Classical Mechanics: From Newtonian to Lagrangian Formulation
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Lecture - 53
Small Oscillation - 1

Hello and welcome back. So, we first finish one remaining problem in variation principle then we will move to a new topic.

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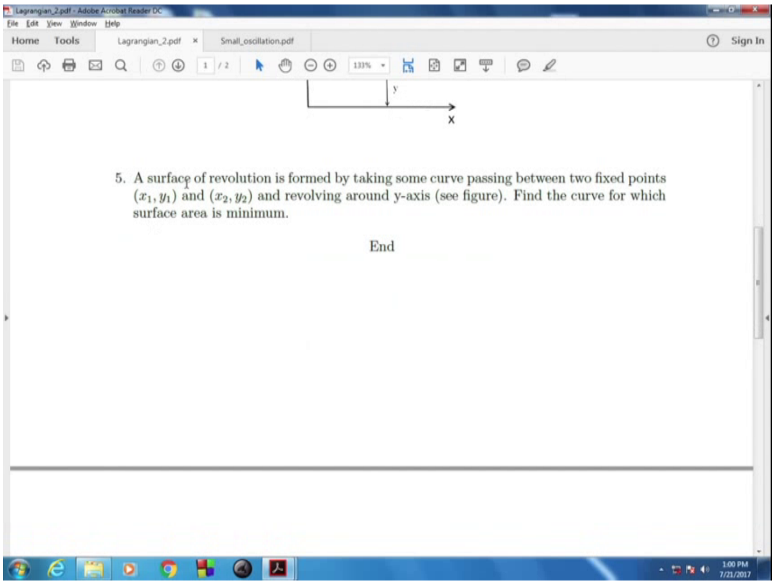

So, the last problem which was due was surface of revolution is formed by making one curve passing between 2 fixed points x 1, y 1 and x 2, y 2 and revolving around y axis. Find out the curve for which surface area is minimal.

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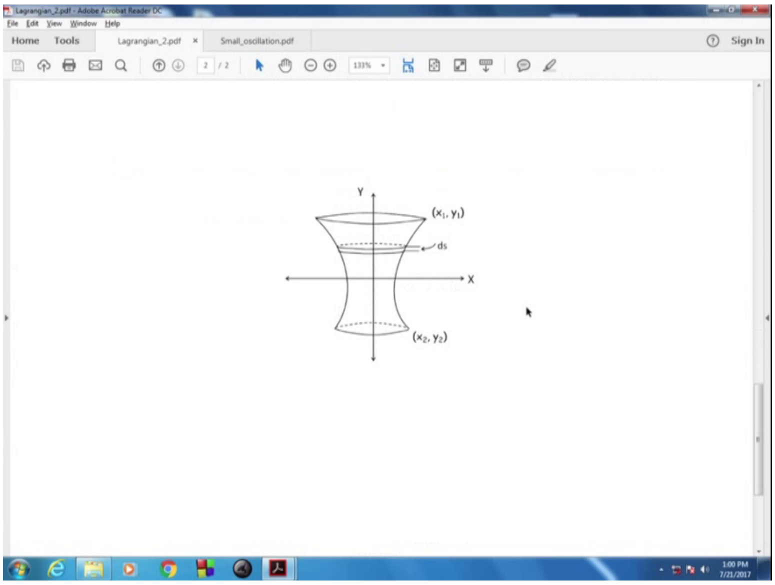

So, this is a picture given here. So, basically you can consider it as if it is a thread and I am holding it at these two points and making it evolve around this vertical axis.

So, here I have given this particular axis as y axis it could be z axis also, I have given y because we typically use so far we have used x and y as the variable.

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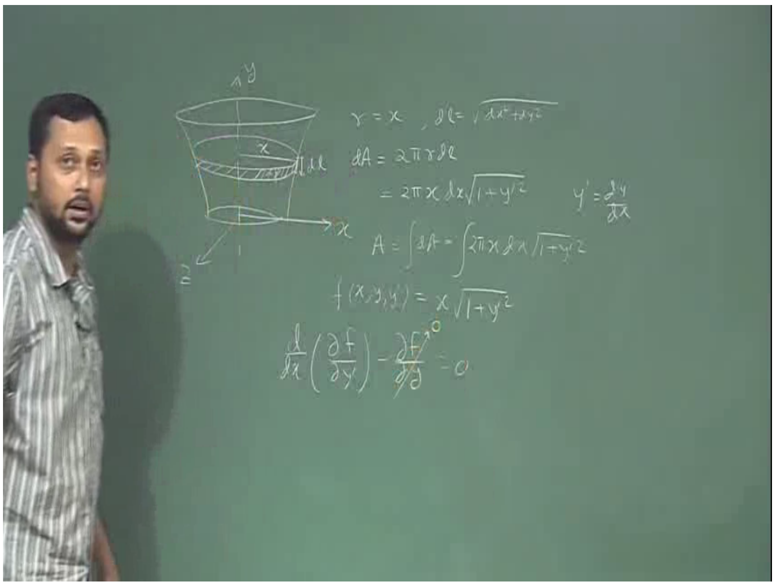

So, in order to solve this let us assume that we have 2 points like this and if I just hold this assuming that there is gravitation present it will look something like this and we know that this shape is called catenary; not exactly like this, but something like this. So, any way, but right now we are and if I now evolve it, let us say this is my axis and if I have to evolve it around this particular axis, it will give you a surface which will look something like this.

But right now this is not for you the; I mean we have to minimize the surface area. So, we have to take a curve which is not this, but some shape which is let us say like this as also given hinted in the figure; right. Now in order to solve this, we have taken this as x this as y. So, what we do is we draw a circle or basically we take a strip of area. So, the instantaneous radius is at this point. So, what we do is we take a slice on the x, y plane essentially it is a 3 d problem, there is an axis here, we can call it the z axis. So, if we take at take any arbitrary radius at any arbitrary orientation the radius will be given by r equal to x square plus z square.

So, what we need to do is we need to find out the surface of this strip and the surface of this strip is given by d A which is equal to 2 pi r d l d l being this length the thickness. So, we take a strip which has a thickness d l and a radius r, but now what happens is as we are I means. So, it is uniform right. So, this surface of evolution is uniform. So, it does not matter if I stay on this point or we go to that point. So, I have chosen a point which is exactly in the y x plane if you do that then you immediately see your r square will be simply x square. I mean r; sorry, it there will be a root here; it is we; it was root x square plus z square which will reduce to simply root of x square which is nothing but x similarly your d l in any arbitrary generalized position. Let us say this position is root over of x or rather not x, but d x square plus d y square plus d z square, but once we take the slice in the y x; x y plane this d l also reduces and there is no z term.

So, essentially what we did here just by choosing this as y axis, this I mean choosing this as y axis and you know taking a slice a taking this length and radius along A in the x y plane; we have reduced this 3 d problem essentially into a 2 d problem with variables x and y, right. So, now, it is easy. So, this we will write x root over. Now I have shown you many times how to write this, then we have to write d x 1 plus y dash square where y dash is equal to d y d x that is it.

And so total surface of evolution A will be integration of d A which will be integration 2 pi x d x root over 1 plus y dash square and this is; this will give us f of x y y dashed as x root over 1 plus y dash square. There will be a factor of 2 pi, but because it is a constant factor and we have to take the equation from here; it will cancel out any way and please note that this function, right. Now the construction the construction which you are we are using this is this as x; it has y prime, but it does not have y.

So, if I to take or sorry; not l del f del y prime minus del f del y equal to 0. We see that del f del y is equal to 0 anyway. Now if is if this is the case then this equal to 0 which implies; this is a constant C and this particular derivative is x root over.

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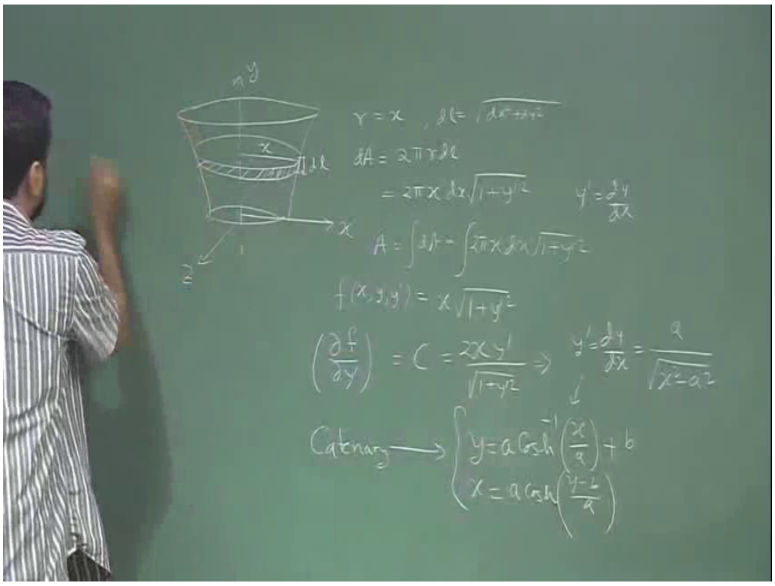

So, it will 2 y prime 1 minus y square and now we simplify this a bit and from here we get the differential equation that y prime which is d y d x is. So, I will just give you the final form is a by some constraint a by root over x square minus a square right.

So, this is where we stand right now and the solution of this will have the following form the solution of this. So, I told you I in the in the previous problem; in last lecture, I could not write the final form of catenary equation this happen to be the final form of catenary equation; I mean not exactly basically it will be a there was a factor 1 over some factor was missing, but the solution of this will actually be y equal to a cause hyperbolic inverse x by a plus some constant b where a and b; a is this constant and b is some arbitrary constant of integration. So, this is right. So, we see that this is y equal to cos hyperbolic function which will lead to a function for functional form of x which is y minus p will come. So, it will be a cos hyperbolic no inverse y minus b by a.

So, you see I mean although it is not exactly a equation corresponding to cos hyperbolic function it is strongly related its inverse of hyperbolic function which can be expressed. So, basically x and y has a relation which is given by a hyperbola so that is; that means, also this curve which will the shape of this curve which will produce minimum surface area is also a form I mean the this is also a catenary, right.

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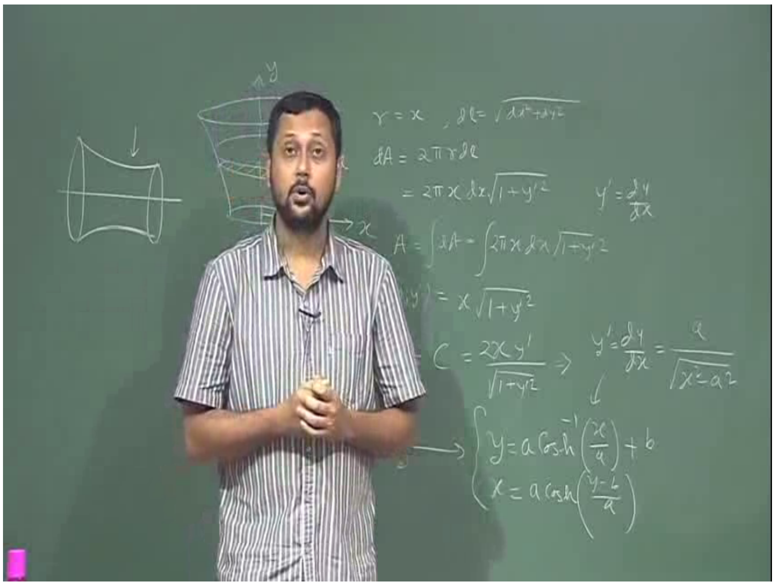

So, for a catenary not only the freely hanging shape under gravitation will be a catenary, but also a I mean a surface of evolution which is produced by rotating a catenary will have minimum surface area the same problem can be solved in a you know in a horizontal geometry as well we can just take the chain like this and try to rotate it and it can be shown that even this surface area will be a minimal. So, this is something which I am living it up to you for an exercise the same problem essentially instead of keeping it vertical we are just we are making the axis of revolution horizontal right axis of revolution horizontal great. So, I think that is it for variation integrals and right now.

So, we have basically finished the chapter on Lagrangian as well the last topic on of Lagrangian was the variation integral of course, any problem which will which we will come; we have come across. So, far during our discussion of Newtonian mechanics except the problem of variable mass can be tackled in terms of Lagrangian's equation of motion. So, essentially Lagrangian's equation I mean Lagrangian's formulation offers you just a new approach a new path way to the same set of equations the equations we are getting are exactly identical of course, we have this kind of variation integral which is somehow related to Lagrangian's you know you know Lagrangian's formulation, but it is not necessarily; it is not necessary that we apply it for only for classical mechanics; these are essentially static problems right the problem when a chain is hanging and so, it is hanging by minimizing its potential energy which is a static problem. So, even that can be solved using variation integrals.

Now, what I am trying to tell you that right now. So, far we have done starting from the beginning of the course we have learnt initially we reviewed the kinetic kinematics of associated with Newtonian mechanics basic Newtonian mechanics then we went into central orbits of course, there were systems with variable masses and all then we moved into central orbit rigid body dynamics in between we had little bit exposure of moving coordinate system which was once again used for rigid dynamics and now we have learnt how to handle the same set of problems with Lagrangian except for the particular familiar problem in which mass is changing as a function of time that even that can be tackled using Lagrangian, but that is lot more difficult.

So far whatever we have learnt is different fundamental aspects of classical mechanics right now the last topic of this lecture is probably the first area in which we will be applying all the techniques we have learnt. So, this is the area which is relatively new. So far the areas we have studied. they were all developed during sixteenth and seventeenth and 18th century, but small oscillation is something which is much new in a sense that if the latest developments of theory of small oscillation using a quantum calculation still going on and people is people are still using it I mean actively using it for analyzing spectroscopic results vibrational levels of different molecules and you know configuration of solids and all.

So, this is a very active field; still now people are using it actively and the mechanism what will be following is a mixture of classical and a sorry the Newtonian and Lagrangian I mean what I mean is the again what I should say is we will be using Lagrangian primarily to begin with, but slowly and slowly; we will be going more into the you know details of matrix diagonalization and see how what insight we can get about the system. And most importantly we will be defining few quantities one is normal coordinate second is normal frequencies and third is normal mode.

So, typically we do this when we will be when we study oscillation normal modes and normal frequencies they are typically define for coupled oscillated systems could it be 2 coupled oscillators could it be could it be more than 2 coupled oscillators, but generally for single oscillator only one frequency is possible and that is the normal frequency. So, it do not state that explicitly, when it comes to coupled system we start you know defining this terms, but let me tell you something strictly speaking normal modes and normal coordinates they are no way related to vibration only.

So, let me start this discussion of small oscillation with a discussion on normal modes. So, typically people do it at I mean; typically, we keep take it take the subject of normal mode towards the middle of the process like when we come to that junction where we have to either you know forget about solving the equations or we have to take normal mode solution and we can proceed, but for me I think I have realized that it is better for the students if we start with the basic understanding of normal modes and normal coordinates.

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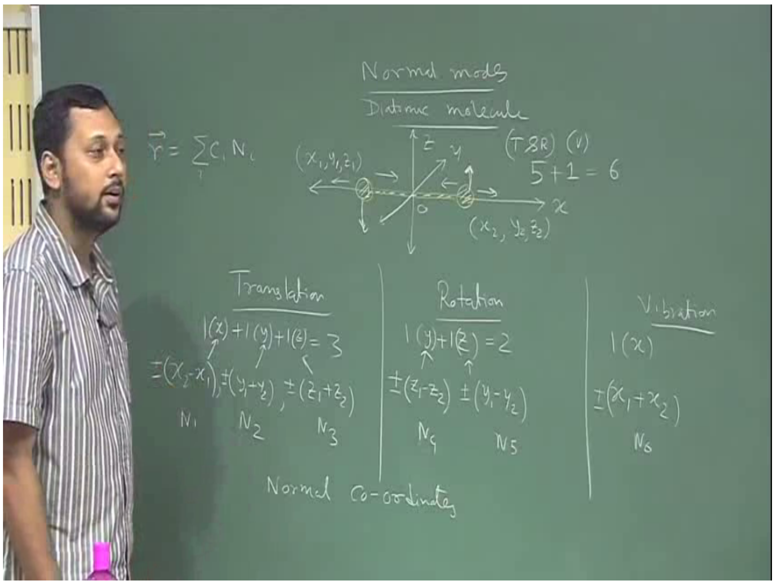

So, we start our discussion on normal modes and let us see let us say we take a diatomic diatomic molecule as a model system; what is a diatomic molecule we have; let us say 2 atoms or 2; 2 atoms joint by a bond. So, right now, we can we can forget about all the you know quantum mechanics associated with it of course, if it is a true diatomic molecule, I mean the if the system is that small there will be lots of quantum effects uncertainty effects and what not, but right now; let us forget about all this let us treat this system classically.

Now, in a system where we have 2 masses sorry; in a system where we have 2 masses and 1 let us say mass less rod attaching this 2; how many degrees of freedom do you expect we typically expect if it is a rigid system, then we expect 5 degrees of freedom why because total degree of freedom will be 3 in 3; 3 plus 3 which is 6, but because there is a constraint condition, if it is a rigid rod which is holding this 2 end plus; we take 1 degree of freedom out of the system and we have 5, but let us assume that it is not a rigid system, if it is a rigid system, there will be no vibration.

So, it is not a rigid system; that means, if it is vibrating how many degrees of freedom, it can be now; it is a; essentially it is a 1 d system. So, if we put the axis here x and z let us say y is out of this plane. So, this is my x, z. And let us draw this y also. So, this is y. So, this is my axis system or rather let us makes it x, y, z it will be easier. So, let us say no actually bit difficult let us make this x. So, y is pointing n and z is up, right; this is the right handed system we follow. So, we 5 degrees of freedom, but now we have vibration and just by inspecting the system, I can tell you that vibration can occur only along; the x axis y and z axis; it cannot vibrate it is a 2 molecule system.

So, you know if it goes up and down like this around z axis; this will not be a vibration, but it will be a rotation, right. So, it can translate along x; it can translate along y, it can translate along z, it can rotate around y it can rotate around z and it can vibrate along x. So, these are all possible types of motion. So, we have translation; we have rotation and we have vibration, right, how many types of translation are possible as I said; it can translate along any of the axis. So, we have 1 plus 1 plus 1 equal to 3 types of translation one is due to x 1 is due to z and the third one is due to y rotation it cannot rotate around x because it is a, I mean although its I have drawn it like this; it is essentially a 1 dimensional system I mean it is so tiny that rotation around its own axis will not be counted.

So, there could be two rotations one along y and 1 along z. So, here also I can draw it like this; how many vibrations only 1 because as I said x and y it can rotate it cannot vibrate. So, there is only one vibration which is along x. Now let us say the coordinate is x 1 y 1 z 1 and for this its x 2, y 2, z 2; the coordinate of the point masses, right. So, we have altogether 6 coordinates and we have altogether 6 degrees of freedom 5 due to translation and rotation and one due to vibration and altogether 6 and we have 6 coordinates. So, 3 plus 3; 6 and in principle the 6 coordinates I mean 6 number of coordinates should be sufficient in order to describe the system describe all this types of motion, right.

Now, what we all we all try to do is we try to find out a combination of this coordinates each of which will talk about one particular type of motion it could a rotation around z it could be a translation around x along x or it could be a vibration along around a along x, right. So, let us try to get the combinations which will gives us the correct; correct description let us first take the translation along x. So, if it translates either along this direction or along that direction both the masses will follow the same direction for a translation it will just move like this or like this the whole system. So, if a translation is taking place along let us say positive x direction; that means x 2 is increasing and x 1 is decreasing.

So, if I take a combination which is x 2 minus x 1 this will always be increasing right because x 1 and x 2 they have opposite sign similarly for combination which is x 1 minus x 2 will give a translation along negative y axis negative x axis. So, what I can do is and now forget about this positive negative business what we can do is we can put a plus minus sign here to take of that and I can tell you that plus minus x x 2 minus x 1 is responsible for a, it will be taking care of any translation which will be happening along x axis now for y axis when a translation is happening along y axis both y 1 and y 2 will increase. Similarly for a translation which is happening along positive z axis both z 1 and z 2 will increase; if it is happening at negative z axis both z 1 and z 2 will decrease.

So, if I take a combination which is plus minus y 1 plus y 2 and plus minus z 1 plus z 2 y 1 plus y 2 and z 1 plus z 2; both of this will give you translation around along y and z respectively is not it think of it x is something different because in x. There are 2 coordinates the origin is here one has a positive coordinate one has a negative coordinate with respect to the origin, but for y and z both changes in the same direction when it moves along z both gain since z 1 and z 2 if it goes along minus y. Let us say along this direction both losses in terms of y 1 and y 2.

So, these are the right coordinates this combinations of coordinate will give you translations around x along x and y and z. Now for rotation around z what happens if it rotates around z, let us say in this direction; this one these molecules you know the y component of this molecule it or let us say about y axis. So, y axis is kind of pointing in to the board. Now if it rotates around along y axis. So, it is this one goes up and this one comes down. So, if this one goes up it will gain in z 2 and if it comes down at the same time it will lose in z 1. So, if I take a combination of plus minus z 1 plus or z 1 minus z 2 this is good enough to give me a rotation around y axis think of it when it rotates around y axis this combination will always increase.

So, either z 1 is going up and z 2 is coming down. So, z 1 minus z 2 will always be positive or the other way around whatever it is this will give you a rotation around y 1 y; similarly a combination of y 1 minus y 2 will give you a rotation around z. So, we have 1, 2, 3, 4 and 5; 5 coordinates already which will 5 coordinates which are combination of this original x; x, y, z coordinates which will purely give you one of the motions I mean which will be responsible for one of this possible motions similarly for the last one which is a vibration see what happens in a vibration in vibration if x 1. I mean this one goes in this direction this mass this one goes in the other direction like this, and when they come out they come out at the same time.

So, when x 2 gaining x 1 is also gaining when x 2 loosing if the value x 1 is also losing the value. So, we need to if we take the combination x 1 plus x 2 of course, there will be a plus and minus will it give what will it give you; it will give you the vibration when x 1 increases x 2 also increases both increases at the same time. Please remember that these are you know, so this is the right combination very good.

So, you see this if I just now level them N 1, N 2, N 3, N 4 and N 5 and N 6; these are the combinations which will give you describe one of this types of motion independent of the other. So, when I am changing; when I am describing a motion which is a combination of let us say translation rotation and vibration I can take a suitable combination of this 6 coordinates; see if I have I am having a slight doubt may be I have messed up with N 1 and N 6; I think these are write, but it might be might turn out that plus sign will give you translation and minus sign will give you vibration which I have to think and if a; if it is wrong then I will get back to you, but that is not the point here point is I can do? I can find out a combination of coordinate which will purely talk about one particular type of portion and that is what we have done that is what we have one here and if for any general motion which is a combination of a rotation a translation and what you call vibration any combination of motion can be represented by a coordinate r which is a linear combination of this Ns.

So, any general displacement of this system can be taken written in a linear combination of this N 1 to N 6 coordinates N 1, N 2, N 3, N 4, N 5 and N 6 and these are your normal coordinates. So, if you have see if you have thought that normal coordinate is something that is only associated with vibration you are totally wrong please keep that away from your head forever normal coordinate is something any coordinate which is actually a combination of Cartesian coordinate which describes one particular type of motion which describes only one type of motion. So, if I am talking about a translation I have to choose between this 3; if I am talking about a rotation I have to choose between here and vibration about here.

So, all these are normal coordinates and a general motion is a linear combination of this important is all these are independent of each other. So, if a translation is happening at the same time a rotation may also happen right similarly a vibration may also take place. So, all this coordinates they are not only you know you know pointing I mean give describing one particular type of motion they are all independent of each other. So, this is a set of coordinate which is also capable of serving as a generalized coordinate because they are 6 in number and they are independent of each other. So, these are called the normal coordinates.

So, now this is the fundamentals of normal coordinate and as I said I will check once again between N 1 and N 6. If I find; if this is not write then I will get back to you. Now what we will do is we will just give you a very brief description of equilibrium before you stop this lecture. And we will continue from there in the next lecture.

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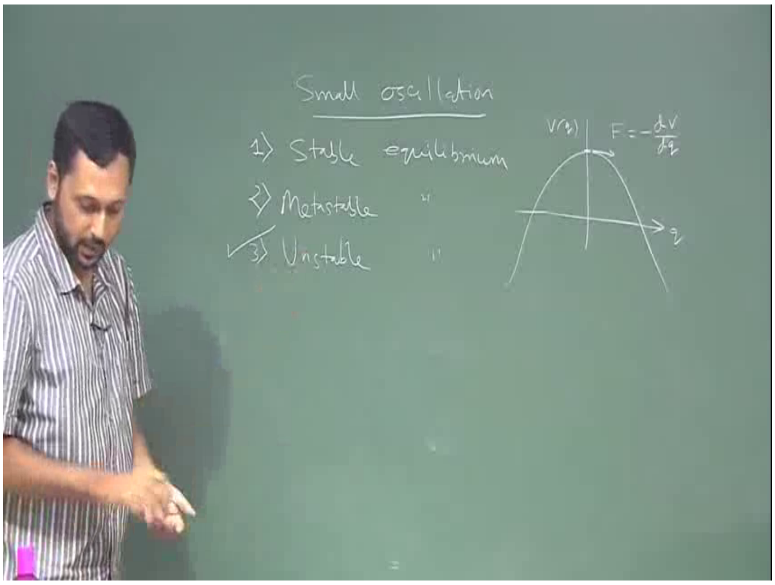

Now, this is normal coordinate which is has nothing to do with vibration as such as in its not no way; it is related only to vibration, but where what we are going to discuss is small oscillation and in small oscillation it is all about vibration, right.

So, what happens is we before we going in going into the details of small oscillation let me tell you; what how many types of equilibrium can exist in a system in a system we can have stable equilibrium we can have a metastable equilibrium and we can have a unstable equilibrium noted this let us take a 2; sorry, 1 d potential function and I will demonstrate; I will just give you an idea of what is stable what is metastable and what is unstable. So, 1 d means there is only 1 parameter. So, let us say we have a our parameter q and v of q if v of q have this shape.

So, we have there is a maxima here now if a system of interest stays at the maxima and if we perturbed it slightly from a maximum position what happens if it comes see the force is always. So, force is the slope of a potential function at any point because F is; as we know is given by minus grad. So, in this case it there is only one function it will be d v d q negative slope. So, here you see the slope is in this direction. So, the force will be driving it if he deviate; it slightly the force which will be generated driving it away from the equilibrium, right.

So; that means, the system will fall it never go back to the equilibrium it is not restoring in nature, but it will be driving away from it from this point. So, the system will fall out of equilibrium and this is an example of unstable equilibrium. So, this is what is meant by unstable equilibrium and then we have another situation in which we can have a potential let us say like this.

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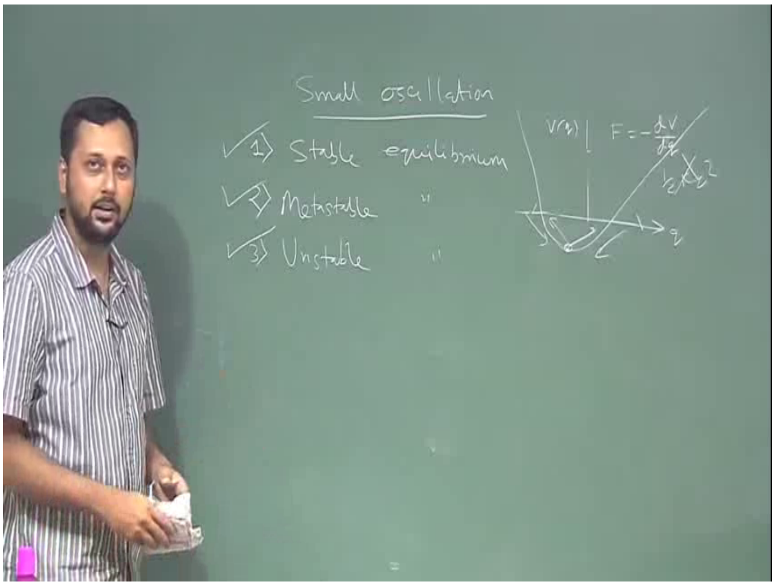

Actually it is not a very good drawing it should be even shallower; you can have a function like this. So, we have a system here we start deviating it from the equilibrium. So, there is a slight slope. Let say or I can actually draw it in a slightly different manner what I can do is I can make it look. So, let us say the system is here; start deviating it from this direction restoration force will bring it back deviate; it little more up to here again the restoring force will bring it back deviate; it up to this point see there is a slight hinge some maxima deviate it up to here and immediately at some point the restoring force will start pointing into that direction.

So, for small deviation and the system will fall out of equilibrium, for small deviation from equilibrium position the system has a restoring force which is driving it back to towards equilibrium. And if the deviation is large it is following of equilibrium. This is an example of meta stable system.

And if we have only one minima something like this whatever where ever you go it does not matter I mean it will continue like this. I am just not giving it a specific shape; no, do not even think that these as a form half k x square or half k q square. No, it is not the case, I am just drawing a general potential which has only one minima and it is extended the other sides are extended till infinity. So, wherever you deviate whichever side you deviate it from the equilibrium this side or that some the restoring force will always bring it back towards the equilibrium. Now this is the example of stable equilibrium.

So, I will stop here, and in the next lecture I will start building things little more mathematically and we will develop the theories of small oscillation.

Thank you.