Classical Mechanics: From Newtonian to Lagrangian Formulation

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Lecture - 60
Small Oscillation - 8

So we are in our last lecture of this long lecture series. And, in we have two more problems to finish in this class. And let us start with the first problem.

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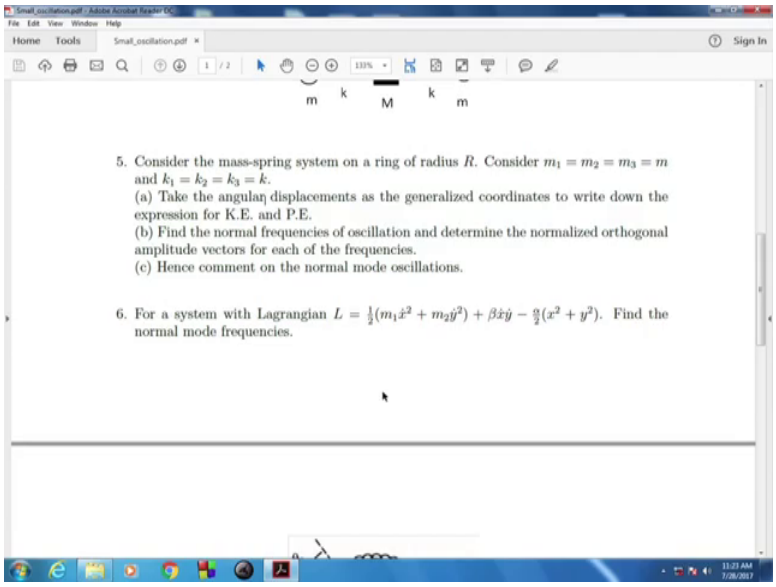

We have a Lagrangian given, which is given by half m 1 x dot square plus m 2 y dot square. And there is a coupling term in velocity, that is; something very unique we do not see that many often very often and there is a potential term. And we have to find out the normal mode frequencies. So, there is not much to be done except some basic calculation.

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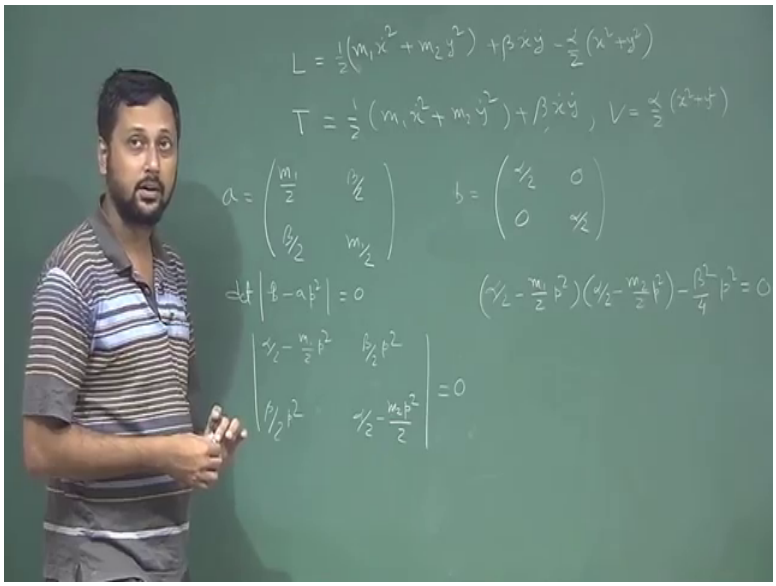

So, let us start with this Lagrangian L which is given by half m 1 x dot square plus m 2 y dot square right plus beta x dot y dot minus alpha by 2 x square plus y square very good. Now first let us separate out the kinetic energy part which will be simply this And the potential energy, which will be V equal to alpha by 2 x square plus y square. So, it is a two mass system where there is a coupling in the velocity, there is a coupling factor in velocity which is very unique. We generally do not see coupling factors in velocities, but think of situation where 2 masses are you know connected by some spring arrangement. Where, the kinetic energy in order to find out kinetic energy you do not have to take individual displacement square, but as a whole you have to take x 1 plus x, x 1 dot plus x 2 dot whole square if you do that or x x dot plus y dots whole square, there will be a cross term coming in.

So, you can think of a situation or I will leave to you as an exercise on metal exercise on thinking which of the physical systems will give rise to this type of cross term. Anyway; so, let us write a which is in this case m 1 by 2, m 1 by 2 and just like we did for cross terms in potential energy we have to take this as beta by 2 x dot y dot plus beta by 2 y dot x dot. So, if we do that. So, we have to write this term as beta by 2 x dot y dot plus beta by 2 y dot x dot see if we do that we get beta by 2 here and beta by 2 here.

Similarly, we can solve or we can write b and here b is the potential energy is diagonal in nature, there is no cross term. So, it is alpha by 2, alpha by 2, 0, 0 very good. Now we

have to find out. So, I choose this problem primarily, because this is something very unique; we do not see that often. Now we have to solve in order to get the Eigen frequencies, we just have to solve determinant b minus a p square equal to 0 right; b minus a p square equal to 0, and if we write it explicitly it will be alpha by 2 minus m 1 by 2 p square beta by 2 p square do not forget that a p square means whenever we have a nonzero term in a a matrix we have to write a p square multiply that with the p square and write that. So, b by 2 p square term will come in the off-diagonal terms also. Similarly, here also we have to write beta by 2 p square and diagonal term will be alpha by 2 minus m 2 p square by 2 which will be equal to 0. So, this is the equation we have to solve in order to get ps.

So, in order to this we have to open the bracket and let us do that it will be alpha by 2 minus m 1 by 2 p square alpha by 2 minus m 2 by 2 p square minus beta square by 4 b square equal to 0. So, this is the equation. And if we; what do we need to do now. So, this is an equation where things will not be. So, simple, but any way it is a quadratic equation there is there is a good that is a good thinking about coordinate equation that whenever we have a quadratic in some certain sense we can always find a solution very easily.

So, let us do one thing let us open up the bracket for this term and m 1, m 2 by 4.

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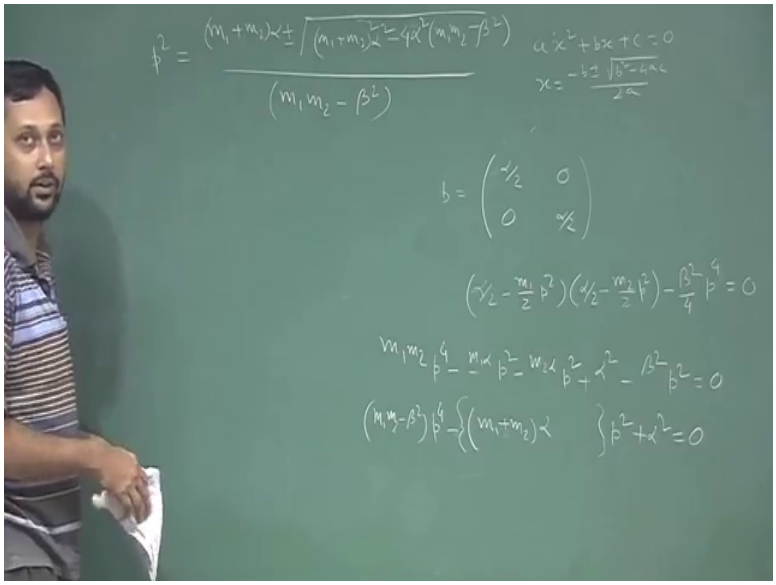

So, between this 2 terms it will be p to the power 4 and between this and this it will be minus m 1 by m 1 alpha by 4 p square between these and these it will be minus m 2

alpha by 4 p square and last term will be alpha square by 4 minus beta square by 4 p square equal to 0. The simplest thing to do is get rid of this force. All force are gone; and then bring all the coefficients of p square together. So, you can write it as m 1, m 2 p to the power 4 minus m 1 alpha or rather m 1 plus m 2 alpha minus beta square right b square plus alpha square which will be equal to 0 right and next is straight forward.

So, we so we do not worry about this anymore; raise to straight forward we simply can write p square, because it is a quadratic equation it will be there was a mistake there was a mistake right sorry; it will be; this is p to the power 4 right, because the last term was; so it will be p to the power 4. So, there will be no beta square here, but it will be m 1, m 2 minus beta square right ok. So, yes this is a once again it is a quadratic equation. So, we have to write as minus b which will be m 1. So, we know that for a x square plus b x plus c equal to 0; we can write x is equal to minus b plus minus root over b square minus 4 a c by 2 a. These are; this is a very familiar form and let us try this out m 1 plus m 2 times alpha plus minus root over m 1 plus m 2 times alpha square minus 4 a c, b square minus 4 a c. So, 4 alpha square into m 1, m 2 minus beta square by 2 m 1 plus m 2 alpha.

So, that is the final form. Let me just see if it is correct. So, first term is fine, second term is also fine, third term is also fine and the denominator is also fine very good. So, it is a denominator has some problem; just give me a second m 1, m 2 minus; this will go to the denominator m 1, m 2 minus beta square. So, it is something that is very ugly, as in we do not have any pattern in it, but if you take simpler I mean assumptions like if you if you get a situation where m 1 is equal to m 2. I can assure you that it will come to a much more simpler form. Anyway; that was not th6e object I mean this is a algebra which everyone can do, but object was to show you what happens if we have a cross term in velocity in the kinetic energy expression. So, how to write this matrices and how to solve for the equation, because most common mistake people would make in that situation is without they would not consider p; p square for the non diagonal terms.

So, that is why I took this example. So, it is not important to you know it is it was not very never important to show you, how this algebra works? This you can always figure out yourself. Any way; so this is the form and let us move to our last problem which will take some time and some understanding, that is a bit complicated problem. And I tell you something we will not find this problem in any of the standard text book, nowhere; you

will find it. So, I had to really dig deep in order to get the get a formal solution of this problem.

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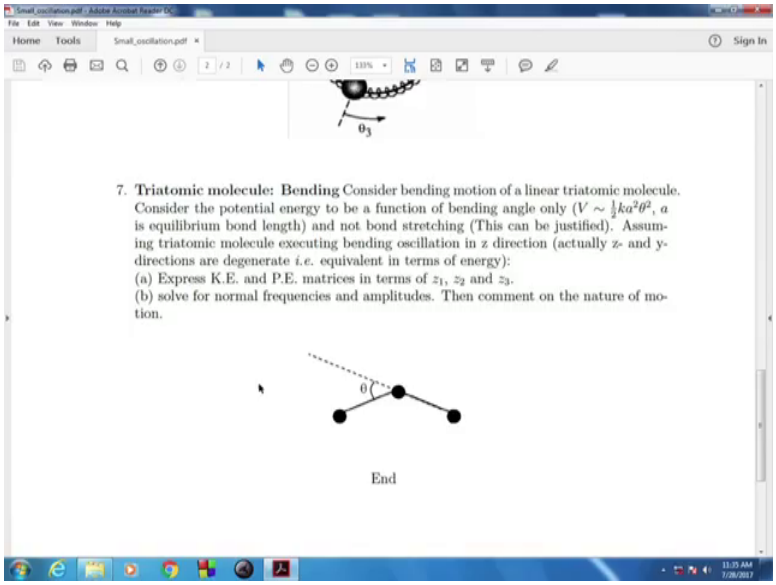

So, problem is once we are considering triatomic molecule, but this time we are considering bending. See; I think I told you already in the class beginning of the class, that beginning of this discussion of small oscillation, that triatomic molecules have stretching modes as well as bending modes. Now, what happens in stretching is covered everywhere? If you look for bending mode of for example, carbon dioxide which is a very common commonly, I mean; very well studied diatomic triatomic molecule, you will gain much insight I mean; at least in the text book levels I have never found anything that it deals with the bending.

There are certain of course, there are lots of research on it, because the bending modes and bending vibration frequencies of carbon dioxide is well reported studied in spectroscopy used in carbon dioxide laser everything, but there is no formal description on how to get to this bending modes; any way, I found it from somewhere that is it is reasonable to consider the bending angle to be the free parameter. Typically, what happens in a bending?

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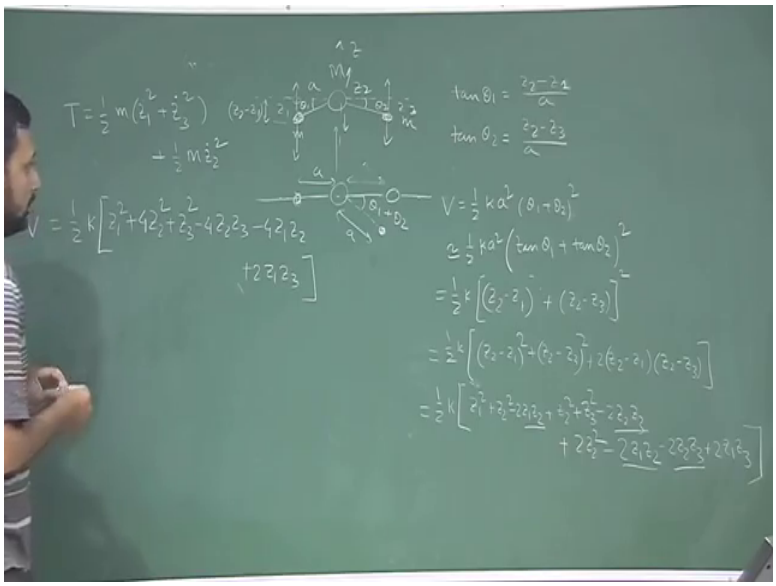

Let us say we have a triatomic molecule, when it bends it also it can also change the bond length, because there are certain you know certain chemical interactions which also which are also slightly orientation dependent. See, if the bending is changing; that means, the relative orientation see; each bonding is a electron bond electronic bond covalent bond.

So, there are 2 electron clouds overlapping, moment you bend it I mean movement a bond is bended, the local configuration of this electron clouds are changing slightly. So, the bond length in practice will change. Although very little, but it will change, but I have I have found somewhere that finally, that actually for bending we can take the; we can think of it as if the major contribution of kinetic energy and potential energy while when it is bending of kinetic potential energy is coming not from the bond change bond length change, but from the angle.

So, it is an angle dependent phenomenon. So, which is very contradictory in terms of the in compare to the stretching modes, where the angle change, I mean angle does not change, but the bond length changes and the kinetic sorry; potential energy is a function of change of bond length only. So, let us go into the problem. So, consider bending motion of linear triatomic molecule. Consider the potential energy to be a function of bending angle only, which is given by V. So, V is approximately half k a square x theta square, where a is a equilibrium bond length and node of not of bond stretching.

Assuming triatomic molecule executing bending oscillation in z direction, there is a note that actually z and y directions are degenerate I equivalent in terms of energy. So, we have to express kinetic energy and potential energy matrices in terms of z 1, z 2 and z 3 and solve for normal frequencies and amplitude, then comment on the nature of motion, ok.

So, the first part you see the second and third part is something that we or part b of this problem is something that we have already done many times for many different types of kinetic energy and potential energy matrices, but part a is something that, probably we have not we have not handled this type of situation. So, let us start with this. How do we define the bending angle? That is a main question. So, this is the equilibrium configuration let us say; without any bending. So, we are not talking about stretching here it is only bending. So, there is no bending angle and the relative angel between this 3 this 3 atoms are 0. Now, let us say we have a situation where we have a bending of the molecule. So, let us say this is molecule atom 1, atom 2 and atom 3 and we have masses small m, capital M, small m.

Now, let us assume that this one has bended; with a angle theta 1 and this has bended with an theta 2. Now, what is the total bending angle? How do you define it? What we can do is? We can consider it as if only; so you see relative angle between this and this still remains 0, there is no relative angle. Now, what we can do is; we can consider this situation, as if this 2 are remaining fixed and the third one has bended to this position with an angle theta 1 plus theta 2.

So, it is again it is essentially it is a 2 coordinate or rather 2; so it is just an alternative representation, we can either think of it that this one has bended by an angle theta 1 and this one has bending by an angle theta 2, but or what we can do is? We can think that this one is stationary and the total bending angle we can convert it to theta 1 plus theta 2; we can translate it to theta 1 plus theta 2. So, we will do that; and if we do that the potential energy V will be half k a square theta 1 plus theta 2 whole square.

Now, also we have to make a assumption very clever assumption here without that we will not be able to translate this into the standard coordinates we are dealing with. So, when a bending movement motion takes place what happens is, molecules are moving up or down, along the z direction. Let us assume this is z, it could be y as well as; I said in

the in the problem note: that z and y directions how do how will you distinguish? It is either x or any of y and z. It could like; if this is the x direction, we can clearly define this as x, but whether it is vibrating in this plane or in this plane or anywhere in this you know on this you know rather any anywhere any I mean; first of all y and z are degenerate I mean we cannot strictly define this is my y axis is my z axis, because according to the cylindrical symmetry of the system; it is totally degenerate. So, we can take any 2 axis perpendicular to the x direction and mutually perpendicular to each other we can define them as the y and z any way.

So, let us say; for now let us consider that the plane it is oscillating in this plane and this axis is my z axis. So, the instantaneous position is given by z 1 and z 2. Also, for I mean right now we are not considering the motion I mean; do not do not think that we are not considering the motion of this one; this one is also moving. So, when this two are coming down, please remember that we have to keep this so for a normal mode vibration or a normal mode small oscillation, the center of mass should stay in the same place. So, if this two are coming down, this one has to go up.

Similarly this two are going up this one has to coming down; this one has to come down. So, let us assume that it has; it has also a z movement in terms of given by the coordinate z 1, so we or sorry z 2. So, we have the kinetic energy matrix; p can be written as half m z 1 dot square plus z 3 dot square plus half capital m z 2 dot square. So, this is the kinetic energy matrix. Now our challenge is to convert this potential energy which is written in terms of theta; theta 1 and theta 2 we have to convert it in terms of z 1, z 2 and z 3.

Now, it turns out there is a way, clever way, we can simply write because it is a small oscillation. We can simply write k a square and replace theta 1 and theta 2 by tan theta 1 plus tan theta 2. This is valid only for small oscillation. We could have taken sign also, but why tan there is a reason, see because it is a small oscillation. So, let us say this was a equilibrium position and this bond length a so this is my equilibrium bond length a and not this, but let us say in this case this is my equilibrium bond length a and, because it is a small oscillation we are not considering any stretching of the bond length, ok.

So, tan theta 1 will be z 1 by a and tan theta 2 will be z 3 by a right or wrong; wrong, because we have also need to consider the relative position of the middle molecule. So, it will not be z 3, z 1, but it will be z 2 minus z 1 minus z 2 by a I mean it could be minus z;

I mean if the sign does not matter really here, and no actually it does because it will be z 2 minus z 1 right and it will be z 2 minus z 3 by a.

So, if this is z 2 this is z 1. So, the gap between this two are simply z 2 minus z 1. Now, this is a way of transferring these thetas into zs. So, we go on and we write z 2 minus z 1 by a or actually what happens is a cancels out nicely. So, it will simply be z 2 minus z 1 minus z 2 plus z 2 minus z 3. So, it will be 2 z 3 minus z 3 whole square, right.

So, if we do that if we open this up. So, it will be 4 z 2 square plus z 1 square plus z 3 square minus, it will be let us see is very very easy to make mistakes. Actually, it was better that way, because this way we are more prone to make mistakes. So, it will be z 1 minus z 2 or sorry; z 2 minus z 1 whole square or plus z 2 minus z 3 whole square. So, if I now open up the brackets it will be z 3 minus z 1 whole square plus z 2 minus z 3 whole square plus 2 z 2 minus z 1 z 2 minus z 3 very good; equal to half k z 1 square plus z 2 square minus 2 z 1, z 2 plus z 2 square plus z 3 square minus 2 z 2, z 3 minus 2 z 2 square minus 2 minus 2 z 1, z 2, 2 z 1, z 2 minus 2 z 2, z 3 plus 2 z 1, z 3 ok.

So, long term and if I write it here, so V will be equal to half k z 1 square plus; so we have oh no so, it will be a plus z 2 square sorry; so it will be a z 2 square plus z 2 square plus z 2 square. So, it will be 4 z 2 square plus z 3 square [FL] 2 z 2, z 3, 2 z 2, z 3 minus four z 2, z 3. So, z 2, z 3 is (Refer Time: 27:22) taking care of and minus 2 z 1, z 2 minus 2 z 1, z 2 minus 4 z 1, z 2 right and what is left is 2 z 1, z 3 plus 2 z 1, z 3.

So, we have this expression for it is a long expression, but, now if we construct the kinetic energy and potential energy matrices, please excuse me; if I have made any mistake, because it is a very long algebraic calculation difficult for me to see you k ow it is very difficult when I do it on pen and paper it is easier.

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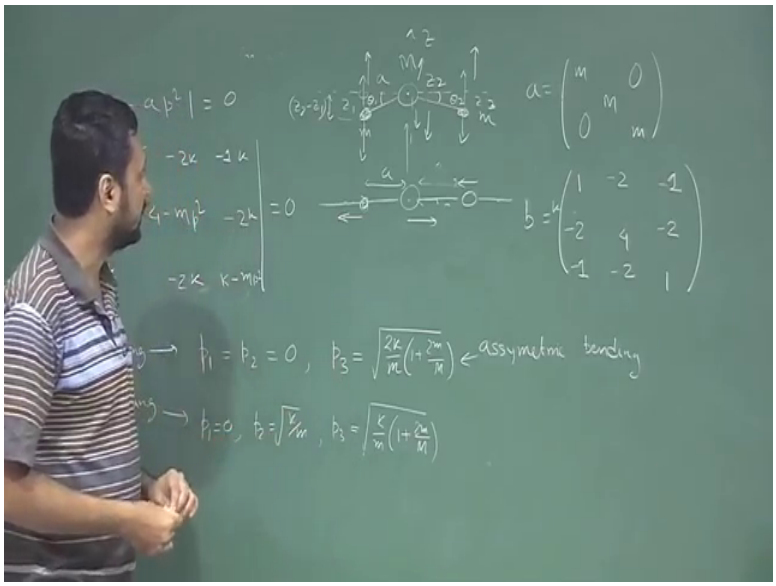

So, a is simply; so we will get rid of the half first, a will be m capital M small m 0 and 0 are p is p is long expression. So, it will be 1, 4, 1 in the diagonal off diagonal we have z 2, z 3 and z 1, z 2 both are with a weight 4. So, we have to break it into two z 2, z 3 plus 2 z 3, z 2.

Similarly, z this will be 2 z 1, z 2 plus 2 z 2, z 1 so, but minus 2 actually. So, it will be minus 2 here minus 2 here. Similarly z 2, z 3 also has this and this will be 1 minus 1 so it will be minus 1 minus 2 minus 2. So, job is half done. We can write this matrices a and b; that means, we have we have almost finished our hard work. Next, is we have to you know calculate the Eigen values.

So, once again b minus a p square equal to 0 determinants; see if we do this it will be 1 minus so sorry there will be a k here. So, it will be k minus m p square minus 2 minus 1 minus 2 4 minus m p square minus 2 minus 1 minus 2 1 equal to 0; it will be k minus m p square and we will have k everywhere. So, that is a long calculation once again. Now we are running out of time a bit. So, what I will do is? I will give you the final answer. So, I think you can do you see you can follow this, you know you can open up this determinant and find out 3 distinct roots now it will. So, happen that we have p 1 equal to p 2 equal to 0 and p 3 will be equal to 2 k by m 1 plus 2 m p m right.

Now, if you compare it with the frequencies, we have got in case of just give me a second; where did I get it where is it right. So, if you compare it with the stretching mode

frequencies, we have p 1 equal to 0, p 2 equal to root k by m and p 3 equal to root over k by m 1 plus 2 m by m. So, this is I am see it is not about absolute comparison of the frequency. It is about this factor and this factor if you remember was responsible for the fact that the center of mass is not moving. So, this is for the asymmetric stretching, when for a for this particular mode of oscillation, when this one is going this way this one is so we are talking stretching now, not bending. For this particular mode of vibration in stretching so we have this for bending and this for stretching right.

So, for stretching, when this two were going in the same direction, this one was going in the opposite direction with a different weight in frequency right. Now this was responsible for the fact; that the even when all 3 masses are displacing this center of mass remains at this position. Similarly, the same weight factor appears here and by symmetry of the problem we can always find out exactly; what is the you know what are the relative positions of center of mass, but without surprise you will find that for this frequency of oscillation, frequency of bending the center of mass remains at the center ok.

Now, they were two 0 frequencies and this 2 corresponds to translations around any of the 2 axis. So, it is a please remember that this is there is no x involved. So, these are all z. So, translation in z and; as I said z and y they are exactly you know they are degenerate axis, because we cannot really define a z and a y precisely. So, with it remains it gives you 2 perpendicular displacements either you know along z or along y mutually perpendicular directions so there is no bending. Only this one corresponds to a bending and of course, the bending will takes place if you calculate the Eigen frequency or Eigen amplitudes or rather amplitude vectors, from this matrix you will find out they are also scaled such that if this two are going up, this one is going down in a manner to keep the center of mass exactly on the same line, ok.

So, this is there is only 1 so we have seen that the there are 2 stretching modes like symmetric stretching and anti-symmetric stretching, but there is only one true bending mode which is an anti-symmetric bending or rather not anti-symmetric; asymmetric ok. So, we have not I am stopping here, because we are not have we do not have enough time to show you the calculation, but only remaining calculation was to calculate the Eigen you know amplitude matrix for this particular frequency, which I am pretty sure you can do it by yourself and you will get no surprising answer to it, ok.

So, that is the official end of this class. It has been a wonderful journey with all of you. And, I hope you were also satisfied with the way we have handled our discussion forum the way you have got your assignments. Assignments have not been very easy all the time, but thank frankly we could not have done better, because and I feel that at some point we have to start attempting you know little difficult questions that is important otherwise we would not learn.

Any way; and also I need to mention the tremendous support I have got from Mister Shubabrotho Ghosh the TA of this particular course. He has been instrumental in preparing all the assignments and exam problem sets. So, exam problems which will be having very soon and also he has helped me a lot in terms of fundamental understanding also I need to mention another of my colleagues name that is professor Shubabrotho Roy he is also offering in the same time, he is offering a; he his course was also running that was on mathematical methods. He was the first one who introduced me to this excellent book by professor Amol Kumar (Refer Time: 36:40) whenever I have difficulties in understanding any of the mathematical expression any of the matrix expression matrix equations, I always consulted him and I always came up he always came up with a very distinct answer any way.

So, I think it is time that we call it a day. Please keep writing me if you have any further questions in on in terms of classical mechanics and I will be happy to answer that.

Thank you all, goodbye.