Classical Mechanics: From Newtonian to Lagrangian Formulation
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Lecture - 06
Systems with variable mass - 3

So we write this rocket equation here. And now it is time to start working on this particular equation.

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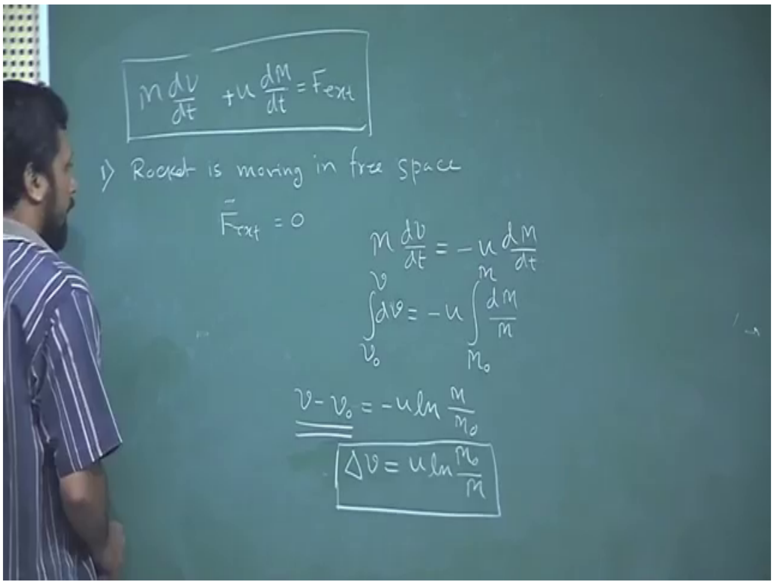

So, we have M dv dt plus u dM dt equal to F external. Now we let us first consider a situation where the rocket is moving in free space ok. So, in that case immediately we see that F external equal to 0, and if we rewrite the rocket equation it will be M dv dt equal to; sorry, there will be a vector term here equal to minus u d M dt no, sorry it will not be a vector equation, because we have already considered the sign of v and u and we have understood that v and u there are working in the opposite direction. So, it will not be a vector equation, ok.

So, we write this equation in this particular scalar form and we try to integrate this. Now once we do that we see integration dv equal to minus u integration d m by m. Now the integration limits will be let us say it is initial velocity is v 0 and at time t it reaches a velocity v, initial mass let us assume to be M 0 and mass is m. So, we do that perform this integration and we come up with the relation that v minus v 0 is equal to minus u lon

M by M 0 or we can absorb this negative sign to write v or and we can write this as the velocity change, we can call this the velocity change denoted with delta v and write u lon M 0 by M right.

So, this is what we have, this is the final equation of velocity as a function of mass I mean it is a function of time because mass is a function of time. So, we have delta v the velocity change as a function of time in the when the rocket is moving in free space, but when the rocket is not moving in free space. So, this is one equation we just keep in mind sorry too high.

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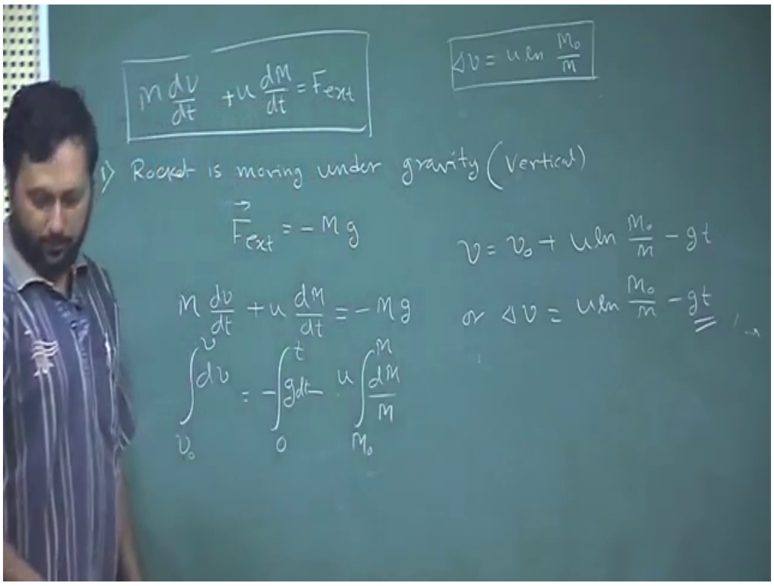

So, we just keep it here and we move to our next topic not the topic basically next situation where the rocket is moving under gravity. So, we are assuming vertical because in principle rocket can go up directly or rocket can go up in an angle like this or like that so, but let us first simplicity we are considering only pure vertical moment so; that means, rocket is just going under the influence of gravity, let us say from earth surface could be from anywhere. So, in general F external will be given as minus m g. Once again please understand that we are assuming that rocket is going in the vertical direction and we are measuring the displacement or velocity starting from the ground level. So, that is why m g comes with a negative sign ok.

If we set our origin somewhere up in the sky then this will come as a positive sign, but this is very important. So, this equation will be m sorry this m will be a capital M here because we are using capital M for rocket mass. So, it is M dv dt plus u d M dt is equal to minus M g. So, if we rearrange this to give dv dt is equal to minus g divided by M minus u by M dM dt once again we take this dt here. So, this one becomes d M by M and integrate. So, 0 to t, v 0 to v, M 0 to M; so it is also another straight forward integration which will give us v equal to. So, I am not doing it completely, but I am leaving on to u it is the very simple integration is just the same minus g t or delta v is equal to. So, it is just the same as the previous one, but we have additional g t term ok.

So, we will just keep this also for future reference. So, delta v is equal to u lon M 0 by M minus g t. So, that is another relation which we would like to keep. So, this is free space and this is under gravity. So, we have two situations.

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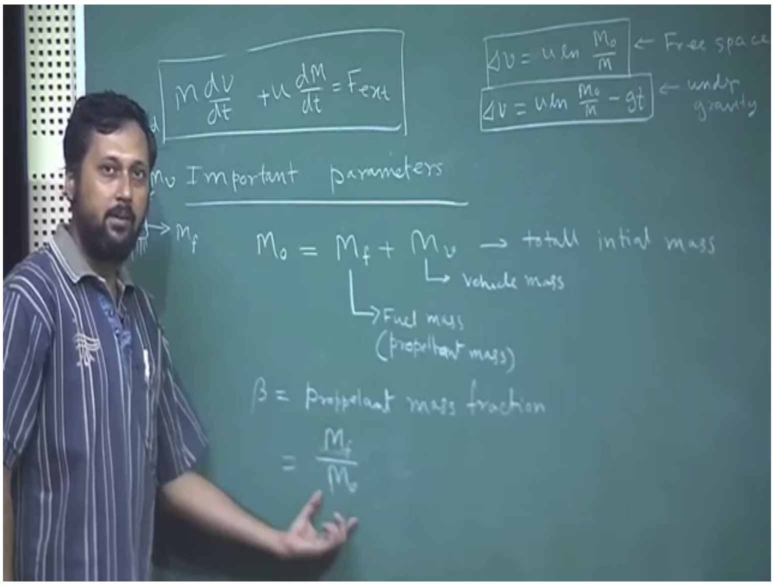

Now before proceeding any further let us introduce some parameters, the parameters that we are going to introduce first one is M 0 which is already there in these two expressions, M 0 is the initial total mass of the rocket. So, M 0 is given as M f plus M v. So, total initial mass of the rocket where M f is the fuel mass also called the propellant mass. So, this is called the propellant mass you will find this term if you search for rocket motion and other things on internet or you would look in to any book, this term is used very frequently and this is the mass of the total mass of the vehicle. So, what happens is in a rocket we typically have ok.

So, in a rocket which is shooting upwards. So, it typically have only a small portion of it is total mass which is for example, a satellite or a missile which whatever thing it is carry in the outer space or it could be a camera, it could be satellite, it could be anything. So, this is called the pay load and so, there will be an empty rocket shell with fuel inside. So, this part is M f and this empty rocket itself plus the pay load is your M v. So, right now we are not separating it making separation between pay load, and the rest of you know empty rocket shell, we are just considering the whole thing as vehicle mass M v and rest of the fuel mass is M f. So, also we introduce a parameter beta which is called the propellant mass fraction, it is called the propellant mass fraction which is given as M f by sorry not m capital M, M f by m 0. So, this essentially tells us what percentage of the total rocket mass is your fuel ok.

So, it is M f by M 0 and typically I can tell you I can give you data I will give data later on typically this quantity is of the order of 80 percent. So, 80 percent of your rocket which starts from earth for example, vertically upwards 80 percent of it is fuel and rest 20 percent we have a pay load and we have an empty vehicle you know that these are all typical numbers it could change depending on how many stages in the rocket you have, what type of pay load it is carrying, but typically this is a good number 80 percent is a good number; now if you look into it.

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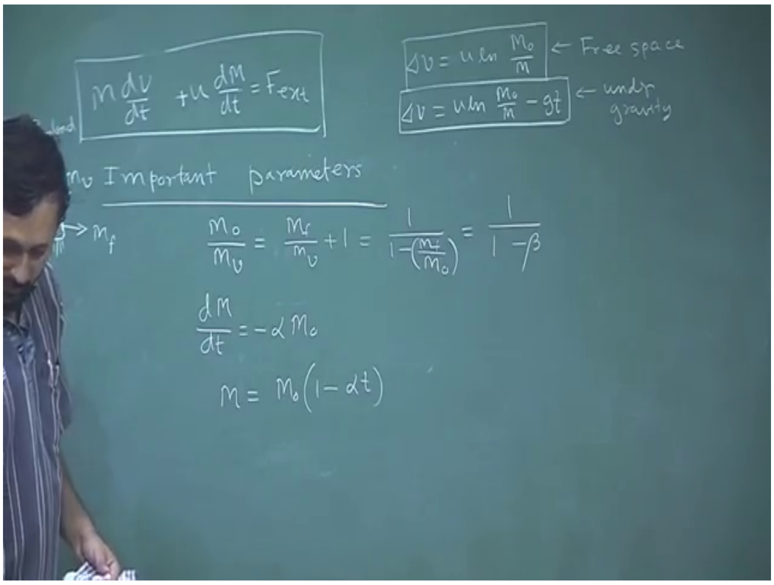

We can also express this quantity M 0 by M v which is important we will see why this this particular issue is important, this will be given as 1 by m sorry sorry it is simply M f by M v plus 1. Now it can be also written as 1 by 1 minus M f by M 0 because it will be M 0 minus M f, M v by M 0 right now this is nothing but your beta. So, we can write this thing as 1 by 1 minus beta. So, this is also please remember that M 0 M v is equal to one by 1 minus beta, where beta is the propellant mass fraction. So, this is the important relation we will just keep it in mind and we will use it later. So, one more important parameter is alpha what is alpha? Alpha is the loss of initial I mean it is a mass loss rate and typically this is given as d M dt is equal to minus alpha M 0. So, this is given in the units of total initial mass of the rocket ok.

So, if the total initial mass of the rocket let us say 50,000 k g. So, alpha is given as something as second inverse. So, it will be some fraction of this initial mass which will be decade per second. Now if we consider this form for mass loss then your m will be M 0 times 1 minus alpha t. So, this will be the expression for M 0 because if you put t equal to 0 m equal to M 0 and t equal to some proper number, then you will get t equal to any arbitrary in time greater than t is equal to 0, you get the mass of the total rocket as a function of initial mass.

So, using this we just using this parameter, how many parameters we have introduced? We have introduced essentially four parameters.

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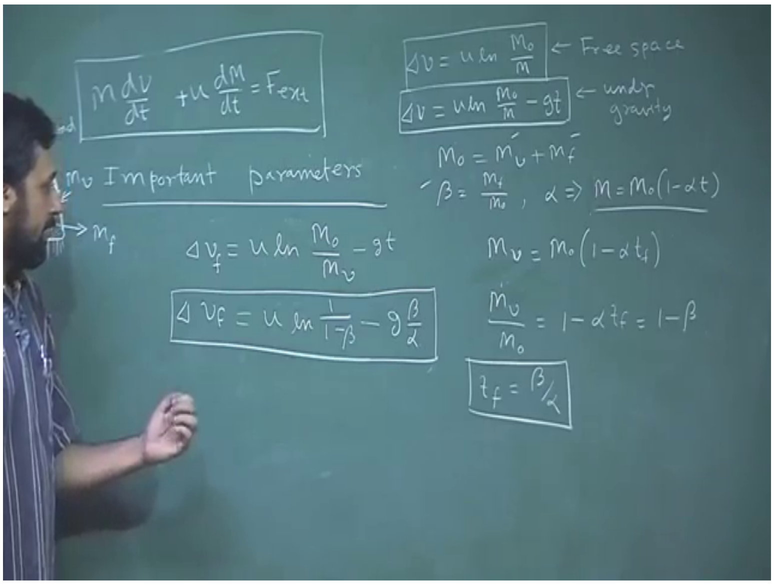

We broke M 0 into M v plus M f, we have introduced beta which is M f by M 0. So, we have used introduced 1, 2, 3 and alpha which is given as m equal to M 0 1 minus alpha t.

So, if we go back to the free space equation free space final expression and try to recalculate I mean try to use this expression and we introduced by it in terms of this new set of parameters and see what we can get. So, delta v is given by u lon M 0 by M. Now what happens when all the fuel is burnt out when all the fuel is burned out M f becomes 0.

So, the final rocket mass after all the fuel is gone is M v that is the vehicle mass. So, in order to get final velocity change, let us say it initially starts with a velocity of I do not know may be it can start from 0, it can start from a moderate velocity of 100 meters per second, it can start from a high velocity of 10000 meters per second, but we are interested more interested into the velocity change. So, the total velocity change which will be given as delta v f for that we have to replace m by M v, that is the final mass of the vehicle once all the fuel is exhausted ok.

And if you recall I we calculated this fraction M 0 by M v and we have found that this is nothing but u lon 1 by 1 minus beta. So, delta v f is equal to u lon 1 by 1 minus beta same those for these expression because this part is identical in this two. So, for rocket motion under gravity we can have similar expression of delta v f, only thing is we will have a minus g t f on it. So, when we are can considering rocket in free space the final change in velocity is given by u lon 1 by 1 minus beta only and when we are can considering gravity into this equation. So, we have an additional ground of minus g t here. So, we have to put t equal to t f in order to get t f, t f becomes the time of flight the total time for which a rocket can fly using it is fuel.

So, happens that t f is also calculated we can also calculate t f from this expression because if we put t equal to t f m should be equal to M v in this expression. So, if we put M f sorry M v by sorry if we put M v here. So, we have M 0 1 minus alpha t f and M v by M 0 is equal to 1 minus alpha t f which is once again is given by what is this expression if you recall it will be simply 1 minus beta right.

So, that gives us t f is equal to beta by alpha fine. So, if we do that if we put this expression of t f here. So, our time flight is given as beta by alpha for motion under gravity. Now this is a very interesting result because when we are not considering gravity it does not matter at which rate the fuel is exhausted, but we finally, get the same in velocity. Moment you introduce gravity into the equation the rate of fuel exhaust is also becoming an important parameter which was previously not there.

So, you see that in free space does not matter if you move slowly I mean by exhausting less amount of fuel or you are moving fast you essentially end up having the same velocity total velocity change for the fuel available, but when it comes to when we have a gravitational pull I mean gravitational pull into this equation, it is also important at which rate you exhaust this fuel and what was the propellant mass fraction. So, there is a competition between this and this fine.

Now, let us. So, you would like to have an expression for the velocity, the sorry sorry for the distance travelled.

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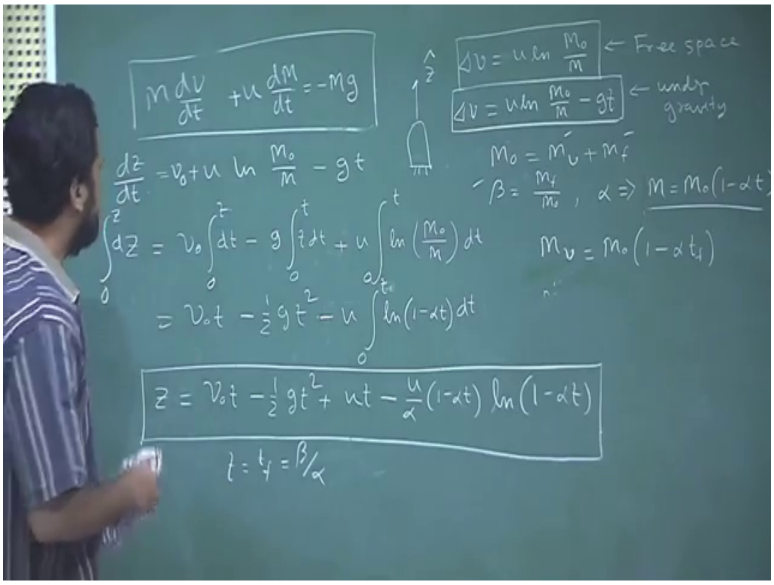

So, in order to gain get this expression we go back to this basic equation and we just replace F external as minus M g and then we try to integrate it and we have already done that which gives us delta v is equal to u lon M 0 by M minus g t. So, this is the exact same expression which I wrote here, now what we do is we introduce. So, we move v 0 to this side and instead of writing v, we write d z assuming that the rocket is moving let us say this is your rocket which is moving along the z direction v is written as d z d t. So, what we are what we are looking for is we so far we have calculated the velocity change and now we want to get gain an expression for the total change total distance covered by the rocket under the influence of gravity.

Now, in order to do that we have to perform this integration, which will be integration of d z from let us say that the rocket starts from the ground. So, at time t equal to 0 z equal to 0 it becomes z here. So, you have v 0 integration dt it will be 0 to t minus g t dt 0 to t plus u integration lon M 0 by M d t. Now first term is straight forward it will give you v 0 t, second term is half g t square third term we have to manipulate once again it will be from 0 to t 0 to t. So, we have to write m the expression for m as M 0 into 1 minus alpha t. So, M 0 will cancel out you will essentially have lon 1 by 1 minus alpha t d t.

So, this is the slightly tricky integration, but it is completely doable essentially oh. So, what we can do is we can simply put a minus sign here and we can instead of writing it like this we can write 1 minus alpha t. So, that is even easier to integrate and once we do the final perform this final integration I am not showing it you can do it yourself you know how to integrate this things, the final expression for z will be v 0 t minus half g t square minus u times t. So, there will be an u t term coming up and there will be a minus u by alpha into 1 minus alpha t lon 1 minus alpha t right. So, this is the final expression for z at any given time t.

Now in order to gain the maximum height which can be achieved by this which can be achieved by the rocket we have to put t equal to t f which is equal to beta by alpha now if we do that we. So, if we do that we immediately.

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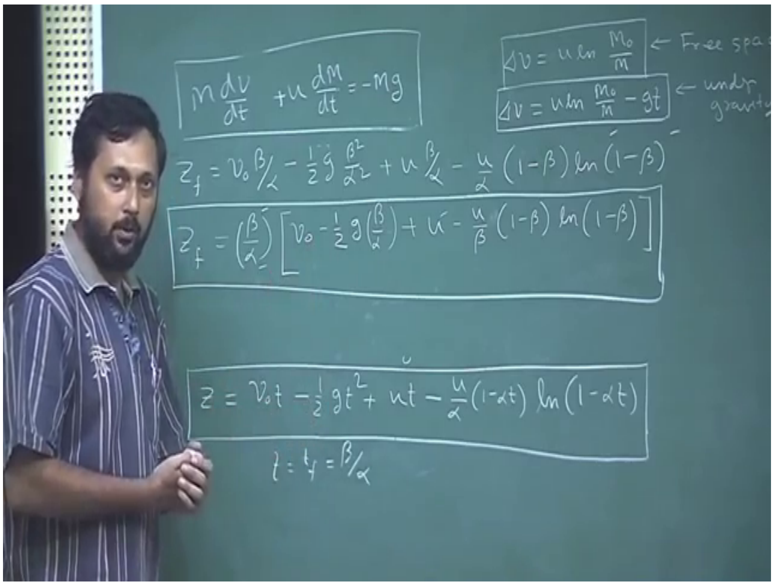

See z f is equal to v 0 beta by alpha minus half g beta square alpha square, plus u beta by alpha minus u by alpha 1 minus. So, beta (Refer Time: 25:53) 1 minus beta lon 1 minus beta; so this is the expression and if we take beta by alpha common.

So, we have v 0 minus half g beta by alpha, plus u minus u beta sorry u by beta 1 minus beta. So, this is the expression for maximum height achievable in terms of the parameters. So, if we put v 0 equal to 0 the initial velocity if we put at 0, then this one gone beta by alpha. So, essentially what parameters do we need; we need beta we need alpha and we need u that is it. So, all every g of course, we know g at surface if we know the value of v 0 is v 0 otherwise we can assume that it to be equal to 0.

So, essentially we need v 0 g beta alpha and u these 5 parameters. And we can give you the final velocity of the rocket space final velocity of the rocket. And we can also give you the maximum height that can be achieved by using this particular rocket.

Thank you.