Classical Mechanics: From Newtonian to Lagrangian Formulation

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Lecture – 08
Central force – 1

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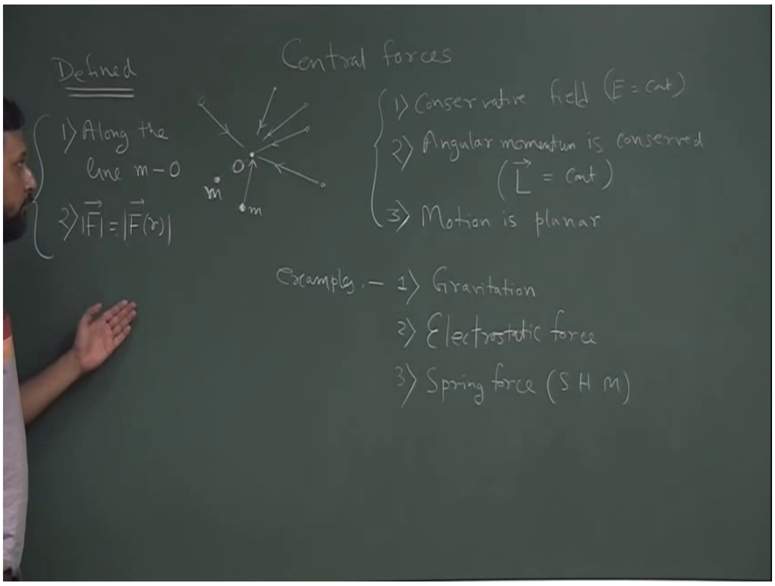

So, today we start our discussion about central orbit. Central forces is family of force field which has the following property. So, let us assume; there is a particular type force which is pointing towards this particular fixed point or pointing away from this particular fixed point, but let us say this there is a force field with the origin o and there particle is there are some particles moving in this system. Now if the force on this particle where ever it is does not matter, if it is here or here or here or here; if the force is always pointing towards this fixed point or let us say from here or if its pointing out of the fixed point that is also valid central field. So, there are called the central forces. Now central forces essentially means that it should always point towards point away from a force field on a fixed point; that means, essentially; it will work along the origin along a line joining the origin and the line joining the origin and the object in question.

So, if this is the point mass in question, then it will be the force will be along this; if this point mass is question, then the force will be along this line if this one is in question. So, it will be along this line if this one is in question, then it will be along that line could be attractive could be repulsive, but it will always be along the line joining this point mass m with origin o.

Now also the magnitude of the force is a function of this distance only, it does not matter; if it if this particle is here or maintaining the same distance, if it is here; the magnitude of the force on this particular m; particular mass point, m will always be one along the line m to o and 2 F will be the magnitude of the force will be proportional to or will be a function of r. So, these 2 properties if certain force has these 2 properties then this is called a central force now there is a very special class of property which is associated with the with associated with the motion of a particle in a central force field and this class of properties first of all central force field by nature is conservative field.

Secondly it will be shown latter on that if a particle is moving under the action of a central force field and if there are no other forces acting on this particle at that time of this motion then the total or rather, I will just write angular momentum the momentum is conserved. So, conservative force field the first point essentially means e is equal to a constant total energy of the force field is a constant. Now angular momentum; momentum is conserved that essentially means L is equal to constant we write L for the angular momentum vector please understand that angular momentum is not a scalar energy is a scalar quantity. So, energy is a scalar quantity. So, energy is constant means e as a number as a number which represents the total energy is constant, but angular momentum conservation essentially means L as a vector is a constant; that means, it is not only the magnitude of that vector, but also the direction of that particular vector in space is a constant.

And number 3 is motion is planer so; that means, the motion of this particle m for example, if it is due to this central force where the origin of the force is o let us say. So, if the motion of this particle is due to this central force alone. Please remember, if there is any other force acting on this particle at the same time which is also possible. So, then we cannot comment on this motion and these 3 sets of properties will not be valid, but if the force acting on this particle is only the central force then we can safely say that the motion will be confined to a plane let us the plane of this board and this 3 properties can be proved will prove 2 of the properties in near future once we learn some of the required mathematical techniques energy conservation can also be proved very easily.

So, let us give some examples very common and we all know about it; gravitational attraction is central force; if we have 2 point masses; let us say, yeah. So, let us say; 2 extended mass bodies I mean masses. So, they the gravitational attractions between these 2 masses are always pointing along the line joining the central of forces centre of masses of these 2 these 2 bodies. Similarly we can also take another very common example that is coulomb attraction or repulsion.

So, or let us try it is not coulomb attraction or repulsion we will write it as electrostatic; electrostatic force which is also an example of central force and number 3 is another example is the spring force; spring force is the force which let us say we have the spring mass system and it is oscillating only in the influence of this spring force then we have simple harmonic motion which is also. So, the spring force will see very soon that this spring force is also a special type of central force. So, these are the 3 very common examples we are all familiar with gravitational electrostatic and spring force and all these 3 are central forces.

So, we first started by defining we have defined central forces by this 2 parameters when the force. So, the force has to be along a straight line joining a force centre and the mass point and the magnitude of the force is a function the distance alone then we have given some properties. So, this properties will systematically one by one in near future and then we have given some examples now as we see that we do not know at this point, but you have to trust me that we know in advance that motion is planar. Now as the motion is planar there is no reason that we. So, so if the point mass is like moving on a particular plane how many coordinates we need to describe it we need 2 coordinates in general in Cartesian coordinate system or spherical polar coordinate system, they deal with 3 parameters let us say if it is Cartesian, then we have x, y and z; if it is plane polar coordinate system then we have r we have theta and we have phi.

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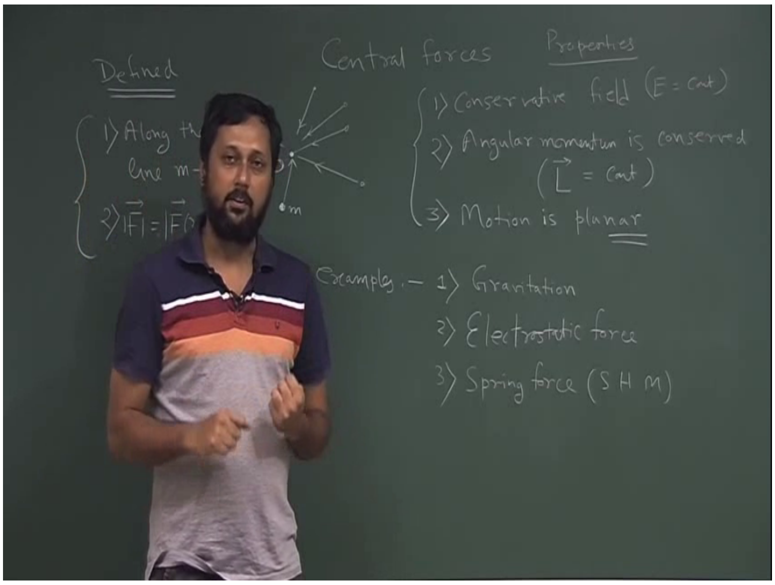

So, in either case, we have 3 coordinate systems, but in this particular case because the motion is a planar motion we can bring it down to a 2 coordinate system and we can just work with plane polar coordinates.

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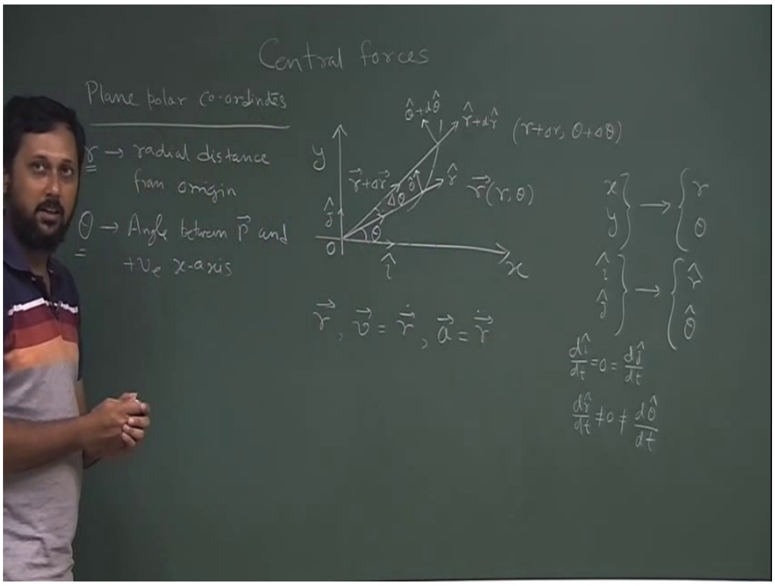

So, let us start our discussions little bit of details about plane polar coordinates and then we will come back and prove this 2 properties of central force one by one now. So, let us say we are we have a Cartesian coordinate system x and y and a particle is moving in some path in this coordinate system. So, if this is the position vector r of the particle at time t and if this one is the position vector r plus d r in time t plus d t then. So, we can always express this r as a coordinate point x y and express this one as a coordinate point which is which can be given as x plus delta x and y plus delta y. So, this is how we represent the motion of a particle or a trajectory of a particle in x y coordinate system.

Now, what we can do is essentially we can introduce a new coordinate system which is called the plane polar coordinate system which will also have 2 parameters, but instead of x and y we will call them r and theta. So, probably you are familiar with it, but still let me go little bit into the details of this. So, let us say instead of writing r in terms of x and y we are writing it in terms of r and theta r being the vector and this is the vector r plus delta r which is given as x plus delta x plus y plus delta y and we instead of writing it x and x plus delta x; we just write it as r plus delta r and theta plus delta theta; what is r and theta r being the distance from this origin to this particular point and theta being the angle it makes with the positive x direction. So, we have r which is radial distance from origin and we have theta which is angle between the vector r and positive x axis. So, r plus delta r will be this distance r is simply this length r plus delta r is this length theta is this and this angle has to be delta theta sorry, sorry, it is so, nicely written first yeah.

So, now we have defined the parameters now if you recall in Cartesian coordinate system we have the unit vectors i cap and j cap, right. Now in this case also, if we are moving into plane polar coordinate systems and we have to write vector equations. So, we need to have unit vectors and so; that means, we have to define 2 unit vectors; one is along one; one is for r and one is for theta now it. So, happens that writing the vector unit vector along r is trivial because direction will be the direction radial direction which will be pointing outward from this origin.

So, we call this like we have i and j. So, let us say when we have x y in Cartesian coordinate system, we have equivalently r theta n polar coordinate system when we have i cap j cap in Cartesian coordinate systems we equivalently have r cap and theta cap in polar coordinate systems. So, r cap being the unit vector along r and theta cap being the unit vector along theta. So, r cap is defined as the radical. So, essentially the unit vectors are we the by definition unit vectors are vectors of unit length pointing to a particular direction.

So, in this case, r cap points towards the radical direction radially outward direction sorry; radially outward direction and theta points to a direction which is perpendicular to r cap. So, this is the direction of theta cap fine. Now what is so different about it I mean again there are 2 unit vectors and i and i cap and j cap are also unit vectors, but the unique property of this is here please understand this we can have different point location in a Cartesian coordinate systems expressed in terms of i cap and j cap. So, that I cap and j cap are fixed in space; we know that whichever point, we I mean which ever vector we write we always know that the i cap component means the component along x and j com j cap component means it is the component along y.

But here as we move on along the trajectory of the particle as we move along the trajectory of the particle let us say when we go from r theta to r plus d r L theta plus d theta your radial the direction of this unit vectors also changes. So, at this point if we try to at this particular point if we try to draw the unit vectors we see that it is a new r cap. So, we just in sorry; nicely drawn it has to be along this line. So, we call this r cap plus d r cap; that means, r cap is not a fixed direction, but it is also a changing right now let us say at time t it is given as r cap in time t plus d t it is given as r cap plus d r cap.

Similarly, theta cap here is given as theta cap plus d theta cap. So, this is a very unique property for this thing because if we take the time derivative of d i cap d t which will be identically equal to 0 as d j cap d t which is not the case here because if we take time derivative of d r cap d t; it will be not equal to 0 similarly if we take time derivative of d theta cap d t it will not be equal to 0. So, when we are finally, we have to. So, for to write the equation of motion of any object in a plane polar coordinate system what we need is essentially is we need first r the position vector then we have to define v which is r dot and then we have to define acceleration which is v dot. So, it will be essentially r double dot. So, I hope you; you remember that dot essentially means d d t and double dot is essentially double dot means d 2 d t 2. So, this is the first time derivative second time derivative.

Now, when we compute this r dot and r double dot in the plane polar coordinate representation we also have to take into account that the radial the unit vectors along r and theta also changing with time. So, there will be additional term coming in here. So, our job is to find out this variation in r cap and theta cap then we can easily write this one the expression for velocity and acceleration now if we if we draw this 2 unit vectors or rather 1, 2, 3, 4 unit vectors separately what we will see.

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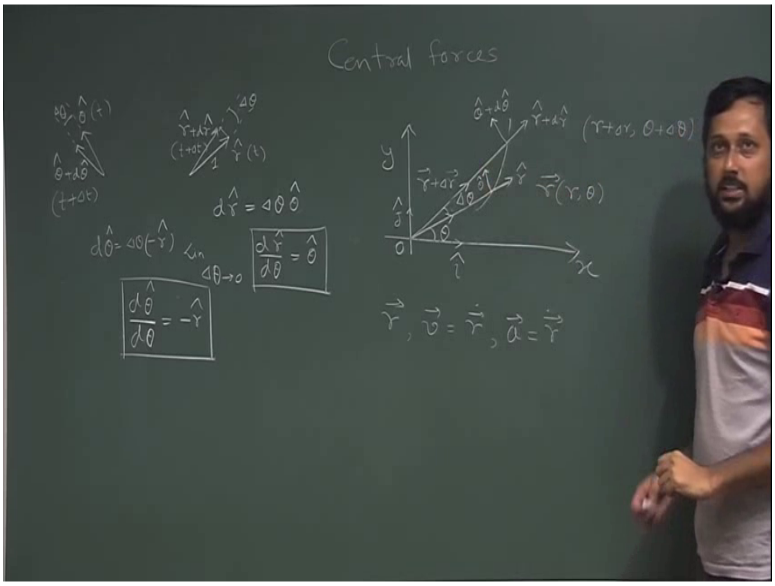

Let us say; this is our unit vector r cap and this is our new unit vector r cap plus d r cap where theta is this angle. Similarly, if we have if we draw theta cap; please remember that theta cap is perpendicular to r cap and theta cap plus d theta cap is perpendicular to r cap plus d r cap. So, if we draw this my theta cap will be in this direction; let us say I am just trying, I am shifting this vector to here and this vector next to it. So, this is your theta cap plus d theta cap. So, this is your theta cap plus d theta cap. So, the angle between them is d theta once again sorry it is not theta, but the angle is d theta.

Now, please try to follow this picture diagram; now we have a change in this is given in time t and this is given in time t plus delta t. Similarly, this is given in time t; this is given in time t plus delta t. So, we have to write an expression for this d r cap which is the change in this unit vector along r in time delta t. So, if we do that we see d r cap is r cap delta theta which is this distance. So, essentially its r d theta r cap is the unit vector I mean the length of this unit vector. So, r this is one because the length of the unit vector is always one. So, this is d theta, but there is a vector direction which is. So, this change d r is in a direction which is perpendicular to r in the limiting case when delta t becomes 0. So, in the limiting case when delta t becomes 0 d theta also becomes 0 and r cap and r cap plus d r cap; they almost fall on each other.

So; that means, d r cap is a direction pointing perpendicular to r cap which is the direction of theta cap. So, d r cap is nothing, but delta theta, theta cap. So, if we now take the limit delta theta delta theta goes to 0 this expression is nothing, but d r cap d theta is equal to theta cap; similarly, if we look into this carefully here, again the change in theta cap is given by d theta cap which is d theta cap equal to delta theta times theta cap length of theta cap which is once again will put one. So, with them we are just putting one here and it is a direction which is perpendicular to theta cap in this clock wise direction. So, please remember sorry anti clock wise direction. So, this is the anti clock wise direction of motion this is the clock wise direction of motion and anti clock wise direction when theta cap changes in this direction is essentially means it is changing along minus r cap direction.

So, we have to put a vector sign minus r cap; once again if we explore the limit when delta t goes to 0; that means delta theta. So, this is theta delta theta also goes to 0 we essentially get d theta cap d theta is equal to minus r cap. So, this is one relation and this is another relation right. So, we have these 2 relations. Now we will see how to how to get the expression for v and a using this relation and also please note that as we as if we move along this radial direction neither of r cap nor theta cap does not change because if we go from this point to this point the radial vector is always pointing outwards. So, that r cap is not changing. So, if and if r cap is not changing theta cap is always perpendicular to r cap that is also not changing. So, we like this 2 relations here we can also write; I will just write it together once again.

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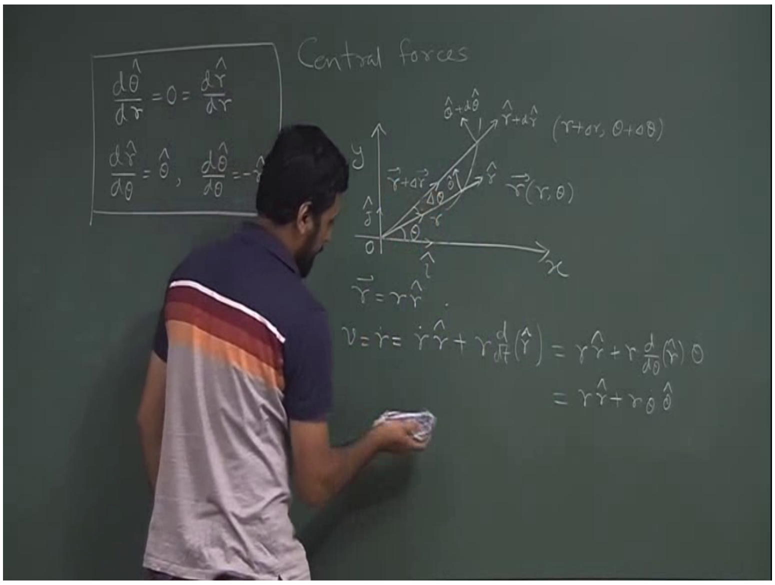

So, we immediately see that d theta cap d r is equal to 0 is equal to d r cap d r. So, they do not change the direction of r cap and theta cap does not change with changing r, but when it comes to theta we have t equal to theta cap and d theta cap d theta is equal to minus r cap. So, these 4 relations we have to keep at hand now if we start with r is equal to r; r cap because r is a vector direction r is a vector. So, there is a magnitude which is this length r and there is a vector direction pointing outwards along r is r cap. So, r is simply this.

So, v is equal to r dot is equal to r dot r cap plus r r cap dot which essentially means d d t of r cap we will just write d d t of r cap. So, what we will do is we will do a; we just do a substitution here r dot r cap plus r d d theta of r cap d theta. So, this is a just we substitute the derivative d d t as d theta d t and d theta if we now perform this from this relation this is nothing, but theta cap. So, this is r dot r cap plus r theta dot theta cap and this is nothing, but theta dot. So, we replace this by the dot we arrange to give you this.

So, this is the velocity expression. So, we get the velocity expression for this spherical sorry plane polar coordinate system as this.

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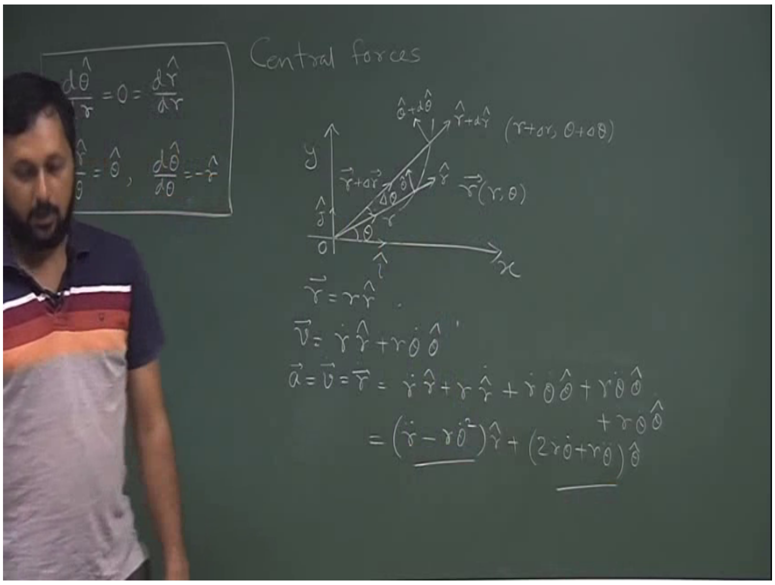

We will write again r dot r cap plus theta dot theta cap. Now we take a which is equal to sorry this is a vector here v dot v which will be equal to r double dot. So, it will be r double dot r cap plus r dot r cap dot plus r dot theta dot theta cap plus r theta double dot theta cap plus r theta dot theta cap dot. So, we have 1, 2, 3, 4, 5 in this derivative can we. So, there is a theta cap which is not visible here sorry just write this term here theta cap got. Now I am leaving it to you to as an exercise to simplify this yes just like previous case we have to replace this r cap dot as d d sorry; we have to substitute for d d t is d theta d theta d t. So, it will be theta dot d r cap d theta and essentially; what we will get is r double dot minus r theta dot square r cap r plus. So, this will be the final expression for a. So, I have one radial component and one transverse component; right.