Classical Mechanics: From Newtonian to Lagrangian Formulation
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Lecture – 55
Small Oscillation – 3

So, we continue our discuss on small oscillation. So, at first, I have to correct myself about the normal modes. So, I had this doubts between N 1 and N 6 for a diatomic molecule.

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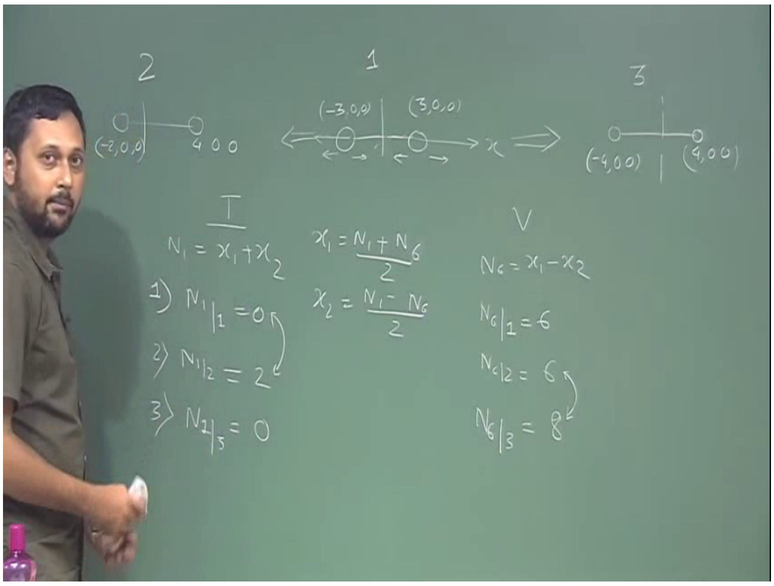

So, we had described this system we took coordinate x 1, y 1, z 1 for this one, and x 2, y 2, z 2 of this one. And we wrote translation notation and vibration degrees of freedom. So, there will be 2 3 translation 2 rotation and one vibration degree of freedom of the system, and for translation the first translation we wrote N 1 which was we wrote a we wrote it as x 1 minus x 2 actually it should be x 1 plus x 2. And vibration should be N 6 which is x 1 minus x 2.

So, somehow, I you know got confused between. This 2 just to show you that these are the right coordinates, let us take origin at the center of this molecule; so, let us say this is my origin. And let us assume that it is placed symmetrically into sides of origin sorry, it is not exactly in the middle. So, let us say this here now it looks better. So, let us assume that this has an x coordinate of minus 3, and y and z coordinate 0 0 to begin with. And these has a coordinate of 3 0 0, let us say.

So, if I now translate it to a new position, now if I translate it to a new position which is let us say towards the positive x axis. So, this is my positive x direction, this is my negative x direction. So now, if the new position is such that this one has become 4 0 0 and this becomes minus 2 0 0. So, when we were initially at this position, what was the value of my N 1? Then actual value of N 1 was initially it was for position 1, let us say let us call it 1 2. So, it was N 1 at 1 was 0, now at 2 it is minus 2 4 minus 2, right. So, it is 2.

So, it has changed. Now let us look at this coordinate. Initially it was N 6 at position 1, it was 3 minus minus 3 it is 6, and N 6 at position 2 if it is 4 minus minus 2; that means, 6. So, this has not changes. Now let us assume situation where the molecules are vibrating. So, let us call it position 3, and in the in vibration what happens? Whether both the either both the molecules go out simultaneously, or they come in simultaneously right. So, let us say they have at this point of time they are expanding. So, an instantaneous position is such that let us say this is 4 0 0 and this is minus 4 0 0.

So, let us let us see what happens to this coordinate. First N 1 at position 3 is minus 4 plus 4 0, and N 6 at position 3 is minus 4 minus minus 4 becomes 8. So, you see there is a change between this 2. Whereas, it is a change between this 2, but there is no change between here and here. So, when there is a translation motion from one when we come to 2 this coordinate changes. When it is a vibration motion from here to here, then this coordinates changes. And otherwise it remains unaltered. So that means, it is a proof that this one is actually a translation degree of freedom this one represents and this one represents a vibration degree of freedom. Interesting to note is that if you now want to express x 1 and x 2 in terms of N 1 and N 2, it is absolutely possible. So, you can write x 1 is which is equal to N 1 plus N 2 or N 6 by 2 and x 2 which is N 1 minus N 6 by 2.

So, the transformation from. So, normal mode as I said at right now, we are taking normal mode as a combination of Cartesian coordinates. So, similarly the Cartesian coordinates can also be represented as a combination of normal modes right. So, this we will keep in mind, we will definitely come back to normal modes in lot more details. So, what we did so far was we kind of able to you know formalize the discussion on small oscillation by writing the kinetic energy and potential energy matrices in a certain form in a in a particular form, and we derived the Lagrangian of the system the equation of portion for a coupled system. But before we go into that let us do some quick let us solve some quick problems quick questions, which does not include any of this tough mathematics which we are going to deal with next actually is a mathematics is not every tough some of it is already familiar to you. These are primarily Eigen value equations that we need to solve, but better before we go there it is better if we you know try our luck with some easier problems, which actually deals it is more conceptual in nature, if there is not much mathematics involved.

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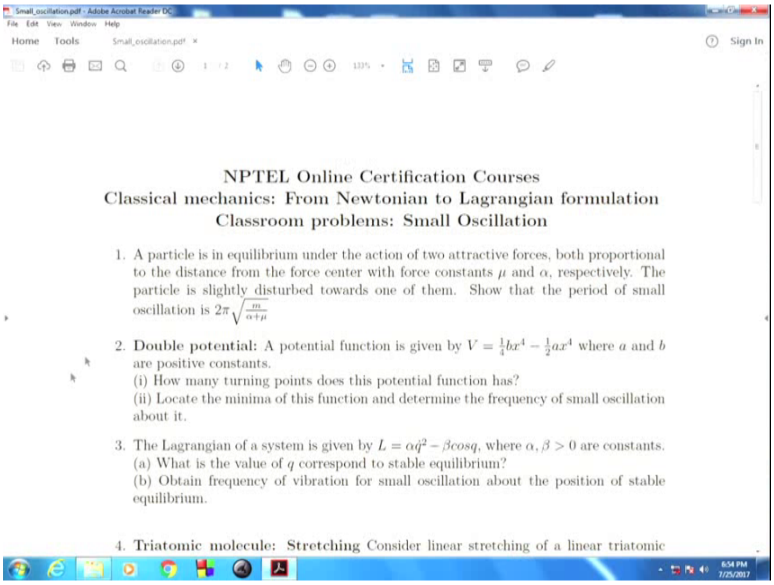

So, the first problem is this is your you know class room problem set for small oscillation. So, the first problem is particle is in equilibrium under the action of 2 attractive forces, both properties though both proportional to the distance from the force center with force constant mu and lambda respectively. The particle is slightly disturbed towards one of them. So, that the period of small oscillation is given by this particular formula. So, how do we solve this? First of all, let us look at the situation.

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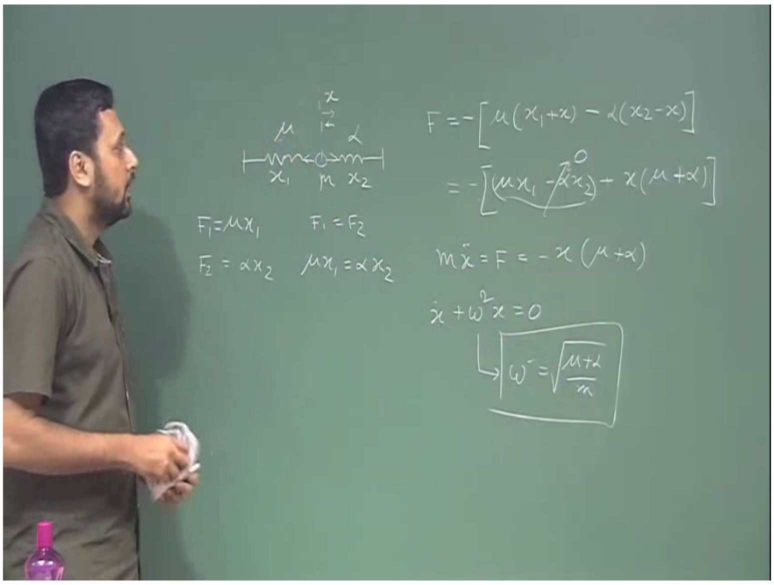

We have one particle which is in equilibrium, with the action of 2 forces 2 attractive forces from both sides. Now and analogy of this system, we do not know what type of forces this, but we can directly draw an analogy by adding 2 springs in 2 sides of this mass m, one with spring constant mu, one with spring constant alpha right. We can do that I mean right. Now we are not I mean I am just taking this an analogy do not take, take this as the spring constant as of now, but we can treat it. I mean we can treat a system in the same manner a system like this in the same manner.

So, when it is in equilibrium a system can be in equilibrium only when the forces acting on it is in balanced state. Now when is balanced it is balanced when we have let us say the distance is x 1 the length is x 1 and the length is x 2. So, as I said the forces are proportional to the proportional to the distance attractive forces and the forces are attracting to both ends. So, we have mu x 1 that is f one and f 2 is equal to alpha x 2. For equilibrium f one has to be equal to f 2 which gives you mu x 1 equal to alpha x 2.

Now let us assume that it is been displaced from it is equilibrium position, which is this let us say by some in one direction. Let us say in the positive direction by small x or yeah, some small displacement x. Right now, if we look at the force equation right down the force equation net force will be in which direction, the net force will be towards the position of equilibrium. Because it is a I mean if a system is slightly disturbed from it is equilibrium, and if the equilibrium is stable, then it will the restoring force will be towards the equilibrium I mean that is why it is called a restoring force to begin with right.

So, it is a stable equilibrium and we want to find out the force. So, force f which will come with the negative sign because it is a restoring force will be given by mu x 1 plus x minus alpha x 2 minus x. So, this is the force towards this direction, that is the force towards that direction this force wins. So, that it comes with the minus sign in the final equation anyway. So, sorry it is mu x 1 minus alpha x 2, and then we have x mu plus alpha right. And by virtue of this relation this 2 term this; this combination will give you 0.

So, the net restoring force when I write the equation of motion it will be m x double dot is equal to f, which will be equal to minus x mu plus alpha right clear. So, if this is the case we can compare it with x double dot plus omega square x equal to 0, and we immediately see that omega square is mu plus alpha by m. So, omega will be. So, this is your answer clear. So, what we did was we just write wrote the force balance equation, from there we found out the net restoring force wrote Newton's law, this minus sign is due to restoring force and from there we just brought it to the familiar form and frequency done.

So, this is how we solve problem number 1. For the next problem we look in to the double potential double world potential a potential function is given by v equal to 1 forth b x to the power 4 minus half a sorry, it will be actually x square. I have to rectify this the second term will be minus half A x square where a and b are positive constants. So, we have to we have 2 parts first of all how many turning points does this potential has, we have to say we have to look into that. And locate the minima of this function and determine the frequency of small oscillation about it.

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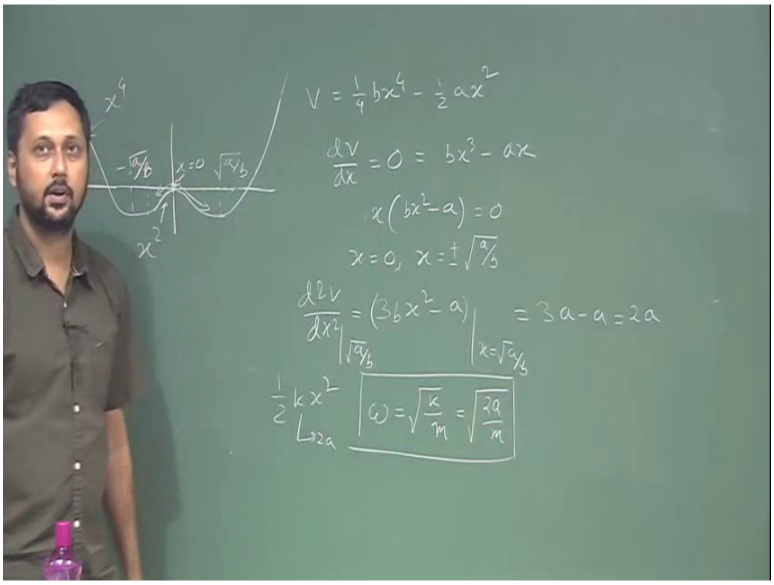

So, there are 2 parts to it any way, let us start with this potential which is one forth b x to the power 4 minus half A x square right. Now if it if you plot this function. Why it is called a double world potential? If you plot this function it will look. So, see this x to the power 4 term it will be dominating in the when x is greater than 1, but when x is less than 1 this term is lot more stronger right. So, we have x square or rather we have a parabolic variation for low x, and x to the power 4 variation in the I mean this term goes down and then it will turn and go up.

So, this is predominantly x to the power 4, and this is x square the for the shorter values right. So, this is more or less your x equal to 1, but anyway that is not important. So, it will be symmetry my drawing does not look very symmetry, but please note that it will be completely symmetric. So, I hope it looks little better. So, this is a totally symmetric case, how many turning points? It has it has 3 turning points one maxima and 2 minimas. And we have to find out the position of this turning point, which is very easy we all we need to do is we need to do d v d x equal to 0. We have to set this equal to 0 and d v d x is nothing but b x cube minus A x, which will give you x b x square minus a equal to 0, right.

So, we have 2 3 solutions one is x equal to 0 and f equal to plus minus root over a by b. So, we know this one is x equal to 0, that we know and this 2 1 is root over b by a another one is root over minus b by a, very good. Then we have to find out the frequency of small oscillation and needless to say that frequency of small oscillation will be valid only for these 2 positions. For this position there there is no small oscillation, if the system is here if you slightly deviate it will go in to one of this minima depending on which side you know give the displacement. And also, we can say that because this 2 minimas are you know they are I mean they are exactly identical to each other we will have equal frequencies in this 2.

Now, in order to calculate this frequency what we need to do is we need to take the second derivative, which will be 3 b x square minus a and we have to equate it we have to compute d v d t x 2 at 1 of this I mean as this 2 will give identical frequencies will just you know calculate at let us say this position. So, it will be x equal to b by a same thing x equal to root b by a, which will be actually it does not matter if you compute I mean if you compute here also, it is A x square term.

So, plus b root b by a or minus root by a it will give you the same number. So, this will be 3 b square by a minus a, is it right? Right 3 b square by a which will be yeah, 3 b square minus I square by a yeah a by b, oh sorry it is a by b, not b by a that is why. Sorry a by b, I am sorry. See this is the problem with a virtual class room, in a real classroom you would have you would have got it before me long time back we just wasted 1 minutes, let us say that makes more sense.

So, it will be a by b. So, yeah 3 a minus a which is nothing but 2 a and if it I is 2 a, then you’re and you see every potential can be brought in to the way especially one-dimensional potential can be brought into the form of half k x square, and in this case k is equal to 2 a and frequency omega will be root over k by N. So, in this case frequency will be root over 2 a by N. So, this is your final answer. So, whichever minima you choose you will have this frequency for small oscillation, fine.

Another problem the third one third one we can do also before we go in to the mathematics. Third problem is the following. Lagrangian of a system is given by l equal to alpha q dot square minus beta cos q where alpha beta greater than 0 and these are constants. What is the value of q corresponding to stable? Equilibrium that is the first question, first part and then we have 2 once again obtain the frequency of small oscillation about the position of stable equilibrium.

So, in principle it is the very similar problem, with the; I mean almost similar problem to the previous 1. We have a potential given potential is not given directly.

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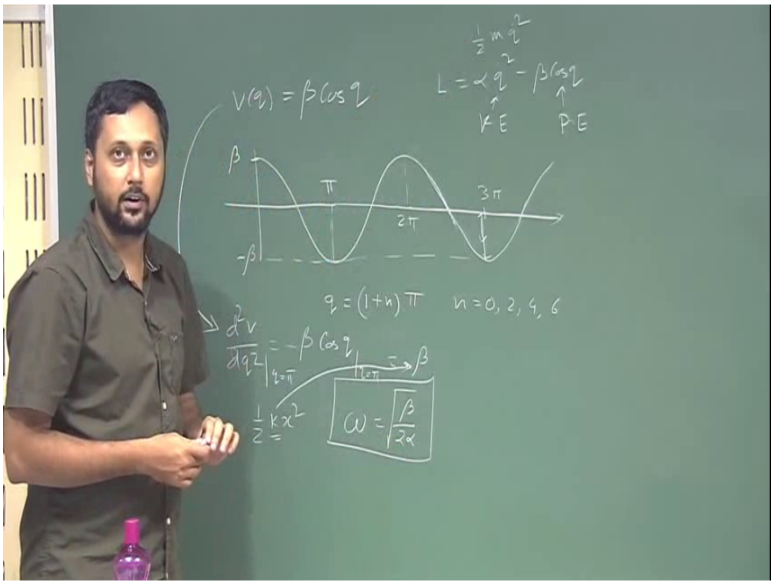

But when you write alpha q dot square minus l equal to alpha q dot square minus beta cos q. It is obvious that this one is the kinetic energy and this one is the potential energy, isn’t it? So, we have kinetic energy here, and potential energy there and if you now take the potential energy, which is the function of q as beta cos q; so, first of all we have to find out; what are the minimal position of this. And there is a very familiar potential right. It is a oscillatory potential it has a value of 1, I mean let us say beta is equal to 1, let us say.

So, if not then we will have a value beta at x equal to 0, then it will oscillate for forever. It is a infinitely long potential. So, there is there is not 1 minima sorry, this this is not a good drawing once again, it is not a good drawing either any way. So, there is no 1 minima to this system, but a series of minimas. First one comes at where does it come it comes at pi right. And the value will be minus theta, does not matter. What whatever is the value as I said the absolute value of the potential is not important we can always set that that equal to 0. What is important is where does it come. So, it is come comes at pi then at 2 pi it is again a maxima at 3 pi, once again it is a minima. So, again my drawing is never very good. So, once again it will have the same amplitude, and it will go on. So, we can say that the minima position of minima is general position is given by q equal to 2, what how do? How do you describe this? You can what you can do is we can write 1 plus N pi right? Where N is equal to 2 4 6? So, it will or 0 actually it starts from 0 right ya.

So, we can or actually it is not needed we can simply say N pi where N is equal to 0 1, no this is actually right. Otherwise we cannot generalize because we cannot, but 0 1 3 like this yeah. So, this is better sorry. So, N is equal to 0 2 4 6 right. So, you put N equal to 0 you get pi put 2 you get 3 pi 5 pi like this. Next and we can immediately see that is all this minimas are equivalent, we does not matter if the system is trapped here or here or in the next minima, it will give you equivalent value for your small oscillation, right.

Now, in order to find out the frequency, we have to take d to v d q to and this will be minus beta cos q. Because first time we take the derivative it will minus sign q and second time you take it will be cos q once again. So, that it will come with minus sign. So, if we value let us say, this as at q equal to pi, what do you get? You get q equal to pi at q equal to pi cos q equal to minus 1. So, you will get minus beta. So, in a for the small oscillation sorry not minus it will be a minus coming or from here it is a minus here. So, it will be a plus beta.

So, for small oscillation once again we can write it as half k x square, in this case k is equal to right. In this case k is equal to beta. So, your frequency omega will be root over k by m which will be beta by m whole root. By the way, now there is no m here, but from this kinetic energy part we see that alpha is playing the role of m, not exactly, because we typically wrote half m q dot square. So, alpha is equal to m by 2 oh. So, we have to write m by 2. So, we have to write m is equal 2 alpha right. So, it will be beta by 2 fine, this is my answer. So, we found out k is equivalent to beta, and we have found out from here that alpha is equivalent to twice of m right. Or rather m is equivalent twice of alpha and we found out the frequency good.

So, we solved 3 problems, which were pretty basic in a sense we do not need any we have only found out the position and frequencies of small oscillation, without discussing much about the nature of oscillation in general and all 3 there essentially one-dimensional systems. So, system of one parameter; so, these are very simple problem which is important for concept build up, but now we have to make things more general. So, let us try that. Let us go in to the theoretical discussion of small oscillation. Now once we do that just give me a second, right.

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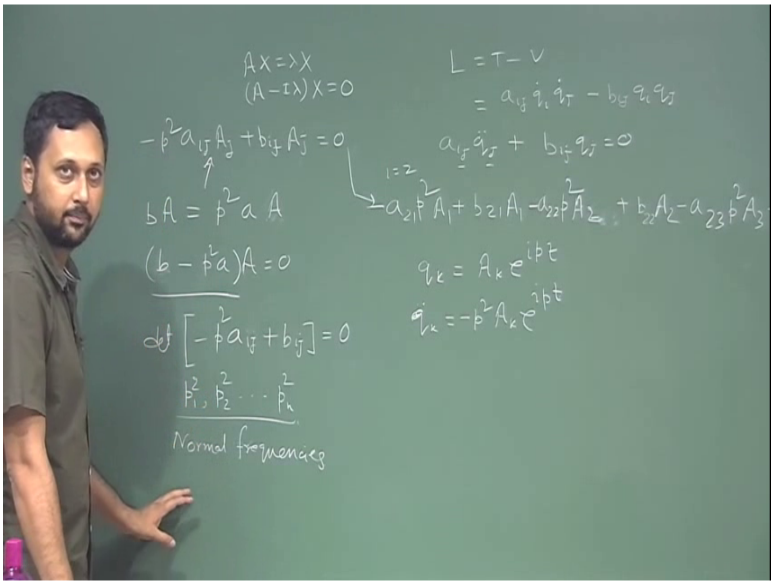

So, what we did already we wrote the Lagrangian L which is t minus v as a i j q j, a i j q i dot q j dot minus b i j q i q j. So, where i j's are dummy indices and there is the summation convention on this. From here we have found out the equation of motion, which was a i j q j double dot plus b i j q j equal to 0. And it is extremely decisive deceptive look because it seems like we have separated out the variables, and we wrote an equation for the parameter I mean particular coordinate q j, but this is very, very, very wrong. Because there is a summation convention involved. So, the j is the repetitive index.

So, actually this equation looks something like a let us say, let us take I equal to 2. So, it will be like 2 1 q 1 plus b 2 1 sorry, q 1 double dot q 1 plus b or sorry a 2 2 q 2 double dot plus b 2 2 q 2 plus a 2 3 q 3 double dot plus b 2 3 q 3, and it will go on like this. As many indices as you can think of I have just wrote one for I equal to 2. Similarly for I equal to 1 there will be one equation I equal 3 there will be another similar equation so on and so forth. And these are all simultaneous differential equations in the in the generalized coordinates q 1 q 2 q 3 up to q N.

Now, and on the right-hand side of each of this equation we have 0. Now this equations will have solution, oh sorry, this equation will not have any solution. These are sorry, but if we take a trial solution of the form right. If I take q k is equal to a k e to the power i p t if this is my trail solution, what we can do is we can write compute q k double dot which will be equal to. So, o p will come once and twice. So, it will be minus p square a k e to the power i e i e t right.

Now, if I substitute that for q double dot. So, what will have is for the first term for example, we will have minus p square and it will be A 1 and it will be simply A 1 similarly it will be a minus sign here. Minus a 2 2 p square 2 2 p square a 2 and this will be b 2 2 A 2. Similarly, this one will be minus a 2 3 p square A 3 and there will be other terms like this. So, this is again this equation is for I equal to 2, for I equal to 1 there will be one such equation I equal to 3 one such equation. So, it is a equation in which it is a simultaneous equation in a and p, right.

Now, please recall that please understand that as are you know for one of the one row of this or one particular equation, we can take a one common. So, we can take A one out sorry, this will be A 1 minus take this will be a one right. Sorry, sorry, sorry it is not the case. So, it will be A one A one, sorry we cannot take common write A 2 like this. But if I go back here if I write the generalized form once again, it will be minus p square a i j a j plus b i j a j equal to 0 right. And this when I open this; this will give you this type of equations simultaneous equations. Now this type of simultaneous equations you know we can.

So, we have the indices I and j running over from I and j running from one to N. So, we have N such equations, and each of this we can group into N such N such quantities of a and b. And what we can do is we can write this in form of a matrix equation which has the following form b A equal to p square A. Where we are not writing it in component form where b is a matrix which we found from the potential energy or kinetic energy. Sorry, potential energy; a is the matrix we found from the kinetic energy and A is the matrix of the amplitudes the column vector, right.

So, this is you know in a; it is just another representation of the same equation here. It is in the component form which will look something like this, and this is in the proper matrix form. Now is not it something similar to the Eigen value problem we have solved. We have solved Eigen value problems which was which has the form let us say A x is equal to lambda x. Now and if you recall what we did was we took a minus I lambda into one side and x and we put it equal to 0. So, similarly we can do the same thing here, A minus p oh sorry, b minus p square a times A.

Where A this capital A is the amplitude column vector, column vector represented by an by I mean representing all the amplitudes for all the you know mass points. So, this equation; this is the set of simultaneous equation. So, like one we have written here. So, there will be N such equations coupled equations. And they will have solution if and only if the determinant of this term vanishes.

So, the condition to have a solution is that so, is the determinate matrix or determinant of the of this matrix equation vanishes. So, the condition will be determinant of minus p square a I j plus b I j will be equal to 0 right. So, this is the condition. Now if this condition has to be satisfied, what we will get? Let us say we find out equations, I mean we open this determinant and we it will be a polynomial of degree N in or rather degree 2 N in p or degree N in p square. So, solving this we will get p 1 p 2 up to p N, right.

So, this will be the solution or rather p 1 square p 2 square up to p N square. Now this will give you us distinct values of the Eigen frequencies. So, these are all frequencies. So, this we can call the normal frequencies of oscillation right. Now from here we cannot really conclude much. Because these are the normal frequencies and we just say write now we know nothing about them. Now in order to you know gain more insight into what are the properties of p and what are the importantly what are the properties of A, these are actually the amplitude matrices though we can call them the normal anormalized amplitudes, right. So, what are these properties; in order to do that we have to go little more into the mathematics. Now, right now we I am not sure if we can go into the details, but one thing I can tell you. Before we can we start that may be in the next lecture I can tell you something.

Let us assume that. So, what is the system we have taken? We have taken a system of N coupled oscillators. Now each of this in a general motion each of this will have some degree of vibration, or it could be I mean it will have some amplitude. Now that amplitude could be 0 for some cases, for certain mass points, but in general there will be an amplitude. Now this amplitude will also have. So, each of this amplitude matrix will also have N number of components. Because understand the situation. Each of this Eigen values these are actually the Eigen values of this Eigen value problem, right.

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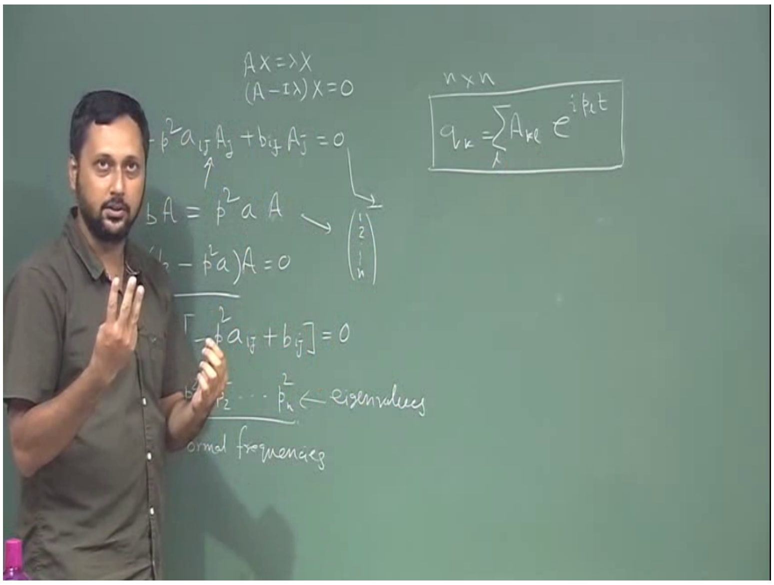

Each of this Eigen values will corresponds to one Eigen vector. One Eigen vector is one column matrix of N.

So, we have for each of this it will something like this we have 1, 2, 3, 4 up to N quantities. That is for each of these frequencies. Now for N such frequencies, we have N such matrices a N such column matrices which will have N number of entries each. So, at the end A the overall description of a actually is spanned into an N cross N dimension, think of think of it just take a 3 by 3 system. Let us say we have 3 mass points coupled. Each mass point oscillates or let us say so that means, the system in general will have 3 you know normal frequencies. Now for each normal frequency, each mass point have one certain type of amplitude.

So, you have 3 quantities. So, for second normal frequency we again have 3 independent quantities right; defining the amplitude of each of these mass full points. So, for 3 total normal frequencies we need to have 3 into 3, that is 9 number of amplitude values in order to get the full description, right.

So, that is why I have it is the dimension of total number of amplitude elements will be N into N right. So, the general solution q k needs 2 indices. It can be written as a k l e to the power i p l t where there is the summation over L. So, we will take this generalized form. When we will take an example, you will understand more in more details why this 2 indices are required or I think you already have a feel because as I said if I have a system of 3 coupled oscillators, for each and 3 frequencies for each frequency we have 3 amplitudes. So, for 3 frequencies we have to altogether 9 amplitudes, right.

So, we will take examples and see. Right now, we are writing this in this particular form. Try to understand that q k represents the amplitude of or the overall displacement of any particular mass point k which is a function of both l I mean k that index and there is another index on which the summation runs. And that index actually represents the variation in normal frequencies. Any way so, we will start from this particular generalized form of solution and develop from here and see.

Thank you.