Classical Mechanics: From Newtonian to Lagrangian Formulation

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Lecture – 59
Small oscillation – 7

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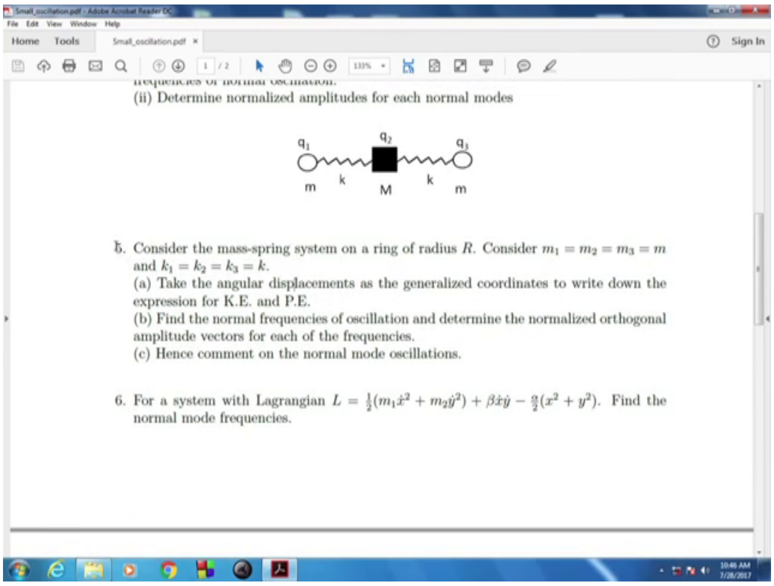

So, we will continue with our discussion on problem solving. We have already solved one problem, which includes normal modes. Now, let us look at what problems we have in hand ok. So, linear triatomic molecule is something that we have already settled now there is a problem number five which discusses this system actually.

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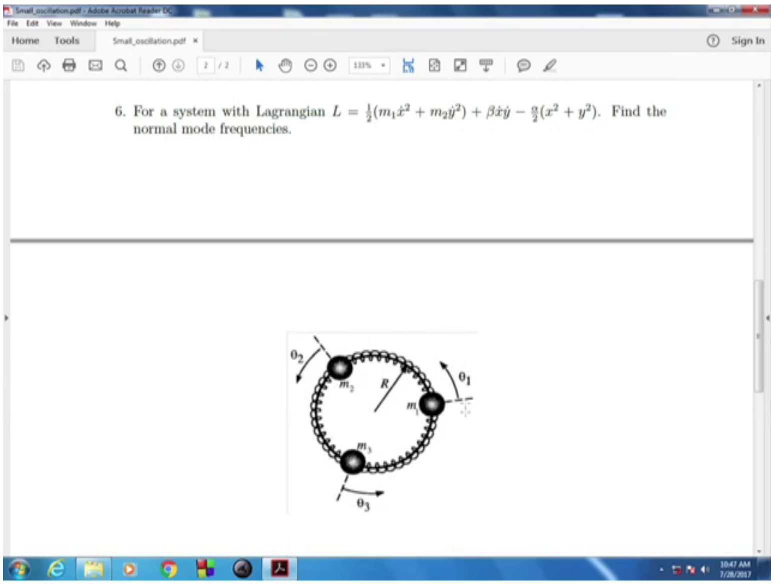

So, three masses or three beads actually connected by rings which are on this moving on this circular loop and their angular displacements are theta 1, theta 2 and theta 3 respectively. Now, we you have to take all the masses to be equal to m and all the you know all the spring constant to be equal to k. And we have to take the angular displacement as the generalized coordinate, we have to write down first that expression of kinetic energy and potential energy, then normal frequencies, orthogonal amplitude vectors and we have to finally, comment on the nature of the motion for different normal mode oscillations.

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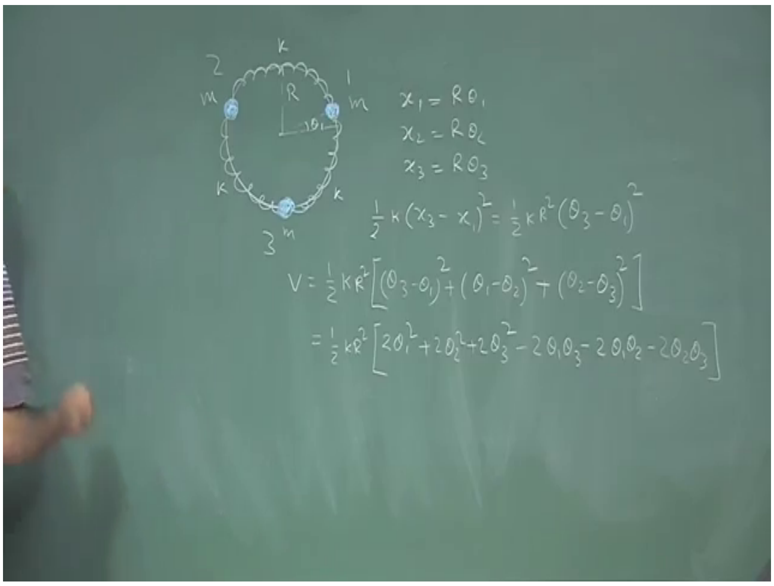

So, let us look at this picture once again. We have spring mass system, which is on this ring. So, they are evenly spaced, so one here, one here and one here, these are beads actually of equal mass m, m and m, and they are connected by springs of equal spring constant. So, what I will do is I will mark them, so that it is easy for you to see. So, these are the beads we are talking about. Let us say the radius of this is capital R. If this is the case and the deviation let us say from equilibrium position which was supposedly somewhere here, let us say theta 1. Similarly, for this one it will be theta 2; the third one will be theta 3 and like this. Now, and the spring constants are k, k, and k.

Now, for springs we do not care whether it is a I mean we do not discuss the spring kinetic energy or potential energy in terms of angular displacement, but in terms of linear displacement. But linear displacement and angular displacement let say if the linear displacement is x 1 corresponding to this one, it will be simply R theta 1. Similarly, x 2 will be R theta 2, and x 3 will be R theta 3. So, for this spring let us say it is let say 1, 2 and 3. So, the spring that connects between connects 1 and 3 for that the potential energy expression will be half k x 3 minus x 1 square, which will translate to half k R square theta 3 minus theta 1 square. So, similarly we can write for the similar expressions for two other springs.

So, what we are doing we are looking at the compression, compression of the spring, but what we are doing is using this relations in order to change this compression into angular displacements. So, we have basically writing this compression of a spring or expansion of a spring which in terms of the angular displacements. Now, if we do that the total kinetic energy v will be half k R square theta 3 minus theta 1 square, similarly for this two it will be theta 1 minus theta 2 square, and for this two it will be theta 2 minus theta 3 square.

So, I hope I could convince you that this is the right way of writing this potential energy. Now, if we open the brackets, it will be 2 theta 1 square plus 2 theta 2 square plus 2 theta 3 square right because each one each for example this one and this one will contribute 2 theta 1 squares this one and this one will contribute 2 theta 2 square like this. So, we have two of both theta 1 square theta 2 square and theta 3 square, and then we have minus 2 theta 1 theta 3 minus 2 theta 1 theta 2 minus 2 theta 2 theta 3 [FL].

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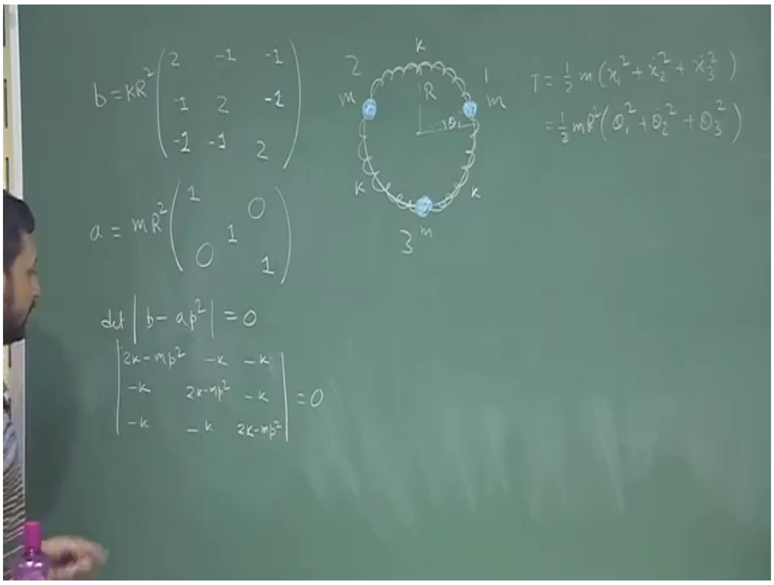

So, I think we have written the matrix I mean that entire expression nicely. Now, it is time to convert it to the matrix. So, b will be given by see there is factor of k R square which will be there. And the matrix form I will just 2, 2, 2 along the diagonal, because all the diagonal elements has a weight of 2. Now, for this terms let us say theta 1 or let us say first start with this term theta 1 theta 2 again we have to break it in a symmetric manner. So, 2 theta 1 theta 2 will be 2 theta 1 theta 2, so sorry 2 theta 1 theta 2 will be theta 1 theta 2 plus theta 2 theta 1, so that the matrix remain symmetric. So, it will be minus 1 here, and minus 1 here and minus 1 here, minus 1 here, minus 1 here and minus 1 here. So, it is a matrix where we have diagonals as two and of diagonals as minus one all of it. I think you have I could make you understand why it is like this if not please try to look into this functional form of v that is very important, how to get the v and what is the function form and try to correlate this two.

Next, so this is your potential energy lets write it for kinetic energy oh sorry I should not have removed it any way. So, sooner or later you have to do it. Now, kinetic energy is pretty straightforward because t will be half m x 1 dot square plus x 2 dot square plus x 3 dot square, it will be half m R square theta 1 dot square plus theta 2 dot square plus theta 3 dot square. So, once again we can wrote this a matrix as, so there will be an m R square term outside, m R square term outside and it will be simply 1 1 and 1 that is it ok.

So, now we have to find out the normal mode frequencies. So, we have done our first bit, we could write the kinetic energies and potential energies in terms of this two matrices. Now, it is time for us to find out the normal frequencies of oscillation. Now, in order to do that we have to solve this determinant which was determinant of b minus a p square equal to 0. By the way, if you like you can write omega also in place of p because in many books many textbooks people have used omega as a standard notation for p. I am using it because it is in the very nice textbook of Prof. (Refer Time: 08:55) notes on classical mechanics I am following this convention, just for my convention it is you do not have to follow this.

Now, it will be R square outside b minus a R, so it will be 2 k minus m p square minus k minus k minus k 2 k minus m p square minus k minus k minus k 2 k minus m p square equal to 0. Best thing would be to get rid of this R square anyway, because this is not contributing anyway, it is a common factor between these two matrices, so it will cancel out. Now, if we open up this bracket, it is a little longer calculation, but extremely doable nothing wrong with it. I will show you may be one or two steps and write the final answer, because this is something that you can try yourself once you know the final answer.

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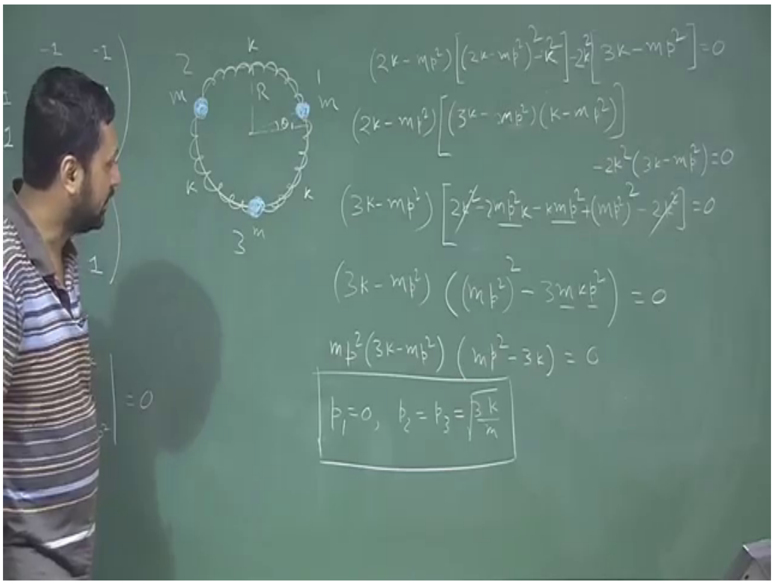

So, if we just open up this determinant I am pretty sure you know how to do it all of you how you know how to open up a determinant. So, if we do that it will be 2 k minus m p square outside, and inside bracket we will have 2 k minus m p square whole square minus k square minus k square and we have second term which is minus k minus k. So, we have to do this minus this square minus k square plus k square plus k times 2 k minus m p square. And there will be a third term which will be again minus k and it will be k square plus k times 2 k minus m p square. Now, you see this, these two terms are identical. So, what we can do is we can simply now add them together put a 2 here, so it will be 2 k and that is it we are good and this is equal to 0 that is the equation.

Now if we have to open it up further, if we do that ok, just give me a second yeah right. So, if we if we do that that is a pain with small oscillation problems we have to solve this complicated algebraic equations, I mean these are these are funs sometime, but sometime it is just you know that answer will be something very simple, but still I have to go through that long steps. But what can you do, this is life, here there was some simplification I did right, so what I did was I opened this one inside before doing anything here I opened this one inside, so it will be 2 k square if I do that and there will be a k here. Now, 2 k square we will be adding to this, so it will give you 3 k square and 3 k will put a k square here minus m p square 3 k minus, so this was one simplification we did I did before hand and anything else no.

And now we have to open up the bracket. So, if you do that, yeah also this one. So, it will be, so this we wrote I wrote as 2 k minus m m p square. So, a square minus b square, so this will be a plus b into 2 k minus m p square minus k right minus 2 k square into 3 k minus m p square equal to 0. So, again we will see that this term will be nothing but 3 k minus m p square and this will be k minus m p square. Now, we can so this is already factorized this term, and we see that 3 k minus m p square is the common factor three k minus m p square that is the common factor. If you take that common from both the terms what is left is this multiplied by this which will be 2 k square minus m p two m p square k minus k m p square plus m p square whole to the power 4 minus 2 k square equal to 0.

I think now it looks more or less nicer right and m p square k. So, finally, you see this two terms are actually two times I mean 2 m p square k and m p square k. So, this will give you 3 k minus m p square and it will be m p square sorry m p square whole square not to the power 4, whole square minus 3 m k p square right equal to 0. Now, what we can do is we can take p square out of this term. So, we can totally factorize it in the following form. So, you can write p square 3 k minus m p square, and it will be m square. So, yeah m can also be taken out. So, it will be m. So, it will be p to the power four p square we have taken out m p square minus 3 k just like this yeah.

So, this two terms are taken out and you see this two terms will give you same root. So, the roots will be p equal to p 1 equal to 0, p 2 equal to p 3 equal to it will be 3 k by m root. So, these are the Eigen frequencies of the system. So, it was not that big as we have thought initially it will be of course, there are some small algebraic manipulation, but this is nothing difficult. So, we have the Eigen frequencies of oscillations. So, we have one zero frequency and other distinct frequency where, but the so there is basically one frequency which corresponds to vibration of the system; and there is a zero frequency.

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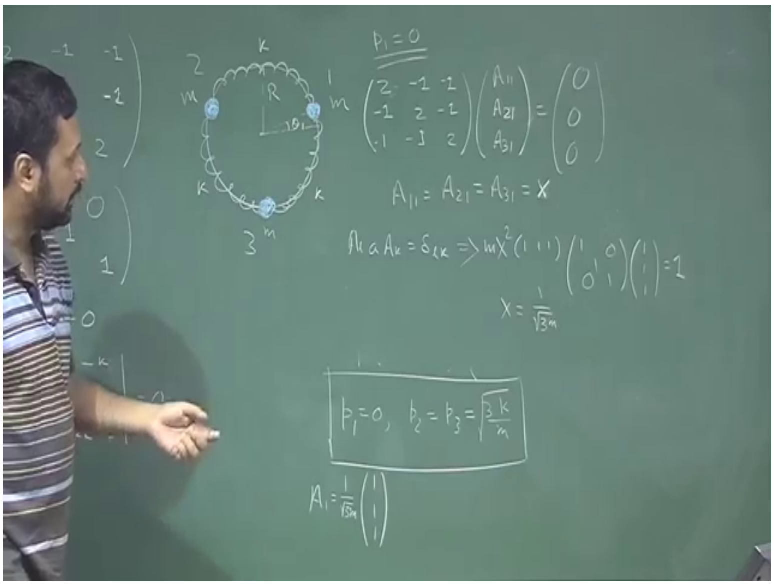

Now, zero frequency if we try to now calculate the amplitudes of each of the terms, if we try to calculate the amplitude. So, for p 1 equal to 0, it will be a straight forward calculation. We once again have to use this equation, but not in the determinant form we have to use it in the matrix form. So, if we do that b minus a p square. So, p is equal to 0, so it will be simply b. So, there is b matrix will be 2 minus 1 minus 1 minus 1 2 minus 1 minus 1 minus 1 2 and we have a 1 1 a or sorry a 2 1 a 3 1 equal to 0 0 0. And if you write out three equations and solve it you do not have to even solve it really if you look at it carefully you will realize it gives a solution that a 1 1 equal to a 2 1 equal to a 3 1 which is equal to some x.

And if we now use our normalization algorithm which is a l a a k equal to delta l k if we do that. So, if we have x square 1 1 1 m m m which is also 1 1 1 essentially. So, basically I can write this as 1 1 1 we multiply this by m and 1 1 1 equal to 1. If we do that what do we get we get x equal to 1 by root 3 m you can do it yourself. So, sorry yeah it is visible this part just shorten a little bit. So, it will be 1 1 1 1 1 1 0 here 0 here and 1 1 1 equal to 1. So, this equation we will translate to this. Now, if we do that we have x is equal to 1 by root 3 m. So, our solution for a 1, I will just write it here a 1 is 1 by root 3 m 1 1 1 fine. Now, for if we put p 2 equal to p 3 equal to root 3 k by m, if we do that just take p 2 equal to root 3 k by m.

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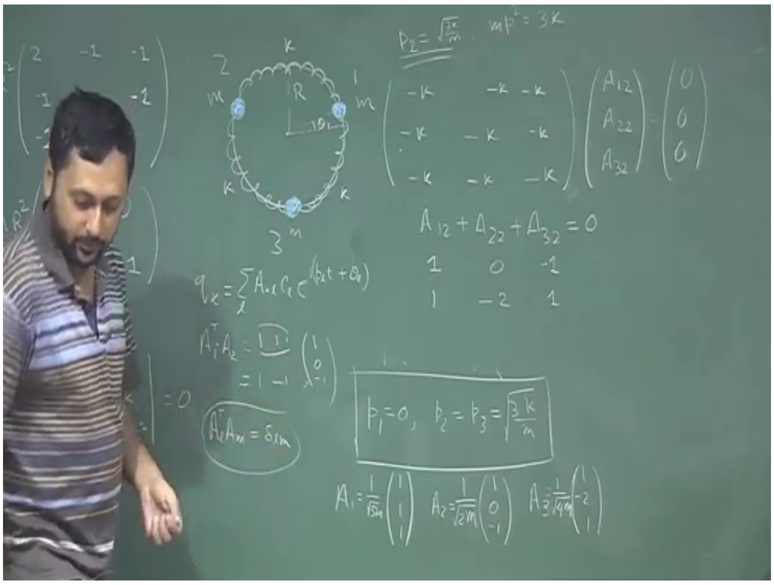

So, p square p square or rather m p square put something m p square will be p square yeah it will be 3 k. So, if you put 3 k here in this equation in or other in this equation multiplied by the matrix form, so you have 2 k minus 3 k minus k minus k minus k 2 k minus 3 k minus k minus k minus k 2 k minus 3 k very good. And we have A 2 1 A 2 2 or sorry it will be A 1 2 1 2 2 2 3, always mix up between which one should I use 1 2 2 2 3 2 we have to use it will be 1 2 2 2 3 2 or wrong just let me think.

So, q 1 will be a 1 c 1 no, no, no, it will be 1 2 yeah. So, c 2 2. So, 1 2 2 2 3 2 yeah this will be the one yeah which will be equal to 0 0 0, sorry I am bit I am always mixing up this indices. So, we have to better if we remember q k is equal to sum over l A k l C l e to the power i p l t plus theta l. If we remember this relation by heart, it will solve the problem. See for when we write the general solution q 1 will have a 1 1; a 1 1 c 1 e to the power i p 1 t plus a 1 2. So, this has to be a 1 2 and third one will be a 1 3.

Now, see this actually these are all minus ks. So, we have a minus k here, minus k here, and minus k here. So, these are essentially collection of three equations all three equations are will give you 12 plus a 22 plus a 32 equal to 0. Now, this is the nontrivial case because as in we can we can choose many combinations. We can choose n number of combinations we can choose you know a 22 equal to 20, a 12 equal to 0, a 32 equal to minus 20 that is one possibility. Similarly, we can choose this to be 25, this to be minus 5, this to be minus 20 that is also another possibility. But their catch is if you go back to the problem we have to find out the set of normalized and orthogonal Eigen vectors. Normalization is something that we can very easily do once we get rid of I mean we have to put this factor in front, so that is not the problem at all, catch is we have to find out orthogonal sets of Eigen vectors. We already have one vector we have to find out two other vectors, which are orthogonal to this one. And also mutually orthogonal between each other. So, the choice is a not many.

So, we can choose one combination, for example to be y and minus y and 0, this is one combination. This is definitely orthogonal, because basically we can x within a normalization factor, we can write this as 1 0 minus 1. So, if we write 1 0 minus 1, we will get. So, if we write A 2 equal to some factor y 1 0 minus 1. And if you compute A 1 dot A 2 or rather A 1 transpose A 2 it will be forget about the factors there will be some factors in front it will be 1 1 1 1 0 minus 1. So, if you do that it will be 1 minus 1 equal to 0, and this is the normalization condition or this is the orthogonality condition not normalization that a l a or a l dagger a m is equal to delta l m, this is the condition for orthogonality and we have to follow this.

Now, so I have not explicitly mentioned this I do not think I have explicitly mentioned this condition well when we were discussing the theory, but generally when we was talking about Eigen vectors in general even for rigid dynamics also, we have to I think we have discussed it in a slightly little bit. Whenever we are finding out Eigen vectors is it is important that we find out a set of orthogonal Eigen vectors in some cases it is not possible to find out orthogonal Eigen vectors, but at least we can you know we can try our best.

So, right now whatever we have we can choose to we have to make two choices which are at least linearly independent I mean these are all linearly independent by the construction of the problem all these vectors are linearly independent. Even if we do not find on orthogonal condition sorry orthogonal Eigen vectors at the beginning we can use something called Gram-Schmidt orthogonalization, which we will learn probably in your mathematical physics lectures, you have already known it using that we can construct a set of orthogonal Eigen vectors. But right now we do not have to do that much we have chosen let us say this one other combination will be this. So, we can choose A 3 equal to some factor z 1 minus 2 1 check between this two if they are orthogonal; check between this two if they are orthogonal, it will be I mean by the construction I am giving you the answer, but this is how we choose it.

Now, for normalization the normalization factor of this will be 1 by 2 two m whereas, for this it will be 1 by root 4 m. So, we have sorry it is not visible, I will 3 equal to 1 by root 4 m vector 1 by root 2 m vector and this vector. So, these are the three normalized Eigen vectors.

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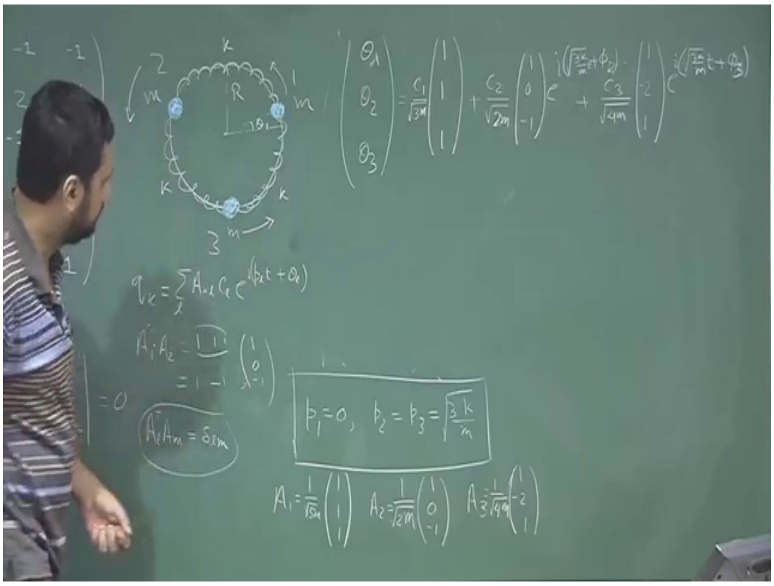

Now we are closing in order to finish this problem. So, the general solution following this construction can be written as sorry theta 1, theta 2, theta 3 I am just writing matrix equation which will include three linear equations C 1. So, there will be a factor coming due to normalization equal to C 1 by root thr3ee m 1 1 1, oh by the way this two orthogonal this two Eigen vectors are although. So, why we had trouble in order to find two mutually perpendicular orthogonal Eigen vectors, because the Eigen values are degenerate. So, you have to choose. And this is zero frequency. So, we do not have any amplitude or phase term 2 m 1 0 minus 1 e to the power i p, this is root 3 k by m plus phi 1 sorry t, t will be here and we have C 3 by root 4 m 1 minus 2 1 e to the power i root 3 k by m t plus phi 2 yeah great. So, this is the total solution of the problem. Of course, theta 1, C 1, C 2, and C 3, phi 1, and actually I should write phi 2 and phi 3 yeah because just to keep the keep it consistent.

So, we have five conditions and we need at least five initial condition five unknowns. So, we need at least five initial conditions to order to get a full description of it. But let us consider and at the end let us consider what are these normal modes telling us. The first normal mode normal with the frequency zero it is actually pure rotation and see that amplitudes are equal. So, when all these are moving in one particular direction with uniform angular velocity that is the situation given by p 1 equal to 0, p or other frequency equal to 9. So, you see amplitudes are equal and it has to be a pure rotation.

Second and third describes the same frequency mode, but with different sets of amplitude.. So, this is for one do not need all this all this ornaments we do not need any more. Now, for this one see one is going this way, this is 1, 2 and 3. Two is stationary, three is going the other way by equal amplitude. So, in this mode, one of the masses, so it is not distinguishable whether we choose this one, this one or I mean which one is the first mass we cannot say. But all we can say that in this particular normal mode, out of three masses one of the mass is stationary and two other masses are going like this or coming back like this. So, they are executing a oscillation around its equilibrium position keeping the second mass constant.

And if you now calculate the what you call the center of mass for this system, see at equilibrium the center of mass will be at the center. If you take originate at here the center of mass will be at the center. Now, I will just leave it to as an exercise to show that the center of mass is not changing in this oscillation. Similarly, for the third oscillation what happens is the first one and the third one they are moving in the same direction. So, if it is moving like this, this one is moving like this, and the second one is moving in an opposite direction with double the amplitude. So, if this one is like this, this one is going like, this one is going like that, so this one at the same time will go double the amplitude in the opposite direction; once again keeping the center of mass at the fixed position at the center.

So, I will just leave that to you as an exercise. So, this is a very good example and why I chose this problem to begin with because if you look at the matrix, the b matrix specially, here all these are connected right. So, we have a connection between first and the third frequency first and the third mass through this matrix, similarly we have a connection between second and third. So, it is a matrix where every it is a potential energy problem where every movement effecting every possible potentials or rather I would say this one is a movement of this one also influences the potential energy of this particular spring, although they are not directly connected. And also it gives you a very interesting concept of finding orthonormal Eigen vectors I mean orthogonal and normalized Eigen vectors in a degenerate case, so that is why I choose this problem we have two more problems and we have one more lecture in hand in order to finish this course. So, right now we have to close this lecture.

Thank you.