Classical Mechanics: From Newtonian to Lagrangian Formulation

Prof. Debmalya Banerjee
Department of physics
Indian Institute of Technology, Kharagpur

Lecture – 09
Central force – 2

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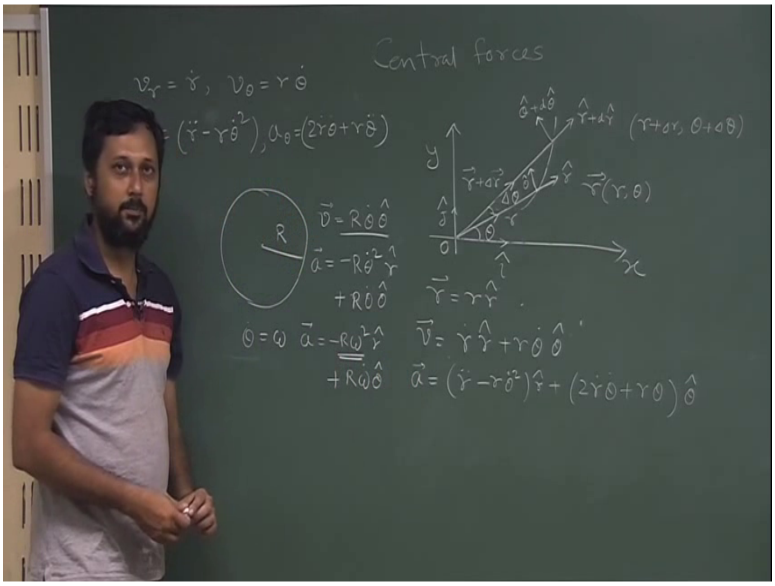

So, we have this two expressions; one for V which is r dot r cap plus r theta dot theta cap, and we have another expression for a, which is r double dot minus r theta dot dot square r cap plus 2 r dot theta dot plus r theta double dot theta cap. So, we write those as components of velocity radial and transverse component of velocity, and radial and radial and transverse component of acceleration, and we just separate it out for future. We will just keep it here in this side for future reference. So, v r is equal to r dot v theta is equal to r theta dot ok.

A r is equal to r double dot minus r theta dot square, a theta is equal to 2 r dot theta dot plus r theta dot. Now what is interesting to note is that, let us assume that particle which is moving in a circular orbit, if a particle moves in a circular orbit, then its radius is not changing; that means, r dot is equal to 0. So, generally if we so, but immediately we see that r theta dot is also changing if it is moving in a circular orbit, then theta is changing.

So, r theta dot is non zero.

So, although it is, there is a the in this particular motion, the radial velocity component is 0, we have a transverse velocity component and the overall velocity vector is a non zero vector. Similarly even if the acceleration due to one of, let us say for the same motion r double dot is not changing. So, we have 0 here, but we still have a radial component of acceleration and the angular component of acceleration is always there, because this term will be equal to 0, and this term will be non zero.

So, we see that for motion on a circular orbit. Let us say if a particle moving on this circle with a fixed radius R, the velocity will be v equal to R theta dot theta cap, and acceleration will be equal to minus r or R theta dot square r cap plus r theta double dot theta cap. Now what is interesting to note here is, there is no radial moment, because it is always on this circular orbit. Now if there is no radial moment, we immediately see that r component of velocity vanishes and we are left with only the theta component of velocity, but in acceleration, although there is no red, no radial movement, we have a radial component of acceleration. And if we write theta dot, let us say theta dot, if we just write it as omega, we immediately see this acceleration we can write as a equal to minus R omega square r cap plus. Sorry simply R omega dot theta cap.

And this is nothing, but the well known expression for the centrifugal acceleration. So, although the radius is not changing, there is no R movement. We have R component of acceleration, which is very evident from this calculation, and also we are familiar with this, because this is the stand. I mean this is the standard expression for centrifugal force fine. So, now, let us look in to the second component of theta. Sorry acceleration little more carefully we just take this.

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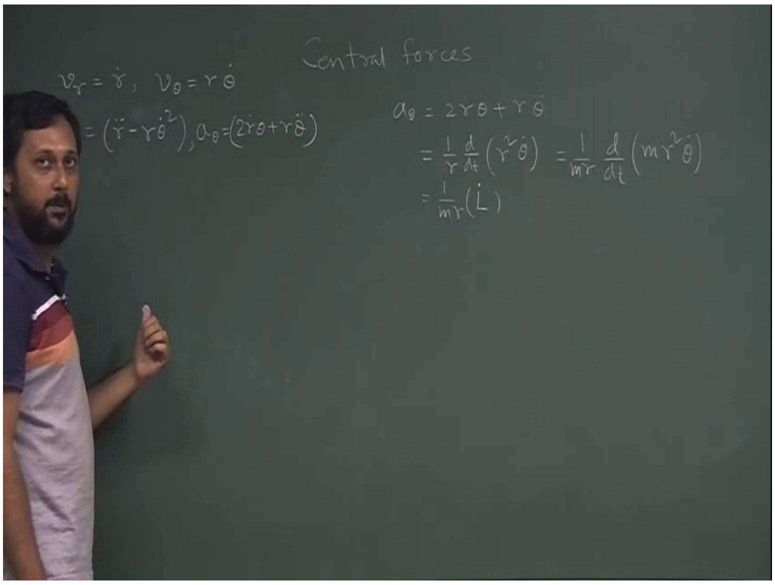

So, a theta is equal to 2 r dot theta dot plus r theta double dot, and this can be written as d d t of r square theta dot. If you compute this time derivative, the first term will be. I mean if you just compute this and divide the whole thing by r, you will get back this expression. Now velocity, look at this velocity component of along theta, which is r theta dot. Now if this is your origin, if a particle has a velocity v. So, it has two components in this particular representation; one component is v r 1 component is v theta ok.

And this v theta which is given as r theta dot is working in this particular direction, which is at a distance r from the origin. So, if we now want to calculate the angular momentum l, which is given as r cross P will be m r square theta dot. Now which will, sorry with this will have magnitude of m r square theta dot, and we simply call this one. So, this can be rewritten in terms of 1 by r d d t of or put an m here m r square theta dot, which will be written as 1 by m r theta dot. Oh sorry it is not visible in this. So, we will just write this as 1 by m r L dot. So, we say that the angular component or the transverse component of acceleration can also be written equally well in terms of the angular momentum. The magnitude of the angular momentum, and of course, it is pointing in the direction of theta dot theta cap fine.

Now, let us try to write the expression for. So, we will just keep it here.

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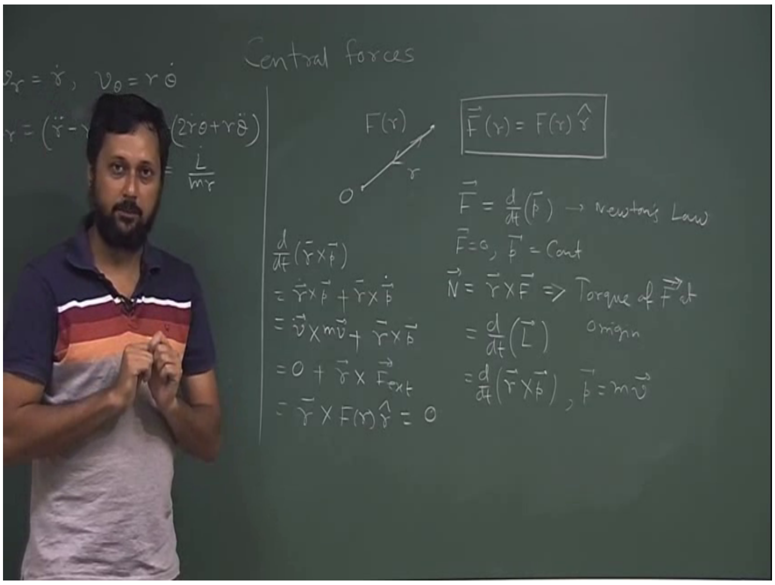

So, we will just write L dot by m r and remove this, now central force which is a force always either along or away from a fixed point at a distance r. So, it is either pointing towards the fixed point, or moving away from the fixed point. And if you remember the magnitude of the force is a function of r only; that is the property of central force. There cannot be any theta dependence, there cannot be any other dependence except from the except, or dependence on the distance of this force. Now we are not talking about the general nature of the force right, could be, I mean. So, sorry right now we are discussing the general nature of the force, we are not talking about any specific force component. So, F r can be written as F r r cap does not matter, whether it is an attractive force or a repulsive force. It has a magnitude, which depends on this distance r only, and it has a direction which is either pointing towards the fixed point o or moving away from the fixed point o irrespective, this is the standard notation we use for central forces.

Now, if this is the nature of the force, then once again we try to write the torque. So, we have newton’s law as F equal to m d V d t or more fundamentally, as we discussed in one of our previous classes d d t of p. So, this is from Newton’s second law, and we immediately see the immediate conclusion is, if F is equal to 0 p is equal to constant. we can write a similar relation in terms of angular momentum, we just have to replace this left hand side with n and that, let us do it actually systematically n is equal to r cross F it is the torque of F, at origin torque of this force F at origin is given as n. Now if we take d n d t which will be d d t of r cross F, and there is a relation that says d n d t is equal to and there is a relation that says n is equal to d d t of L, L being the angular momentum, which is defined as r cross p being the linear momentum ok.

So, if we have torque equal to 0, we immediately see L is a constant, and let us do one more step, and we will immediately realise that n has to be equal to 0 r cross p is equal to r dot cross p plus r cross p dot. Now please remember p is equal to m v, assuming that the mass of the system is not changing. Please remember we are, please understand this we are back into the systems, we are in the description, where mass is the constant. So, there is no time derivative for the mass term here. So, angular momentum m the linear. Sorry the linear momentum P dot is nothing, but this is v, this is m v, this is r cross m V dot. So, there is no time variation of mass. Now V cross V will always give you 0, and we have r cross. Yeah r cross. Now m V dot is p dot which is F external. Actually we do not have to write this step, we can simply write this as p dot, and this is r cross. Now using this form which is once again equal to 0 ok.

So, the right side of this equation will always vanish, which essentially means the. Sorry the time derivative of this angular momentum will always vanish for a central force, because we have this particular form of the force, which is along r. So, r cross r cap will always give you 0 right. So, that is from the basic vector calculus. So, cross product of two vectors pointing in the same direction will always vanish. So, if that is the case we immediately see that for a central force, force field d d t of L is always equal to 0.

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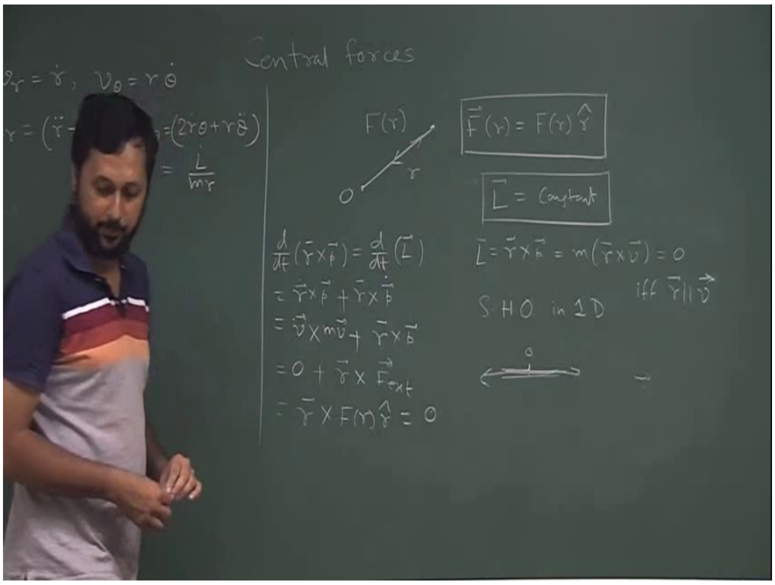

Which essentially means L is equal to r constant, and this is the vector equation d d t of. So, here we are computing d d t of l, which is r cross P and that gives us L is a constant. Does not matter what is a particular form of this function F of r does not matter, because it is pointing only towards r, because of this particular form of central force, L will always come out as a constant. So, this is a very important result which essentially tells us that the angular momentum of a particle moving under the influence of a central force is 0. Sorry is constant. Now there is very special case, where L can also be equal to 0, because L is given as r cross p, which is m times r cross v. Now if for the particular case for a particular type of motion r and V is in the same direction, then this term will also vanish. I mean this term will vanish, and we will have L equal to 0, if and only if r is parallel to v. Now that happens when we have simple harmonic motion, or simple harmonic oscillation, if you want to call it, what happens in a simple oscillation in 1 d.

We have a particle which is, let us say this is our force centre o and a particle, which is moving along this line like this. So, r is along measured either. I mean measure along this length or this length, and v is also in this range. So, r and v are always parallel. So, we get a special case of simple harmonic oscillation, where L is equal to not only a constant, but it is always equal to 0. So, that is special case.

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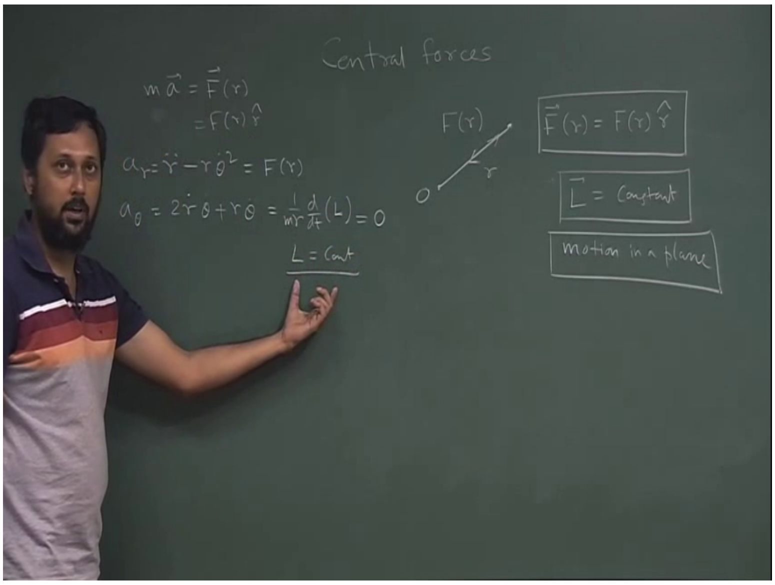

So, in general if we have a motion under central force; that means, L is equal to constant. Now this L being constant, immediately tells us that the motion is also a planar motion, how L is equal to r cross p right. So, let us say we have a. The vector r is the position vector, and P is the momentum at any given point. Now L being constant that essentially means that it is not only the magnitude of the vector l, as I have discussed already in this class; that is not only magnitude of the vector, but also the direction towards, which is towards which is pointing is a fixed direction in space. Now if you examine this term P is the momentum, linear momentum of this particle, which is moving under the direction of the central force, and r is the radial vector, which connects that to the origin. Now if and only if r and p lie in the same plane, let us assume this situation, it is a some locus of the particle. So, this is your r and at any given movement the velocity vector is pointing towards the tangent of this locus ok.

Now, r cross v. So, r cross p essentially means m times r cross v. So, r cross v, if it has to have a fixed direction, it is necessary that r and v they lie in the same plane, if not the magnitude might remain the same, but if it is not in the same plane, the direction of r cross v will has to vary. Now if they stay in the same plane, then r cross v has a direction in this particular case, which is pointing inside this plane, if v is. Let us say if the particle is moving from this direction to this. I mean this way. So, v will be like this, and r cross v will be pointing outward of this plane, but irrespective L is either out of the plane or inside this plane, and if the direction has to maintain r and v has to be on this particular plane of the board. In this case if there somehow leaving out of this plane, the direction of L will also change. So, we see that as r is given. Sorry L is given by r cross v L equal to constant means r and v has to be in the same plane. So, that essentially means the motion has to be a planar motion ok.

So, we immediately see that the two properties of central force, which is namely motion in a plane, and L being a constant vector is a immediate conclusion of F being, having this particular from F r times r cap. So, this three are essentially related to each other, and also the last property or the third property, which I also stated; that is an energy conservation. We will see that, because these central forces F r, I mean there is a, these are all conservative forces by nature, and there is a potential function associated with it. It is obvious that the energy will be conservatives.

So, essentially proved all the properties of central forces, now let us try to look at the equation of motion, but before that we also need to prove another very important aspect of. Let us first do the equation of motion, then we will come back to this. So, equation of motion is m a equal to F of r which is F r r cap. So, if we go by the component form, we immediately see we have two equations which is r double dot minus r theta dot square, which is a r equal to F r r cap or r cap is not needed, because we are writing the component equation, and a theta equal to two r dot theta dot plus r theta double dot, which can be written as one by m r d d t of l. We already proved that L is equal to constant that. So, no surprise that this whole, this is equal to 0.

So, even this equation itself implies that L is a constant, because a theta essentially we have seen that it can be written as one by m r d d t of L. Sorry there is no vector here. So, this essentially tells us L is a constant, but please remember that this L is a magnitude only. So, it only gives. So, this particular equation, if we consider that d d t of L is equal to 0, it only gives us half of the information, the complete information lies here, where we say not only the magnitude, but also the vector is equal to constant fine, but now if we try to solve this two sets of equation. This equation essentially does not give us anything new, because it only tells that magnitude L equal to constant, which is also basic property of the central force which we know already. So, we will just leave this from the question.

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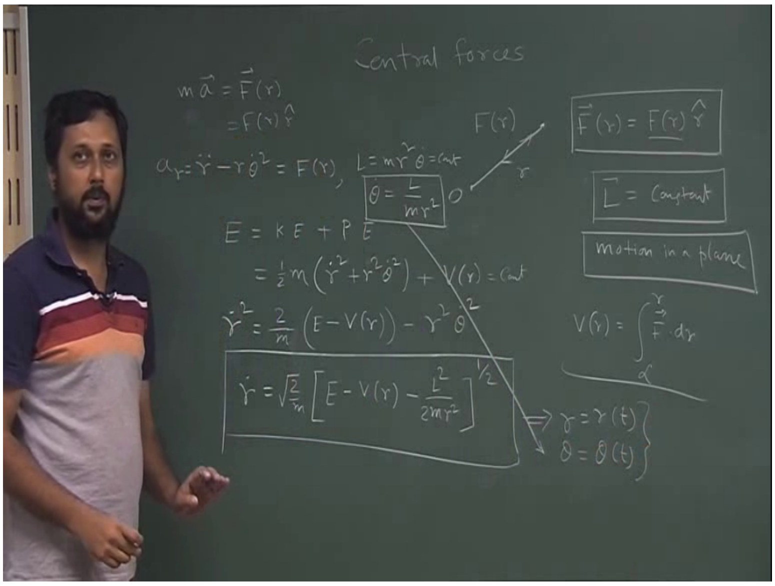

Now, if we try to solve this equation, we have a second order differential equation in theta, and first order differential in, sorry second order differential in r and first order equation in theta, first order differential in theta. It is extremely difficult to solve in this particular form, what you can do is. We can try to write the total energy, which is a constant as a sum of kinetic energy plus potential energy, and kinetic energy will be half m V square V square is r dot square plus r r square theta dot square plus potential energy is V of r, where V of r is the potential function, which is given as integration F dot d r from infinity to r. So, potential by definition is the work done from bring the, bringing this particle in question from infinite distance to a distance r that will give you the potential at a point r.

So, this is a V r, assuming that we know that functional form of F r. We also have knowledge of V r and this whole thing is a constant, because E is a constant. Now if we slightly rearrange this equation here, and please keep in mind that L is equal to m r square theta dot is a also a constant. Now if we rearrange this bit what do we see? We can immediately see that r dot square is equal to 2 by m theta dot square. Oh sorry 2 by m E minus V r minus r square theta dot square, and if we substitute theta dot is equal to L by m r square from this relation, we see that r dot is equal to root over 2 by m E minus V r minus. I will just give you the final form s square by 2 m r square whole to the power half.

So, we have one equation for r dot and one equation for theta dot. So, in principle we can first in principle. Again I am not doing, trying. I am not asking you to do it or I am not doing it myself, either we can integrate this equation, because this is E is a constant V r is a function, which we know from this relation L is a constant. So, we have a right hand side which is function of r only, and in left hand side we have d r d t. So, we can integrate this equation to give get r as r of t, then we can put it back in here, and we can integrate this equation to have theta as. So, this equation will give us theta as theta of t, and the trajectory is solved. So, what we need to know. We in order to solve the equation of motion we need to know the total knowledge of r and theta in terms of time right. So, if we can integrate this equation, and this equation one after the other in order to get r as r of theta, and theta as r as function of t, and theta as the function of t, we essentially solve the problem completely.

But as you can guess by looking at this integration this particular form may be under certain circumstances. This integration is easy to solve, but in general it is a very complicated equation to solve, and then we have to get not only this, we have to get the functional form of r plug it back in here, and we have to solve for theta. So, it might be easy, if when we have a computer simulation. I mean when we are running a computer simulation that might an easy way, but at least at this level, when we are trying to solve problem in the class room, and trying to get an insight of the things. This form is not the best form of equation, say equation of forces we can have for central forces.

So, in the next class, what we going to do is. We are going to develop a, or rather we will start with this existing equation, and we will try to modify it in a smart way, so that we can get something, which is more easily understandable and more easily solvable.