Classical Mechanics: From Newtonian to Lagrangian Formulation

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Lecture - 54
Small Oscillation - 2

So, we are back and we will be discussing Small Oscillations.

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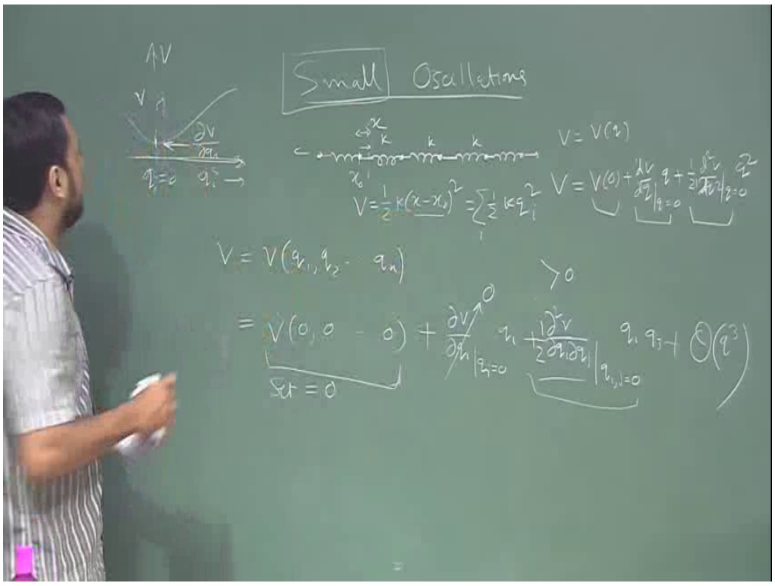

So, we have learned little bit about normal modes normal coordinates and we have learned the terms like stable equilibrium, unstable equilibrium and meta stable equilibrium. Now, out of this 3; we will be discussing two as in we will be discussing primarily stable equilibrium, but also under the approximation of small oscillation that is very important. We can also deal with systems which are closed to a; in a meta stable equilibrium.

So, first of all what is the term small means? See small and big; they are all relative; you can call a length you know we can call this much small compare to a kilometer, but this is huge when we are talking about a millimeter. So, these are all relative. We need to have some scale in order to define small and large or small or big whatever. So, why does this scale in this system there I mean in this problems which were in the context of small oscillation. First thing is we have to define what is small. So, let us assume that we have a system which has you know different degrees of freedom as in let us take a more specific example we will be dealing with systems which are coupled oscillating systems. So, let us say we have a system which is it is an infinite spring mass system; we have masses in this points and in between we have springs and we want to study the oscillatory behavior or the frequencies of oscillations and what not for this system.

Now, what is the procedure what we can do is if we go by you know Newtonian mechanics Newtonian dynamics we can find out the equations of motion we can do the same thing using Lagrangian, but that is not the question here the question; here is how what is small, right. So, I have taken this specific example because this is a system which is very well known to all of us let us say one all this spring constants are k. So, we now that the potential energy of a spring which has the; which has the spring constant k can be given written as half k x square where x is the compression or extension of that spring, alright.

So, each of this mass at equilibrium when the system is in equilibrium if there is no vibration present in the system there is no potential energy right. So, moment you start deviating the system from equilibrium these deviation is your x not the individual coordinates if that equilibrium coordinate is x 0 what you can do is you can write the equilibrium coordinate as let say x 0 and this coordinate you can write x. So, you have to substitute this with x minus x 0 square which will be your half k q square.

So, you have generalize coordinate or coordinate of interest is the difference or deviation from equilibrium position right. Now, if it is a system where we have not only one q, but we have for each of this masses we have you know q 1, q 2, q 3, q 4 for each of this masses I can assign an individual coordinate. So, here the potential energy V is a combination and we know that energy is additive we have one particle with energy one you add another particle e to in to the system with another particle with energy e 2 in the system the combined energy goes up by e 1 plus e 2. So, same holds for potential energy. So, if this is your V 1, then they will have V 2, V 3, V 4. So, total V is a sum over all these q i squares, right.

So, this is a general norm where this is a general form of potential. So, in effect V is a function of q 1 q 2 up to q n; n being the number of molecules. Now n can go very high it can we can have 2 mass spring we can have 3 mass spring 10 twenty and in the continuum limit we can have you know almost in principle we can have Avogadro numbers. Avogadro number is 10 to the power 23 which is the huge number, but important thing is we are dealing with the discrete system here if it is a continuous system this particular model of small oscillation. I mean small oscillation is actually an interface between a small system which is purely classical in nature and there is a large system which can be treated is a as a continuous medium.

So, in between we have a phenomena we have a we have a domain in which the masses are distinct each point masses can be identified individually, but their number is large as in large; it could be anything between 3 to 100; 200, but for of course, for our purpose we will be dealing with systems with degrees of freedom of 2, 3 or 4 because we have to do the calculation on pen and paper, but in principle we will be developing a formalism which can be applicable for any n; n number any way.

So, let us come back to this what is important is V is the function of q 1 q 2 up to q n. Now if q is equal to 0; that means, each if each of the mass points are staying in their potential I mean in their equilibrium position then we have potential of V where all the qs are 0. So, it will be it can be written as V 0 0 0, right. Now, one way of dealing with small mathematically is if we take a Taylor series expansion of a function around for a particular parameter and if I keep if I can approximate this. Let us do it if I have a function of let us say V single parameter half k q square.

So, what I can do is I can do a Taylor series expansion or rather I would say any function I do not know if its k q square also I have a function V of q. So, what I can do is I can do a Taylor series expansion around q equal to 0. So, it will be 0 plus V sorry del V or d V in this case d V d q at q equal to 0 times q plus half d to V d q square q equal to 0 q square so on and so forth.

So, the concept is here we try to see the; you know we try to compare each of this term with the first term V 0 and also with the initial value V. So, we have the matrix or the measure of smallness is how small this full term is compared to your initial 0th order term right. So, this is the matrix, what we follow and typically what happens is even if we take the quantity q to be very small; even if these derivatives are not very small by itself the value of derivative first derivative evaluated at q equal to 0 value of 7 second derivative evaluated at q equal to 0. They are individually not very small, but if q, I have taken as a number which is really small then q square becomes smaller, then q; q to the power q; q will be even smaller than q square so and so forth. So, the each weight of each term becomes diminishingly small.

So, it is typically what we can do is we can you know keep may be a up to first order term or sometime we keep up to second order term now we can do the same procedure here and we try to you know derive a Taylor series expansion of the function V around the equilibrium position. So, in this Taylor series expansion this will be the first term second term will be del V del q i at q i equal to 0 multiplied by q i this is a summation term for each of this individual qs; we have to have a partial derivative right and then the second term del 2 V del q i square evaluated at q equal to 0 and we have q i.

So, actually that is a mistake actually we should not write partial derivatives in like this in summation convention; what we should write is q i q j evaluate at q i j equal to 0 q i dot or here sorry q i q and there will be a factor of 2 coming here, right. Now if it is a potential function. And please remember that for a spring system where we have taken an example; here, it could be any system actually, but the special in this special example q equal to 0 is a stable equilibrium position. That means, if I deviate any of the point masses or all of the point masses slightly from this equilibrium position; they will get a each of it will gain a restoring force and we will try to come back towards the equilibrium.

So, if this is the case; it is not a; if it is a stable equilibrium and for stable equilibrium; we have already seen that this if this is your V; this is your q this has to be a point minimum point not a maximum point a maximum point is a non-stable equilibrium minimum. So, for stable equilibrium it has to be here in principle we can also have a meta stable equilibrium which has a local minima at this particular point and if I keep it small-small as in it will not if the it is a it is. So, small that the restoring force is always towards the equilibrium position even that is a valid description. So, that is a valid; there is definition of smallness for which we can the amount for which we can deviate from a meta stable equilibrium and gain a restoring force which will bring it back to the equilibrium.

So whatever it is, if it is either a stable equilibrium or a meta stable equilibrium both cases it is a minima could be a global minima could be a local minima, but if it is a minima; that means, this term is uniformly going to 0, because we know that the first derivative at a minima for any function vanish, right. And it is a multi dimensional function I am just drawing a 1 dimensional version of the same function, but it is a multi dimensional function of many qs. So, it will have a very complicated you know we cannot even draw it in a; in the 2 d length, but what we can think of like we did draw the configuration space; we can also try to draw the energy landscape where all the qis will be along the x axis at the overall energy V potential energy V will be along the y axis.

So, in this you know complicated hyperspace this will also have some kind of a minima either local or global fine. So, in both cases the del V i or del V del q i term evaluated at this particular point if I am taking this as my x equal to 0 forget about this axis which is arbitrary. So, we can take it here. So, this is my x or rather q equal to 0 all the qs are 0 here, right. So, this is my position of local or global minima and del V del q i is equal to 0.

Now, another property is the second derivative at a minima is always greater than 0 right that is something we have learned in our during our schools school life itself when we learnt the find how to find maxima and minima using derivatives we have seen that the second derivative is always positive at a minima. So, this term is 0 unanimously because we are or and let us assume that this smallness. We have to again define the small part smallness is such that the magnitude of this term itself is very small compared to this term. So, if this is the case we can truncate it here only we do not have to take the third derivative and above.

So, of course, there will be more terms which is of the order of q cubed we are just ignoring them now this is 0 and this term the coefficient of q i q j is positive definite and this one is a constant term and we know that if I it is totally our choice whether we take the 0 of you know 0 of energy here or here that is totally our in our hand, if I can set this one is as my ground level; I can set this one as my ground level or I can say this is my ground level 0 is in our hand. So, what we can do is we can set this term equal to 0 and this is a. So, this is 0 by choice this is 0 by definition and the coefficient of this term which is which you know which is a second derivative at evaluated at a local or global minima is greater than 0.

So, this potential function although we took an example of a spring mass system where we now that the potential function has the form half k x square, but forget about it.

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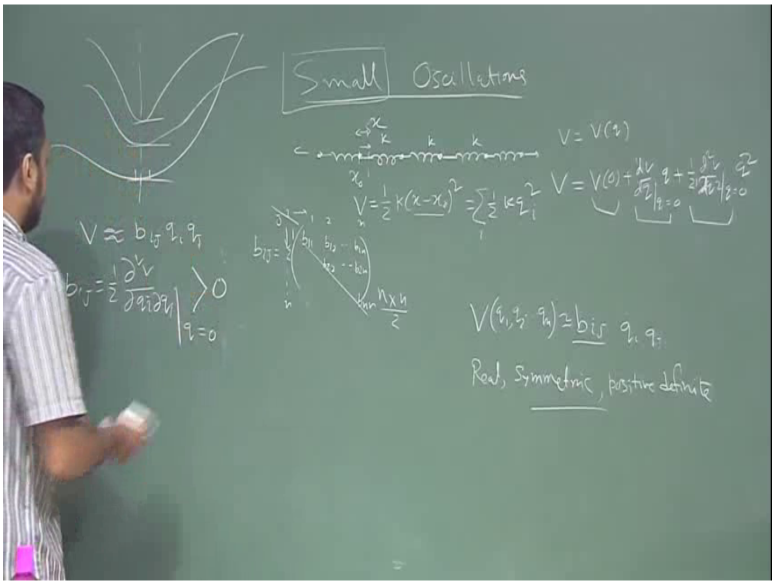

If we have any potential function worth of any arbitrary shape like this or you sorry oh sorry like this or like this or even asymmetric you know like this; if I evaluate it around this minimum point and if I approximate it for small oscillation small as in small such that we are not deviating much from the equilibrium this condition is satisfied the condition is this quantity whatever it is has to be much greater than this quantity, but again we are setting this to 0; do not mix up this 2; we are setting it to 0 just to shift the overall level.

So, if this the each of this expansion each of this potential when expanded in a small interval around the equilibrium will give you a quantity which is approximately equal to half k or rather b i j q i q j where b i j is equal to half del 2 V del q i q j del q j q i q j evaluated at q equal to 0 is greater than 0. So, this is your definition. So, essentially what we are left with only this term and this we can just write b i j. So, V under the approximation of small oscillation V q 1 q 2 q n is approximately equal to this and we can neglect the higher order terms, right, b i j is this once again it is a tensorial quantity because this is a I mean depending on the values of i and j the value will change. So, we can represent b i j in a matrix form.

So, we have let us say i values going in this direction and j values going in that direction. So, I equal to 1, 2 up to n j equal to 1, 2 up to n. So, each of this element it will be b 1 1 b 1 2 b 1 n similarly b 2 1 b 2 2 b 2 n b n 1 b n 2 b n n. So, it is a square matrix positive definite each of the element has to satisfy this criteria. So, it is a positive definite and must be real because a potential function in is a real quantity I mean potential is a real function the Taylor series expansion of a real function cannot be imaginary.

So, we have a matrix which is real symmetric and positive definite why symmetric because you interchange in the definition you interchange i and j nothing is changing. So, b 1 2 is has to be equal to b 2 1; so, in principle if I just write one corner of the; or upper triangle of the matrix or this part and put 0s here or should not put 0 actually just describe only half of the elements. So, it is a n by n matrix. So, if I have n by information about upper n by n divide by 2 number of elements I get the total description of the matrix we have seen this type of matrices real symmetric and positive definite not positive definite is something new, but your moment of inertia tensor was also real and symmetric, right.

So, we have this now in similar manner if I try to describe the kinetic energy kinetic energy of each system t.

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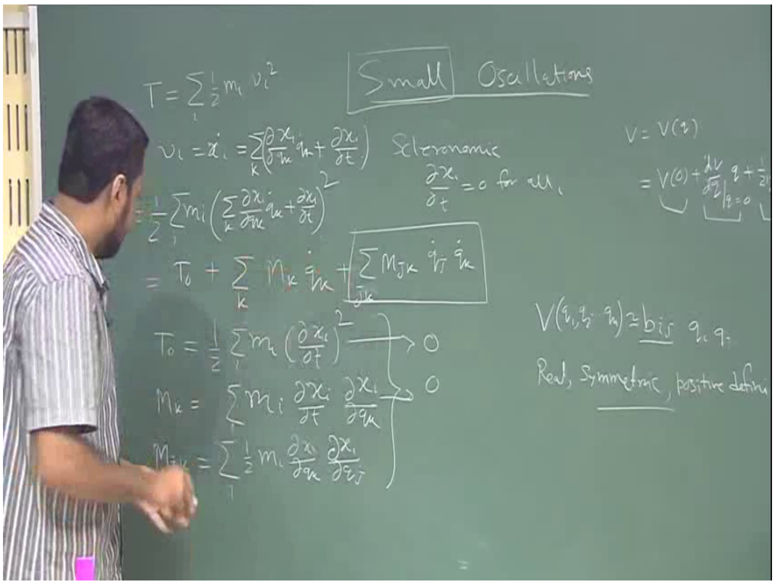

Total kinetic energy of the system will be a sum over i or rather it will be its just a minute; it is there is a form here right sum over i half m i V i square, right. So, where V i is the velocity of each individual system you can also write it as q i dot square. Now if I do that it will very long why I put it like this once again, I show you because just please remember when we wrote this V i was the velocity derived in Cartesian coordinate system.

So, V i is r i dot is equal to V i right now r i sorry; V i equal to r i dot. Now what is r i dot if you recall r i dot was d r d T which will be given by sum over k del r i del or del rather write it x I, it will be matching with your previous notes del x i del q k q k dot plus del x i del T is it not; this was the definition of velocity and in terms of generalized coordinate please remember T was originally defined in terms of vis and in principle which is given in Cartesian coordinates in principle we have to bring it in terms of generalized coordinates.

So, in order to do that what we have to do is we have to write this as half sum over i m i sum over k del x i del q k q k dot plus del x i del T whole square. So, if I open the brackets; please understand that I am talking about a general system. So, I have to do a general construction. Now if I open the brackets, it will give you half I will just write the final form of it because otherwise it will take little longer; it is a very simple straight forward calculation u k dot plus sum of a k j k m j k q j dot q k dot.

Now, what is T 0? T 0 is given by; I will just open the bracket and you will get all these terms T 0 is given by sum over i m i del r i or del x i del T whole square whereas, m k is given by half sum over i or sorry no half here sum over i m i del r or del x i del T del x i del q k and m j k is some over i half m i del x i del q k del x i del q j right. So, these are the 3 terms which will be coming in the kinetic energy expression.

Now, out of this if you are talking about scleronomic systems where constraint condition is not changing as a function of temperature function of time both these terms will vanish this term. So, if we are dealing with just scleronomic system then del r or del x i del T is equal to 0 for all I like this is the case, then immediately we see that this term and this term will give you 0 leaving behind only the last term. So, you can you can see that this is also a tensor and the last term. That means, only this term will survive which is quadratic in generalized velocities and with the coefficient m j k which has this particular tensorial form then this is also once again if you look careful. So, your total kinetic energy T will be equal to this.

So, I am just getting rid of all this. So, going by the construction; we do not even need j k we have T equal to this. Now this is the tensorial quantity which is also a function of coordinates q k and q j. Now this will this will be the serving term which once again can be expanded; so, m j k sorry; not a good way of writing.

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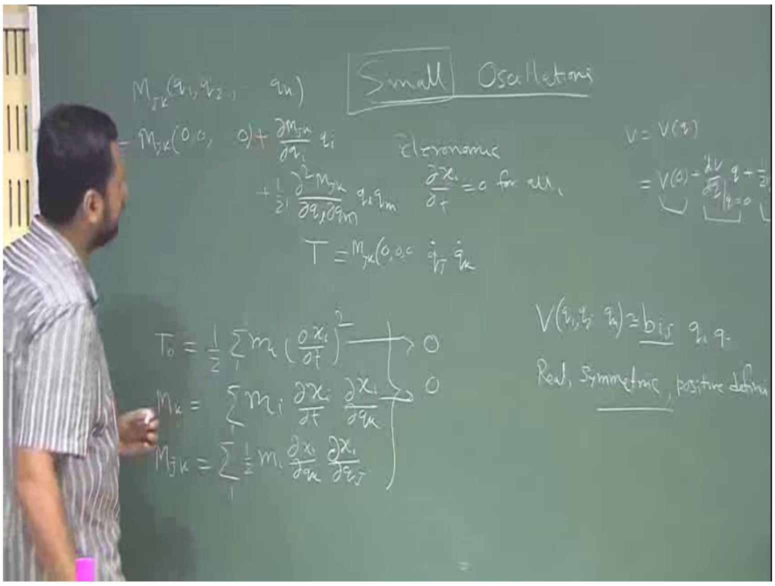

So, M; capital M j k which is a function of q 1 q 2 up to q n can also be expanded in a Taylor series. So, the first term will be m j k evaluated at 0 then second term will be del. So, del m j k del q i; q i plus 1 by 2 factorial del 2 del q k or q l del q m q l q m like that, but this quantity is now we are discussing a case of small oscillation once again. Now if the displacement is small, we can safely assume that the velocity is will also be small. Now if the velocity is the small we already have a quadratic term in velocity which will be even smaller.

So, if I now multiply another factor which is dependent on q or q y. So, it will go as you know it is I mean it is very vague, but it is smallness squad small into small. So, if I add a small another I mean multiply it with another term which is again small. So, it will be cubed going as something cubed of small very number cube root or sorry not true, but cube of some small number and this will be even higher quadratic in some small number. So, only in this entire expansion only the first term for a small real small oscillation only the first term will service the other terms will be really-really vanishing. So, we can approximate this with simple m j k or rather what we can do is only this term will survive in the series and we can write this term as a j k.

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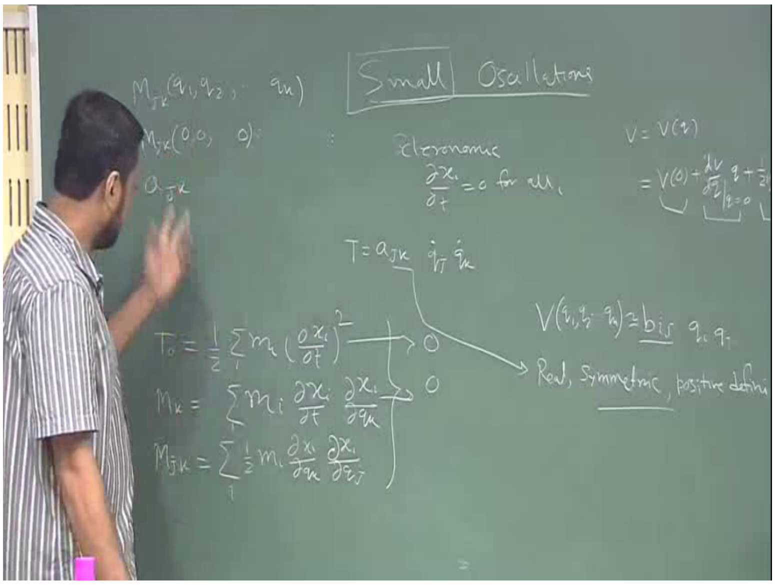

So, T you will be simply given as j k multiple by q j dot q k dot. So, a j k is the first order approximation of this particular tensor now it sounds very complicated, but finally, you will be writing it only in terms of generalized coordinates. So, whatever coefficient the velocity factors will have will come into your; a j k. So, you do not have to worry about all these mathematics; I always thought that it is necessary for me to show you that simply writing in terms of I mean if we start by writing in terms of generalized velocities it might not be the correct description. We have to do this transformation, we have to get rid of the first 2 terms of the transformation third term will remain even the third term will have a tensorial coefficient which has to be expanded in a Taylor series to get you the correct mathematical expression.

In reality when will be I mean we have to find out you know that kinetic energy and potential energy matrix elements there it is very easy because we will be looking at the expression and picking out the terms. So, that is something which is relatively easy, but once again a i j also has this following properties; it is a real matrix real because kinetic energy has to be a real quantity it has to be a positive definite quantity because if there is any motion and there is any energy; energy cannot go negative a kinetic energy cannot go negative potential energy can potential go negative because you can have a maxima, but kinetic energy whether it is there or the lowest value of a kinetic energy could be 0 and symmetric because if you; once again, if you interchange j and k in this expression or in this expression nothing changes.

So, it is a real symmetric and positive definite. So, we have for small oscillation if I have a system which is scleronomic in nature that is one condition we have to follow. So, it all boil downs to this we have a system where which is scleronomic in nature around stable equilibrium and deviated only slightly from the equilibrium.

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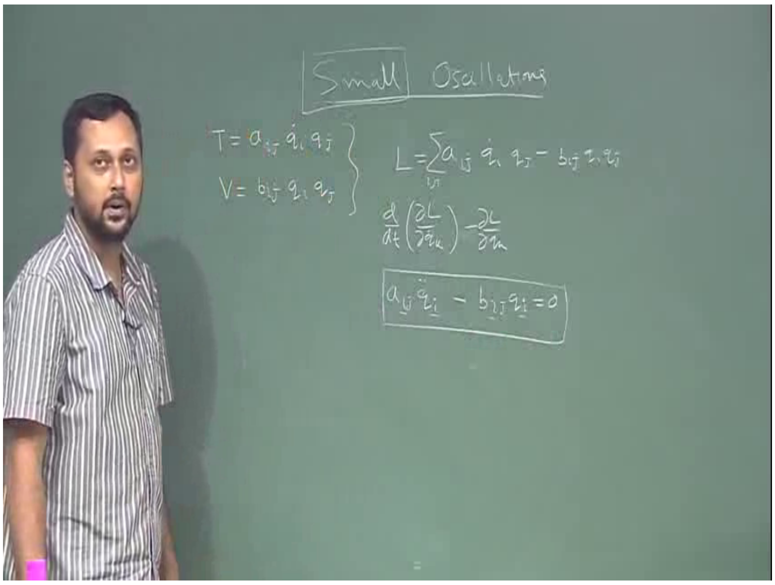

And we have T given by a i i j q i dot q j dot; I wrote q j q i mean j k, but does not matter because these are dummy indices we can change them we are we are free to change them and this is given by b i j q i q j.

So, the Lagrangian of a coupled oscillatory system which is oscillating around its stable equilibrium position and where the constraint condition is not a function of time that is very important your T or l can be written as a i j q i dot q j dot minus V i j q i q j very good. So, this is your general from of Lagrangian and from here we can derived the differential equation write we can do that.

So, please remember that form is once again for the parameter for the coordinate q k minus del l del q k very good and once we do that it will give you equation which will have a i j q it will be q j. So, I have to choose the k-th ordinate. So, it will be q i double dot minus b i j q i equal to 0 very good. So, we have in principle solved the problem because we have all this we have the equation of motion for the each of this coordinates and we are good, but look at it very careful this is a system of coupled oscillation there is summation involved in this expression there is the summation over both i and j running here also there is the summation over i running there is the summation over i running into the system. So that means, it is not a single equation, but a system of; I mean, but it is a very long coupled equation.

So, what we are end; what we have ended up with if we try to you know basically find out the individual equations of motion for each of the coordinate we will get an equation of motion which includes all other coordinates each of the mass point with ordinate q m or q j; we get an equation which includes all the coordinates, right. That means, if a system size is anything more than 3 or 4.

Let us say it is almost impossible to solve this by hand. So, you need to have some other technique other method in order to solve this coupled system of differential equations which will take up in the next lecture.

Thank you.