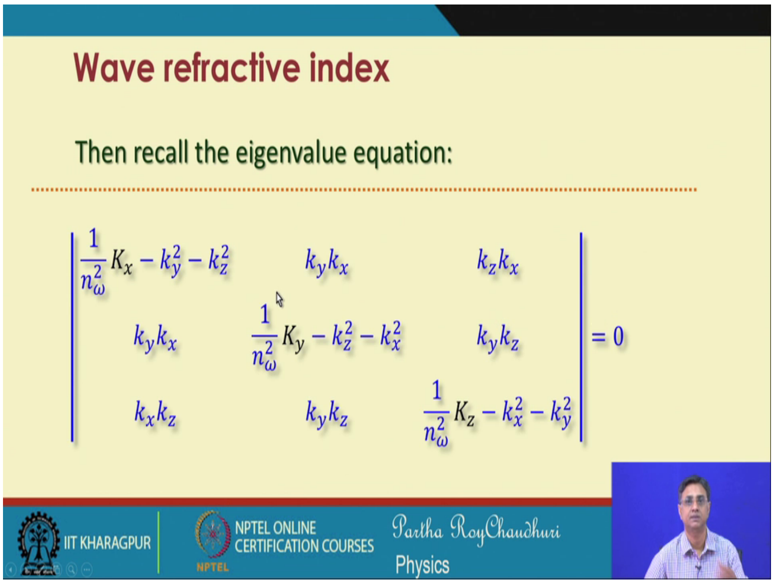

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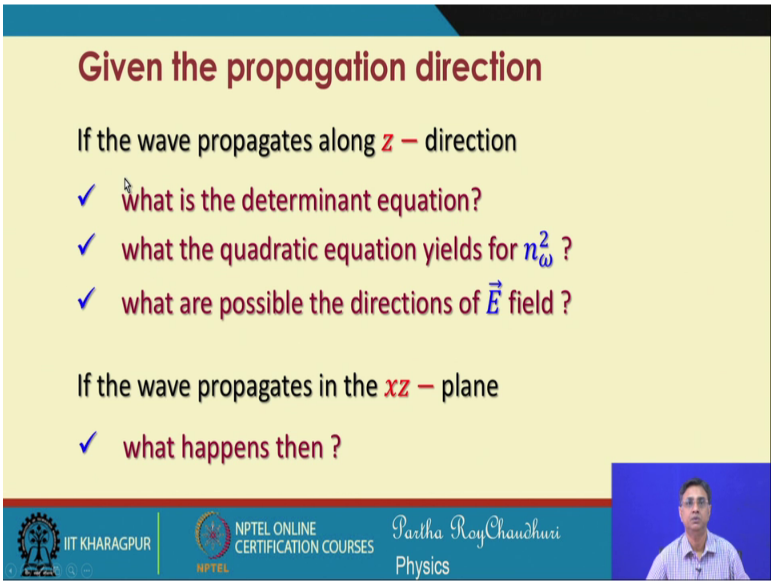

So, now we will consider the propagation of the wave along the z direction. So, given the direction of propagation that is along the z direction now, we will see how this determinant equation simplifies by plugging in the properties of the values of k x, k y and k z. Then we will look at the quadratic equation which is a result of this determinant and the solution of the quadratic equation that is the two values of n omega, the wave refractive indices. Then knowing that how we can extract the electric field direction from there; so, and then there after we will also discuss the case of the wave which is propagating along the zx direction.

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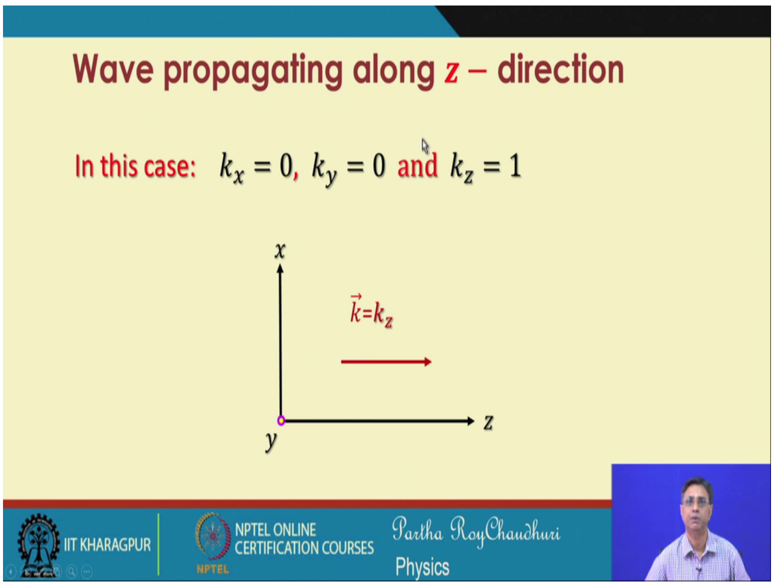

So, when the wave is propagating along z direction then you have k z is equal to 1, the only component which survives because there is no component k x or the component k y both put equal to 0. Therefore, so the wave is travelling in the z direction we should look at the schematic representation of the wave propagation.

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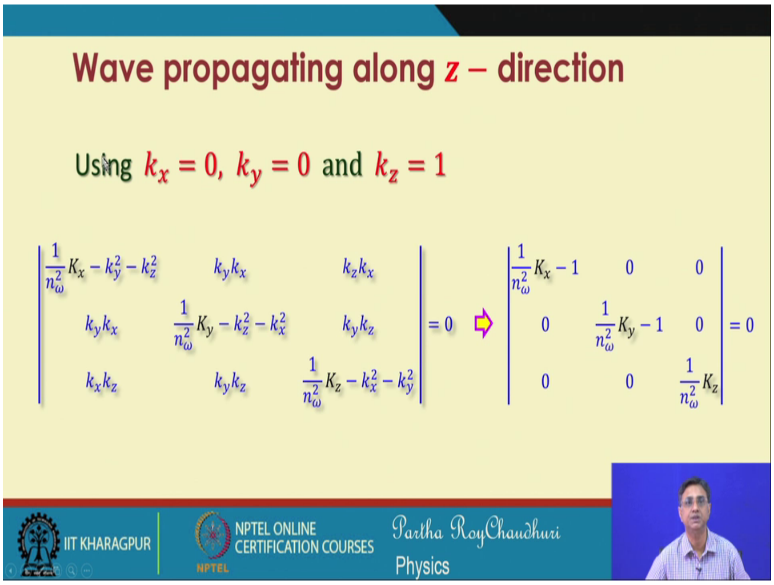

So, we use this values k x equal to 0, k y equal to 0, but k z equal to 1. This equation will take the shape of this because k x equal to 0. So, this will make this quantity this term 0, this term also 0, k x will make this term 0, k x will make this term 0, k y will make this term 0. So, 1 2 3 plus 3 so, all 6 of diagonal terms will be equal to 0.

So, you have determinants now for this case that is the case of wave propagating along the z direction which is which will look like this, that the diagonal elements of the determinant will be put equal to 0. So from these equations, if I open the bracket then I can write in this form this quantity into this into this minus 0 will be equal to 0.

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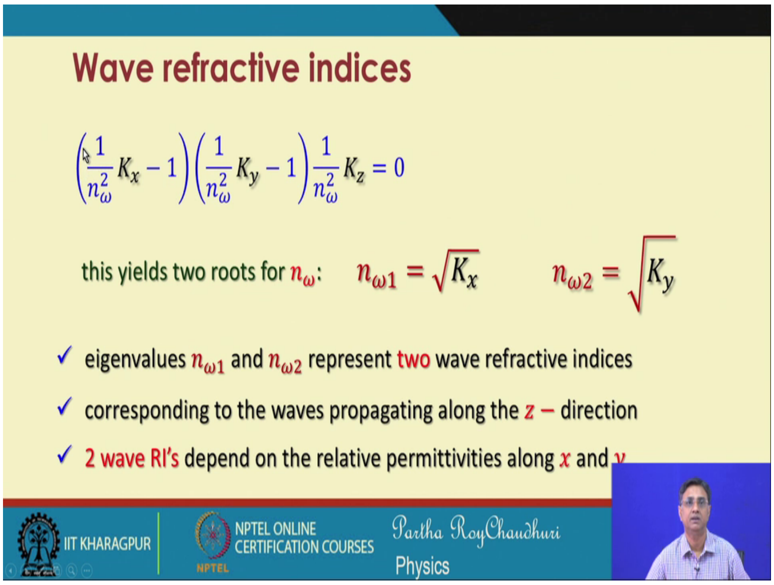

So, I have done that 1 by n omega square K x minus 1 into 1 by n n omega square K y minus 1 into 1 by n omega square K z equal to 0 because, this quantity does not have any minus 1; so, this into this into this the product of these 3 quantities. So, if you if you now multiply this will yield two roots of n omega which will be n omega because, anyway this quantity will be put equal to 0 or this quantity will be equal to 0 or this will be equal to 0.

But, in any case n omega is not 0 in general because we are looking for a finite value solution and K z cannot be 0. So, this quantity cannot be 0; so either this is equal to 0 or this bracketed quantity will be equal to 0. So, this the first one yields that the value of n omega must be equal to under root of K x and from the second one we find that the value of n omega 2 will be under root of K y.

So, we have two refractive indices for the case of wave propagation in the z direction and the wave is propagation the z direction, the refractive indices that is seen by wave is under root of K x and under root of K y. So, simply it indicates that the refractive indices are associated with the permittivity of the medium along x direction and along y direction, the wave is propagating along the z direction.

So, the eigenvalues n omega 1 and n omega 2 these two values are the two refractive indices seen by such a wave which is travelling along the z direction. And, the corresponding waves that represents that is that see is the refractive indices values of this and these are the two directions of the polarisations. The 2 wave refractive indices depend on the relative permittivities along x and along y. So, the wave is along z direction, but the refractive indices are due to x and y direction. So, this is very interesting.

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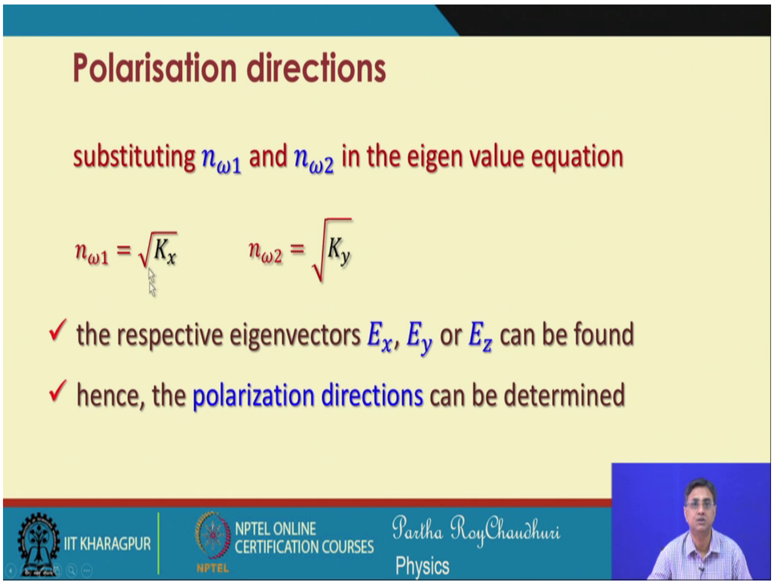

So, substituting the value of n omega 1 equal to this and n omega 2 equal to this in the eigenvalue equation we can find out the respective eigenvectors E x, E y and E z. So, this discussion is very interesting to know that given the refractive indices, then how we can obtain, we can find out the eigenvectors that is the direction of the E x, E y or E z whichever is surviving. and these directions will be called the polarisation directions.

So, by knowing their refractive indices we can calculate the direction of the electric fields and hence, we can also calculate the direction of polarisation of the wave when the direction of propagation of the wave is given.

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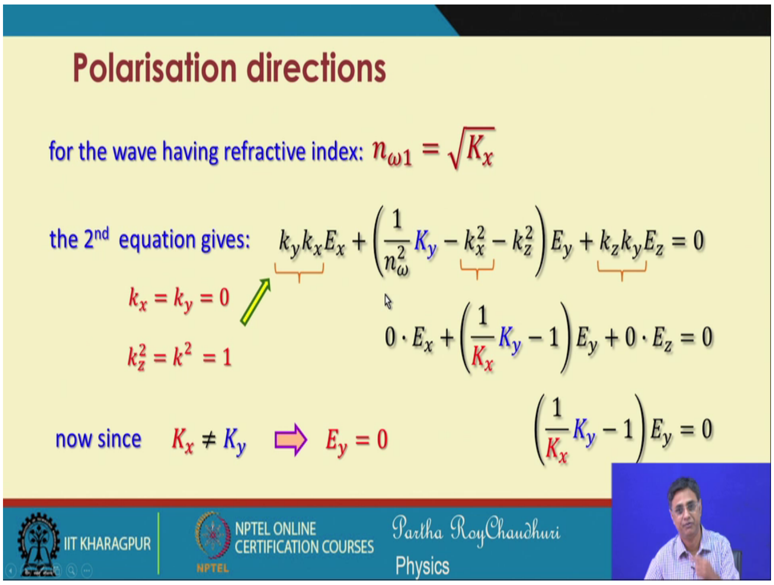

So, let us first use the value of n omega 1 equal to the under root of K of x that is the permittivity along the x direction. So, if we use this value in the second eigenvalue equation remember that second eigenvalue equation look like this. So, k y k x E x, but out of this because k x and k y are 0 so, this quantity will become 0. So, 0 into E x, but now k x square is also 0 so, this minus k z square k z square is equal to 1 so, this minus 1.

So, I have written exactly that, but n omega square I have use the value of n omega 1. So, that is equal to 1 by K x into K y minus 1 into E y equal to 0 and for the third terms, since k y equal to 0 so, we have 0 into E z equal to 0. So, this is the reduced form of the first equation where we have used the appropriate values of k x k y k z and also the n omega 1; the value of the n omega, then only this quantity has to be 0. If it is so, because K x and K y in general they are not equal; that means, E y must be equal to 0.

So, let us understand this situation that a wave is travelling along z direction and its see is a refractive index which corresponds to the permittivity along the x direction, then the field E y is not there. And because, it is propagating along the z direction, so therefore E z is also not there will see that. So, the only possibilities that E y is not equal E x is not equal to 0. So, this consequence of n omega 1 is equal to under root of K x the permittivity along the x direction is that E y cannot be existing. So, E y must be equal to

0. So, the wave is travelling along the z direction, the refractive index seen by this wave is n x and there is no E y. So, it is only the E x which is surviving.

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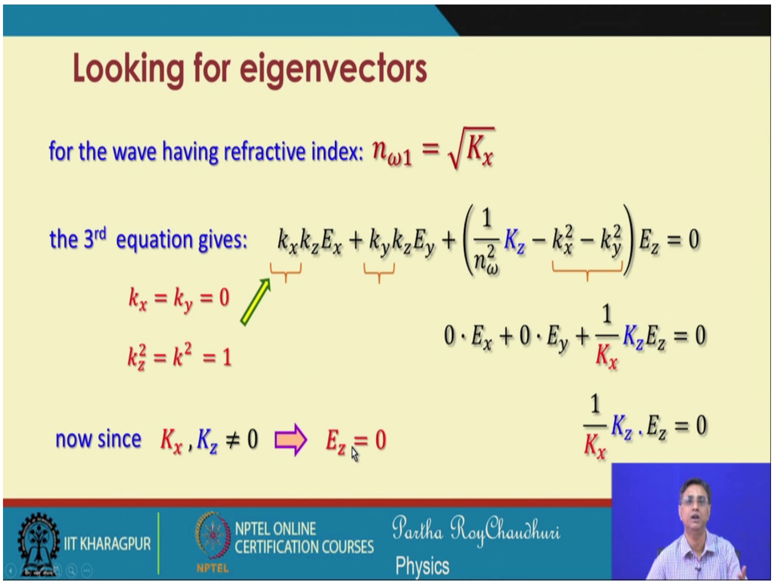

So, let us use this refractive index values the wave refractive index values when we plug in to the second third equation. Again this quantity is equal to 0 because k x equal to 0, k y is al is also equal to 0 so, this is equal to 0; so, 0 into E x 0 into E y and here this quantity and this quantity both of them or equal to 0. So, if we write this equation we get 0 into E x 0 into E y plus 1 by K x K z E z equal to 0.

That means, K z by K x into E z equal to 0; it simply tells you that because K z and K x they are whatever might be the value of this they cannot be 0. So, E z must be 0. So, for a wave we have seen two consequences one is E y cannot be 0, E z cannot be 0 the wave is travelling along the x direction. So, the only possibility is E x not equal to 0.

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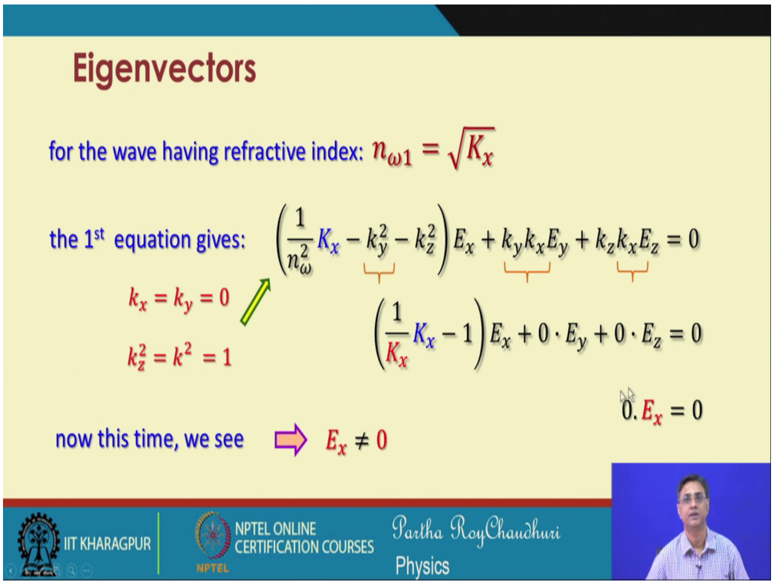

Let us see that when I use the first equation and I use the same refractive index values a wave refractive index then because this quantity is equal to 0, this is also equal to 0 and this equal to 0. So, in the same way we can write this equation that K x by K x minus 1 E x 0 into E y plus 0 into E z equal to 0. As a result this quantity because, this is 1 minus 1 equal to 0 so, 0 into E x is equal to 0; that means E x not equal to 0.

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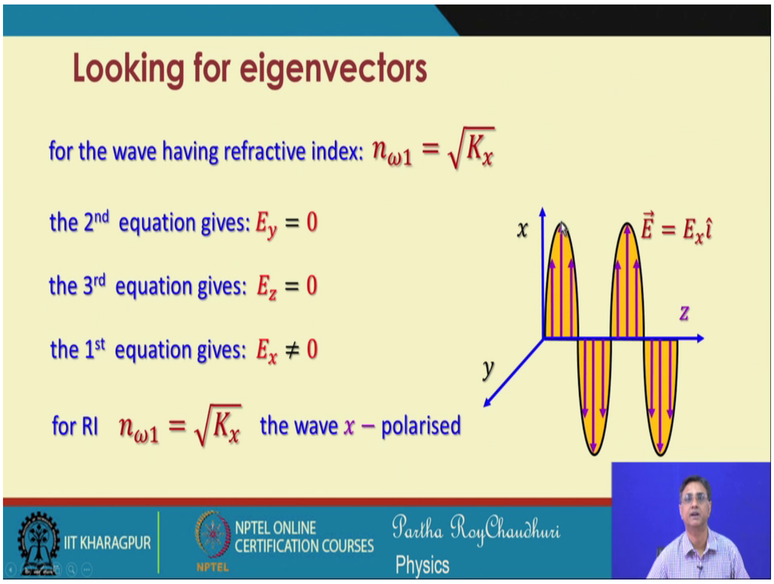

So therefore, therefore for a wave which is travelling along x direction, I find that the refractive index seen by this the wave refractive index seen by this waved is due to the electric field which is along the x direction. And the value of the refractive index is due to the permittivity for value of the medium along the x direction. So, the second equation gives E y equal to 0, third equation gives E z equal to 0 and the first equation gives E x not equal to 0.

So, you have only E x associated with the wave which is travelling along z direction, if the solution you assume is due to the permittivity along due to the refractive index seen by the wave along the x direction. So, such a wave this called x polarised because, the displacement vector or the electric field vector, they are along the same along this direction. For the wave having refractive index n omega 2 is equal to under root of K y, that is if I use the other solution that is the other root then an if we proceed in the same way that is k x is equal to k y equal to 0 this quantity become 0; we can write 0 into E x and similarly this quantities 0, but this is equal to 1.

So, we can write K y upon K y minus 1 into E y and for this quantity because k y equal to 0, we can write 0 into E z equal to 0. It simply means that this quantity 1 minus 1 which is equal to 0 into E y equal to 0; that means, E y cannot be 0. We can use the refractive index value in the other 2 equations, that is equation 1 and equation 2 to identify that in such a situation it is only E y that is not equal to 0, but E x and E z they are equal to 0.

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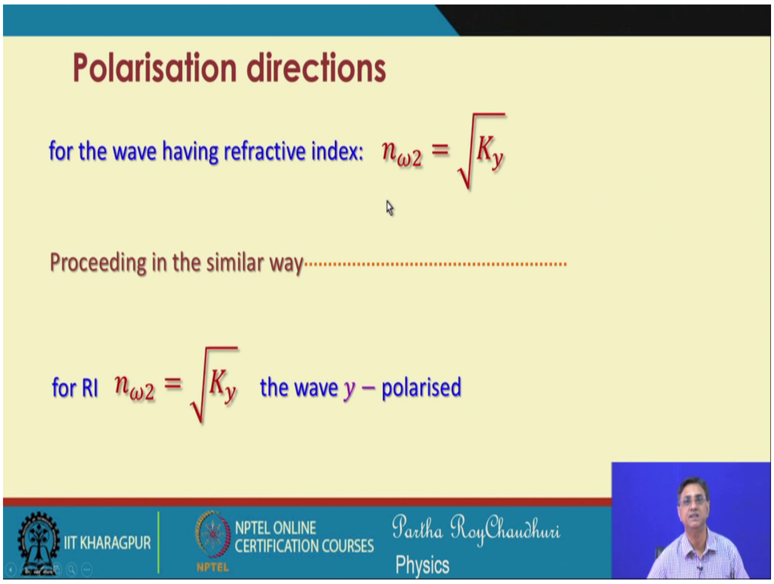

So, when you consider the property when you consider the solution for the wave refractive index n omega 2 is due to the permittivity along the y direction, then it is only the y field y electric field that survives and the propagation is along the z direction. So, the refractive index n omega 2 equal to K y it gives you a y polarised light.

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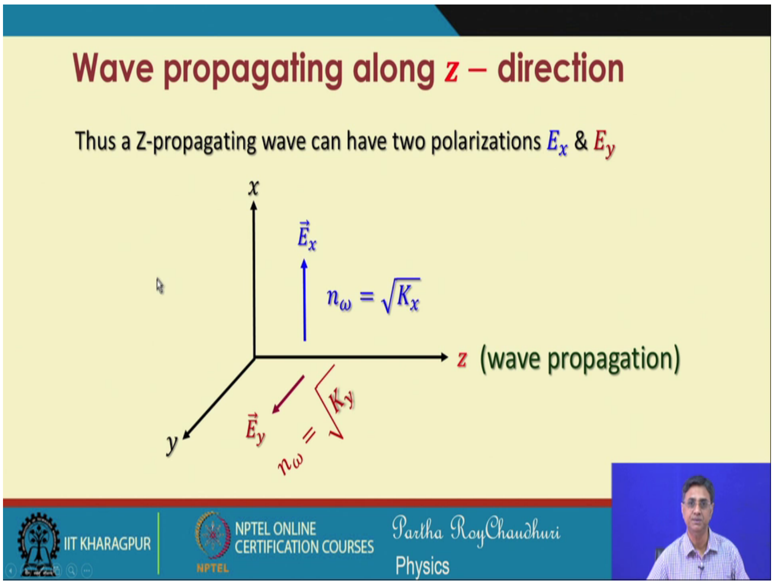

So, we can now find out that when the wave is travelling along the z direction, this wave can see a refractive index wave refractive index which is along the x direction. And, given by this permittivity square root and the electric field is along this x direction. This wave can have because, it has two solutions this wave can have a refractive index value associated with is with it is n omega equal to under root of K y and the electric field is directed along the E y.

So, this wave which is travelling along z direction can have two polarisations that is E x and E y. And, the corresponding refractive indices n omega 1 and n n omega 2 are given by these 2 equations. So, when the wave is travelling along travelling in the z direction with the electric filed E x that is this is the x polarised wave and this is the y polarised to wave. So, there are two possible polarisations associated with the wave which is travelling along a direction z.

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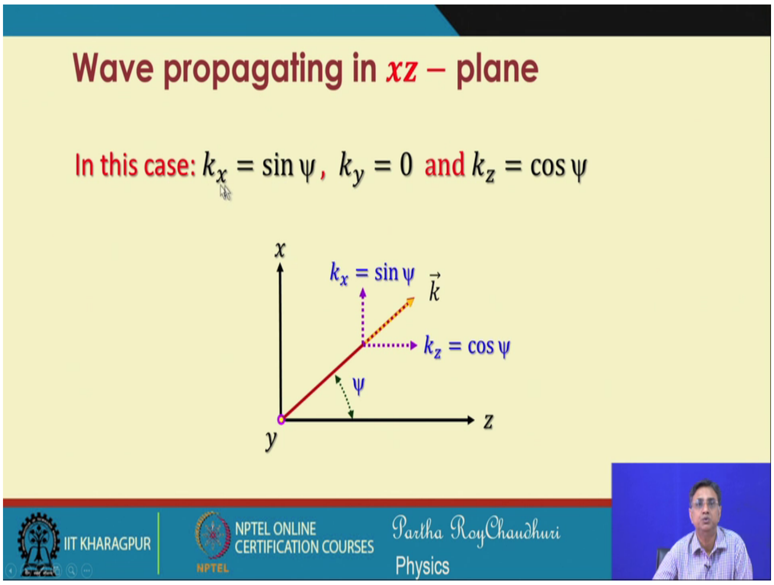

And now, we will consider the other situation that is when the travel wave is travelling in a in a general direction in the xz plane, that is it makes an angle of it makes an angle of phi equal to theta plus phi where theta is the angle of the point angle that the pointing vector makes with the z axis and phi is the angle that the direction the ray direction and the wave direction they make with each other. That is the wave direction is given by k vector whereas, the ray vector that the ray direction that is the pointing vector is given by S. So, this angle is phi the same notation that we have used earlier.

So, in this case when the wave is propagating along xz plane we can write that because, this is your k vector we can decompose the vector along the z direction, this is the angle psi. So, k z will be equal to cosine psi and k x will be equal to sin psi. So, now we will use the value of k x and k y and k z in the eigenvalue equation that is the determinant equation.

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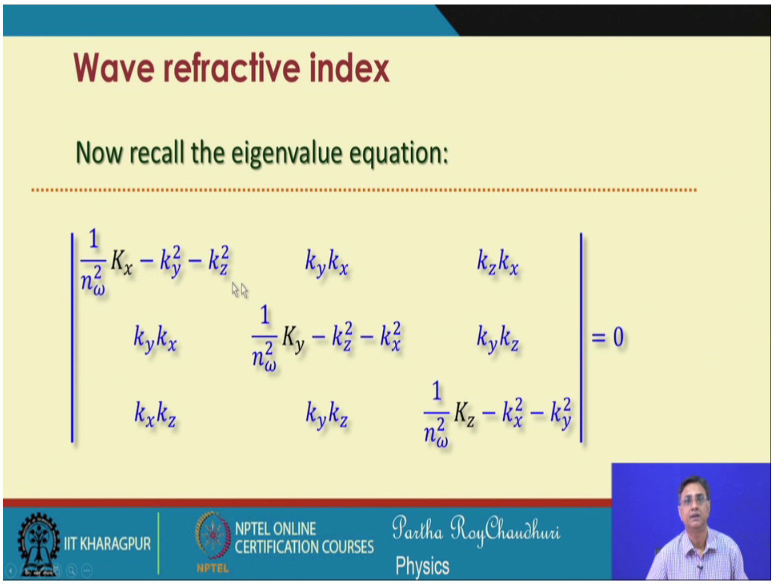

So, if I do that this K x k y and k z we plug-in to this equation, then it will take these values that 1 by n omega square K x, capital K x minus cosine square psi 0 sin psi cosine.

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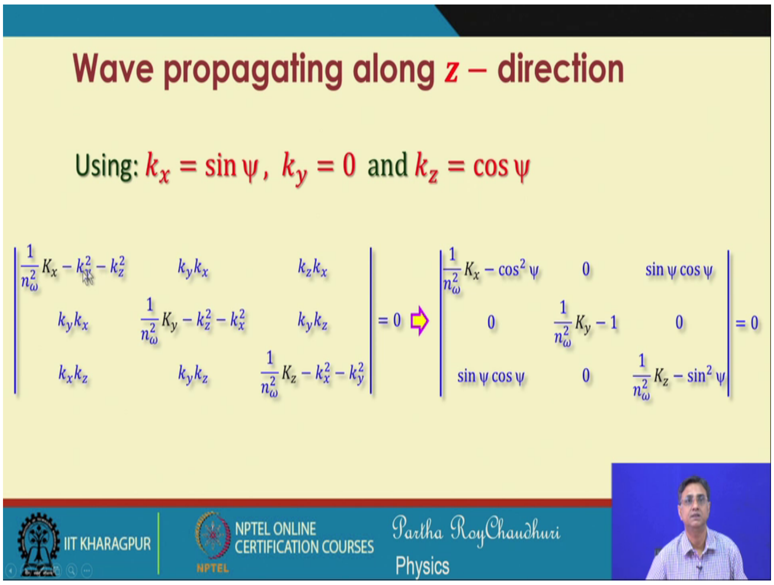

And similarly, this you can see the diagonal elements which are almost symmetric this x x component and z z component; they are the same. But this y y component is slightly different because, this k z and k x they are having the values whereas, k y this 0.

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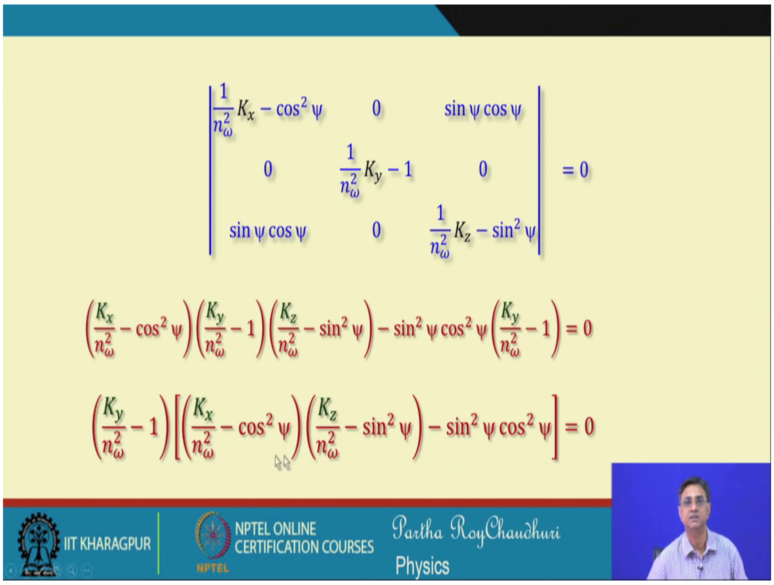

So, from this determinant equation if you open up this determinant we can write this equation in this from this into this into this minus 0. So, that is what I have written K x by n omega square minus cosine square psi and this. So, this into this into this I have written and there will be one more value that is for this into this.

So, we can write this equation because K x by n omega square no, K y n omega square minus 1 and K y by n omega square minus 1 it appears in both the terms. So, I take them out and we put these remaining quantities in the bracket that is K x by n omega square minus cosine square psi and this remaining part of this equation that is sine square psi cosine square psi equal to 0.

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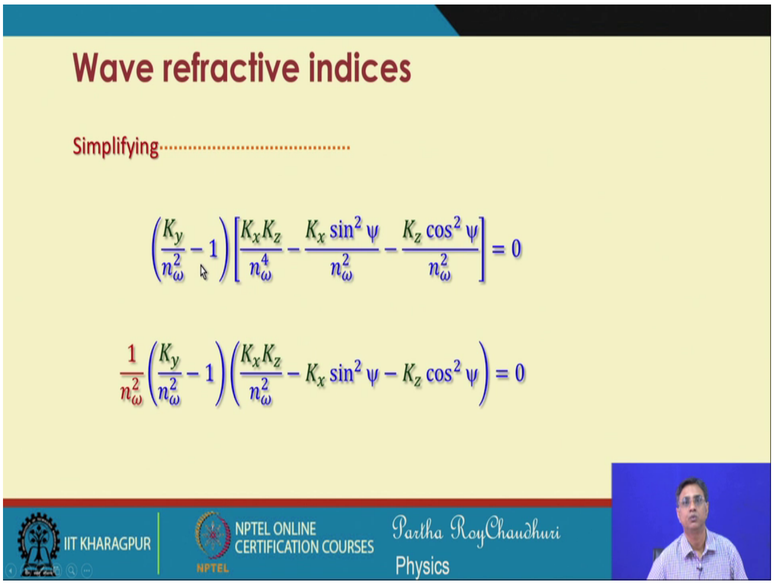

So, this reduced equation if I simplify then if I simplify this equation that is I again multiply this it will give you cosine square and sin square product which will cancel because, you have minus and minus input give you plus. So, sin square psi and cosine square psi will identically, when is there will be only x by n omega square sin psi K z by n omega square cosine psi with minus sin multiplied by this quantity.

So that is, what it appears that K y by n omega square minus 1 into this quantity is equal to 0. So, we get an equation which involves K x and K z because the wave is travelling in the x z plane. So, 1 by n omega square K y by n omega square minus 1 this quantity could be 0 or this quantity could be as well 0. So, we have two different possibilities the possibility that this quantity will be equal to 0; will give you a solution for n omega that is K y by n omega square minus 1 is equal to 0.

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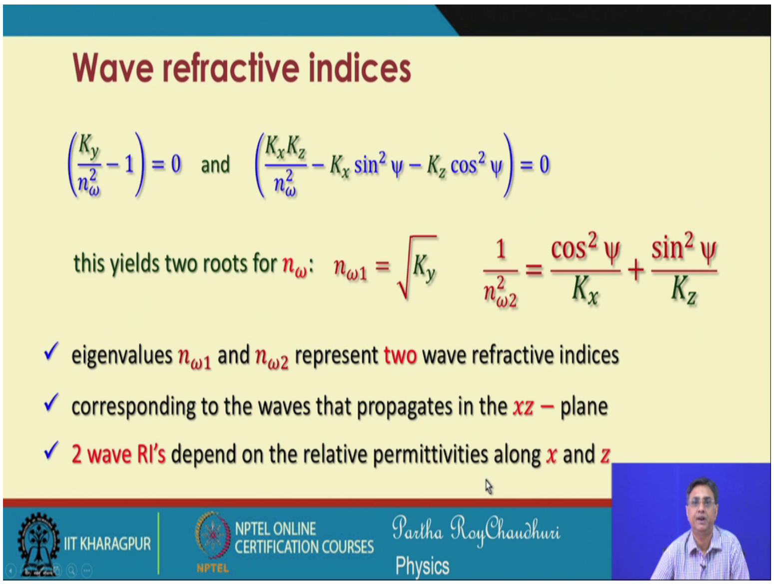

This will give you simply a solution that n omega 1 equal to K y. We will see what does it mean and in the case of this quantity becoming 0, we will see that n omega square 1 by n omega square will be equal to cosine square psi by K x plus sin square psi by K z. It is not simply the vector sum or the some of the square root of the permittivities. But, it enters into this equation in a very specific way in terms of cosine square psi and sin square psi.

So, the second root depends on K x and K y which involves the sin square and cosine square function of the angle that the wave makes with the ray direction in general in the xz plane. So, by doing this we have seen that there are two eigenvalues attached with this wave which is propagation in the xz plane. These are n omega 1 and n omega 2 and these two values are the two refractive indices seen by the wave. Corresponding to these two waves that that is propagation along xz plane there are 2 different relative permittvities corresponding to x and y x and z directions.

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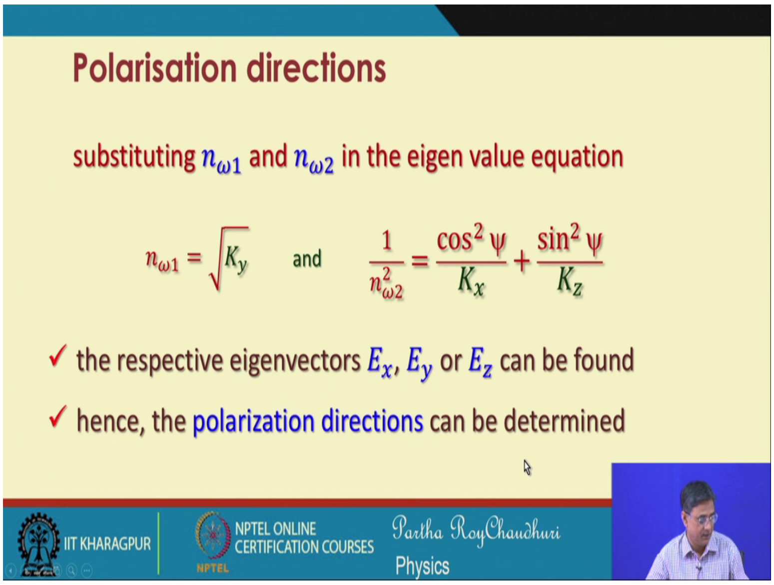

So, let us see that when you substitute this n omega 1 and n omega 2, these two values the two solutions of the determinant equation in the eigenvalue equation; then we can find that n omega 1 is equal to this and n omega 2 is equal to this quantity. So, these are the two respective eigenvalues for the wave which is propagating in the xz plane. And, the eigenvectors corresponding to these will be E x, E y and E z which can also be found out by the same wave, we have done it for the z propagating wave. And hence, the polarisation directions can also be determined by doing the solutions n omega 1 and n omega 2.

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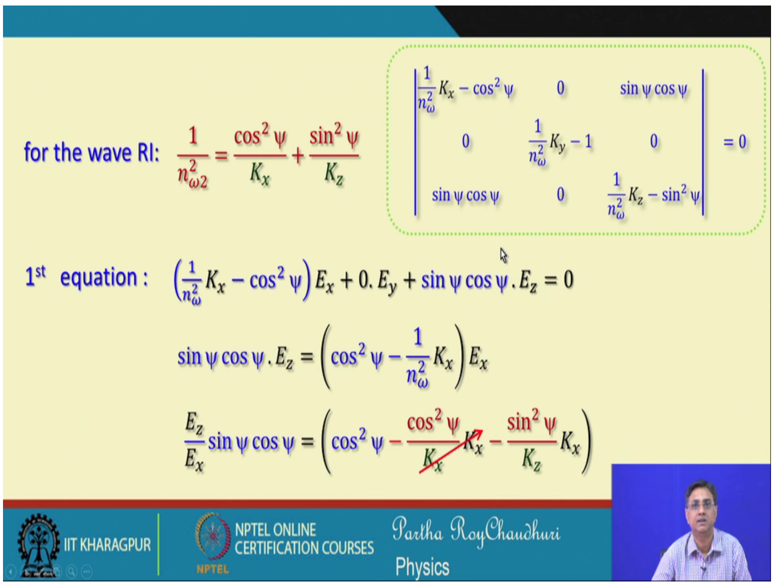

Let us try to look into that for the wave with refractive index n omega 1 that is the permittivity due to the y axis due along the y axis of the medium. Then if I use this solution in the second equation, you remember the second equation. So, we will we will get 0 into E x 0 into E x because, this equation itself is 0 and you have E x outside. This quantity into E y will be surviving and this is 0 into E z is equal to 0. So, in the second equation I have used the value of 1 by n omega square is equal to 1 by K y that directly tells you that K y by K y minus 1 into E y equal to 0. So, if this quantity 0 so, y cannot be 0 the wave is y polarised.

That means when the wave see the refractive index which is due to the refractive index of the medium along the y axis so, then E y field exists. So, E y field it is the direction of the electric field that decides the refractive index of the medium, that is seen by the wave when travelling along that medium. So, this waved E y which as a corresponding refractive index n omega 1 is y polarised.

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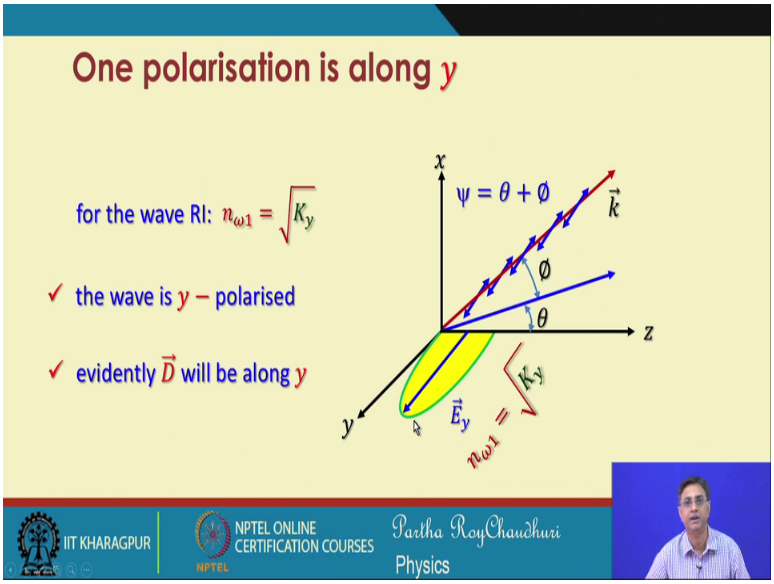

So, for one way for one solution of this wave is very straightforward that is that is when the refractive index is n omega 1 equal to under root K y, then the wave is y polarised. The wave is y polarised and it seen the refractive index K y and it makes an angle so, the other polarisation you will be this. Even though the wave is travelling along these direction, the direction of polarisation, the direction of the electric field vector is along the y z plane. And it is pointing along the y direction and the value of the refractive index is square root of K y.

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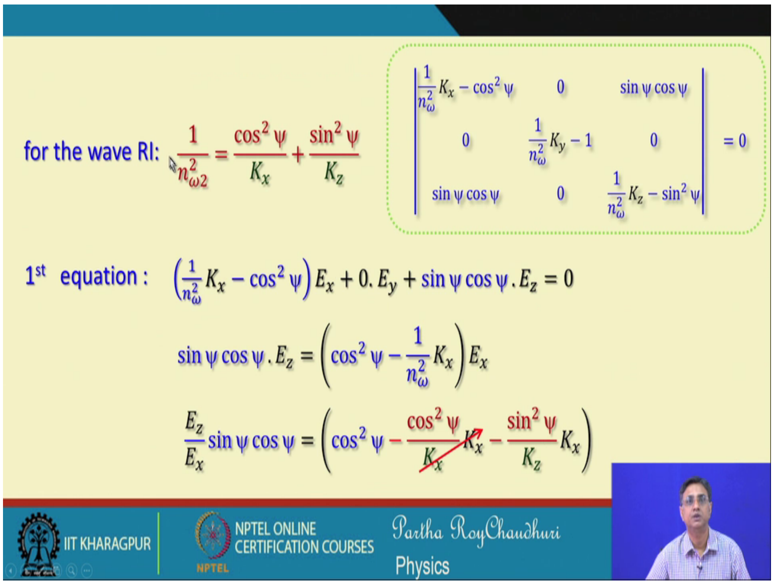

Now, for the second solution that is 1 by n omega square is equal to these if I now substitute into the first equation, you can see that this equation into E x second one second term is itself is 0 into E y and third term you have this. So, if you arrange this equation sin x cosine x into E z equal to this into E x. So, E z by E x into sin x cosine x you can see that this quantity identically vanishes, where you use the value of 1 by n omega square equal to this. So, they will vanish.

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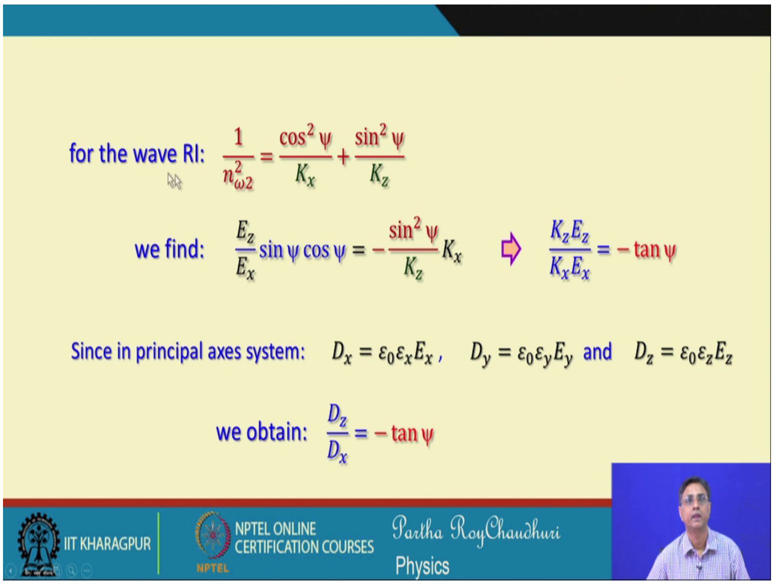

Therefore, for this value of the refractive index we find that E z by E x sin x equal to this and E z E x by K z E x equal to tan psi. So, it means that this quantity is the component of the dielectric displacement vector along the z direction and this is the component of the displacement vector along the x direction. So, they are inclined to each other at an angle of minus tan psi which is an interesting finding.

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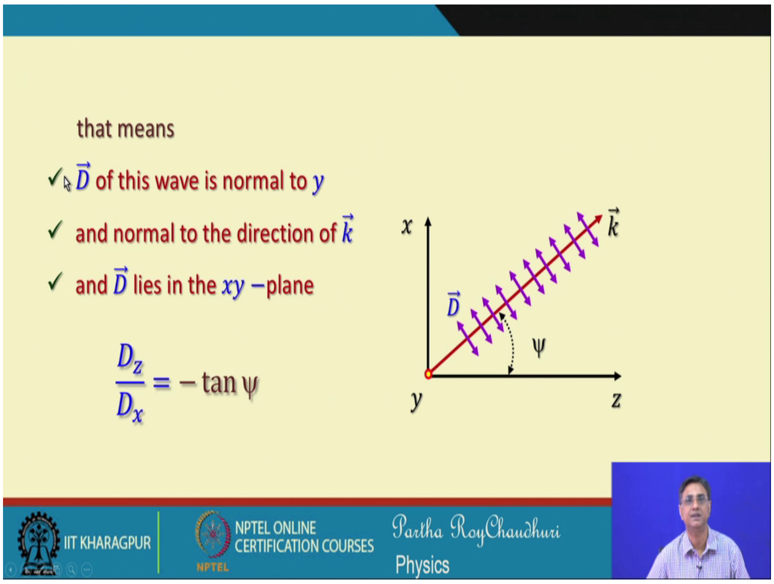

So, you can see that when the wave is if you consider the solution the solution, the second solution that is 1 by n omega square 2. This solution will give you a polarisation, where the electric field vector the displacement vector lies in general in the x z plane. And, the D z and D x if you decompose these into two components D x and D D x and D z, this is this will be D z.

So, because it will be directed in the negative direction of the z so, D z by D z will give you this value of minus tan psi. So, the displacement vectors are inclined at an angle of psi with the direction of propagation of the wave.

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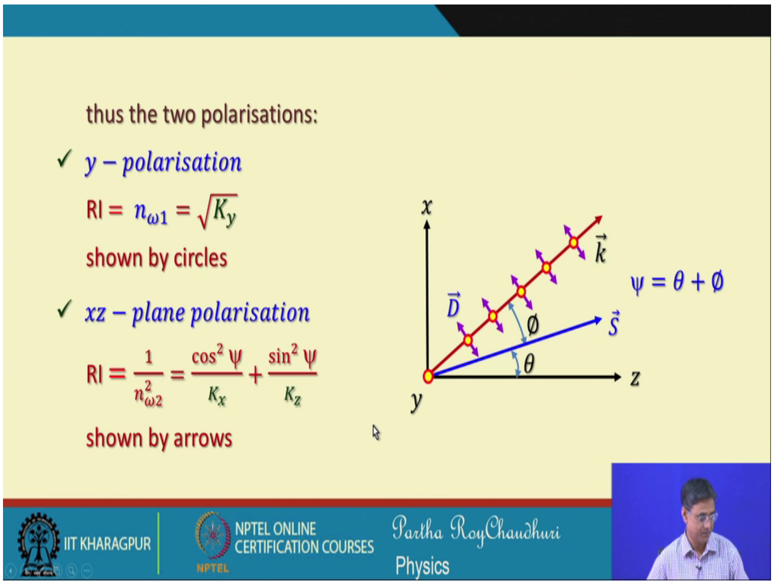

So, for y polarised wave you have a very specific case that it is directed along the y direction, but for the other polarisation which is in the in the x x z plane. So, thus yellow circles and this arrows denote the two polarisations corresponding to the two refractive indices seen by this wave.

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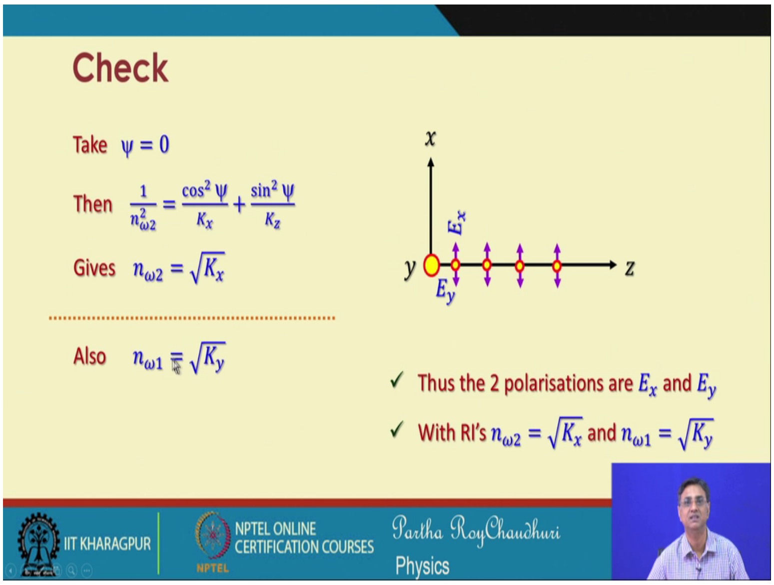

So, that is a check point that if you take psi equal to 0 then you can get this quantity equal to 1, this equal to 0. So, n omega 2 becomes exactly this. So that means, if you take that wave is propagating along z direction you end up with the earlier possibilities that the 2 polarisations are along x and y direction and these are the 2 refractive index indices.

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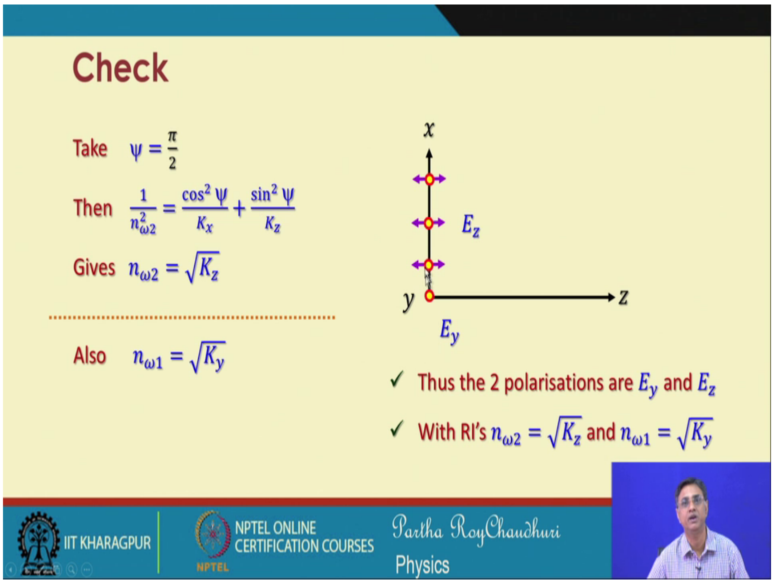

And if you put psi equal to pi by 2, that is the wave is now travelling along the x direction then putting this value this quantity will become 0. So, you will end up give n omega 2 equal to under root of K z n omega 1 equal to under root of K y. And, again you will get these 2 polarisations which are the eigen-polarisation that is it is along the z direction, another is along the y direction. So, these are the two different polarisations.

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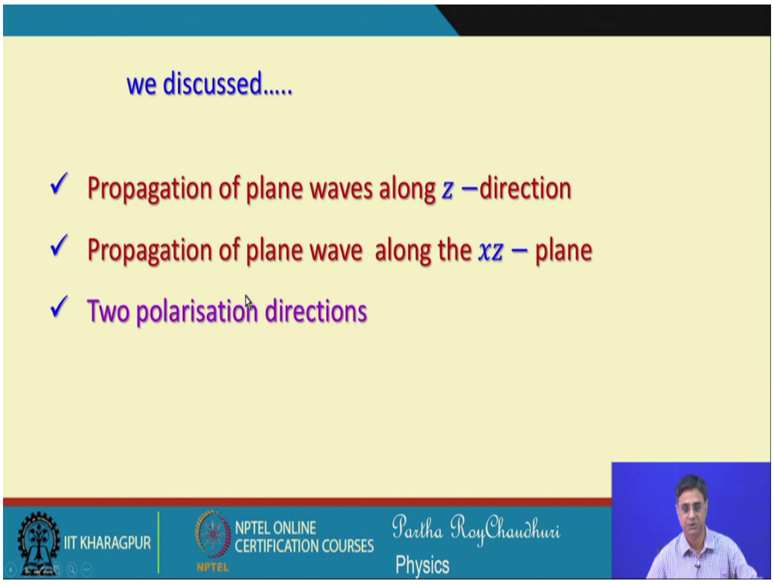

So, by doing this we could actually identify that what will be the direction of polarisation. What are the direction of the electric field vectors and a slightly different situation, when the wave is travelling along the xz plane what are the two different directions of the polarisations. So, in the next section we will continue with this.

Thank you.