Classical Mechanics: From Newtonian to Lagrangian Formulation
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Lecture - 52
Lagrangian Formulation - 10

So, we are back and we start with the quick review of whatever we have done so far.

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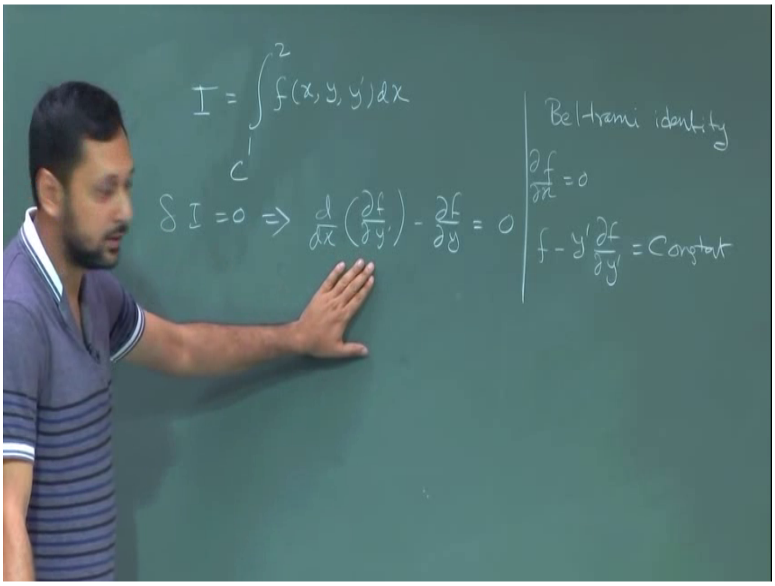

So, what we have done is we have seen that this integral which is taken between 2 stationary points of any function x, y and y dashed d x. If between 2 fixed point, if this integral has a stationary value; that means, if it is following a path; by the way, these are all path integrals, so it is following some paths c. If this curve is such that this integral reaches a minimum or a maximum value that technically means delta I equal to 0, then we have seen that this function f will satisfy this equation which is in case of a; if f we choose as a Lagrangian and we replace x with time y with q and y prime with q dot then this actually gives you the Lagrange's equation of motion this we have seen.

So, that is a fundamental principal of variation and also we have derived something called Beltrami identity which was the end of the last class which says if f is not an explicit function of x the parameter x here. That means, if del f del x is equal to 0, then f minus what was it y prime del f del y prime is equal to a constant c. So, it is a constant.

So, this is the Beltrami identity now this two: this one we have already used in certain cases this one will take up and in the very next example.

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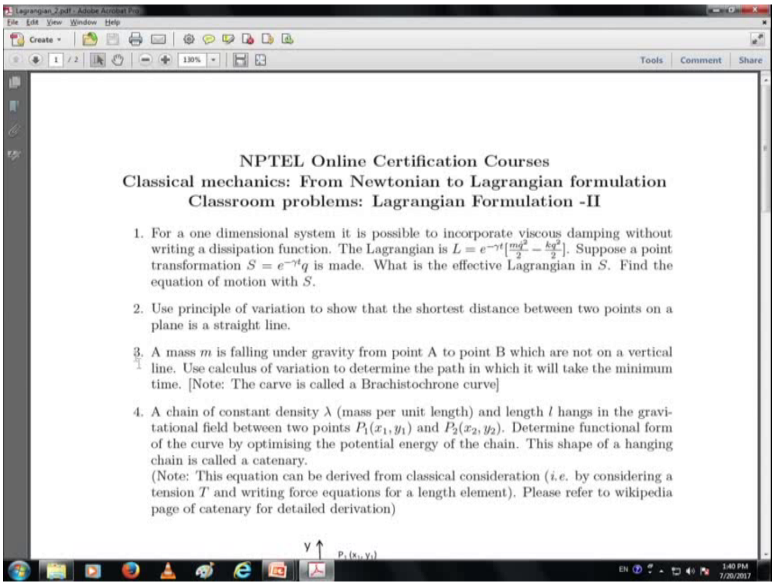
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Classical mechanics: From Newtonian to Lagrangian formulation
Classroom problems: Lagrangian Formulation -II

1. For a one dimensional system it is possible to incorporate viscous damping without writing a dissipation function. The Lagrangian is $L = e^{-\gamma t}[\frac{m\dot{q}^2}{2} - \frac{kq^2}{2}]$. Suppose a point transformation $S = e^{-\gamma t}q$ is made. What is the effective Lagrangian in S. Find the equation of motion with S.

2. Use principle of variation to show that the shortest distance between two points on a plane is a straight line.

3. A mass m is falling under gravity from point A to point B which are not on a vertical line. Use calculus of variation to determine the path in which it will take the minimum time. [Note: The carve is called a Brachistochrone curve]

4. A chain of constant density λ (mass per unit length) and length l hangs in the gravitational field between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. Determine functional form of the curve by optimising the potential energy of the chain. This shape of a hanging chain is called a catenary.
(Note: This equation can be derived from classical consideration (*i.e.* by considering a tension T and writing force equations for a length element). Please refer to wikipedia page of catenary for detailed derivation)

So, let us be back with the; let us go back to the problem set; the problem number 3 says a mass m is falling under gravity from point A to point B which are not on a vertical line so; that means, points B is not exactly below point a, but there is a gap; I mean there is a you know horizontal shift. So, we have to use calculus of variation to determine the path in which it will take minimum time. So, if I try to draw a picture of figure it will be something like this.

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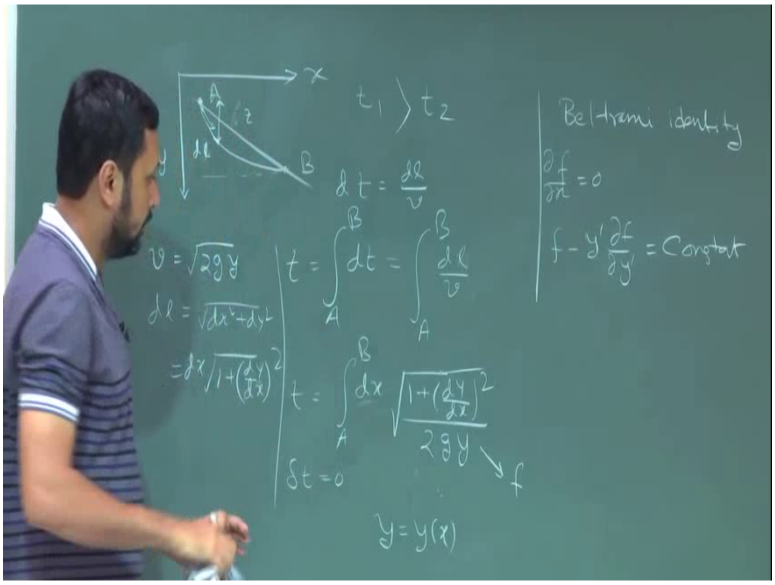

I have this is my vertical direction. So, we have 2 points A and B, right and there are not exactly on I mean B is not on a vertical line. So, it could be anywhere, but not exactly at this point; it could be anywhere and a particle of mass m is travelling from A to B good. So, now what is the time; what is the path along which the time taken will be a minimum?

Now, intuitive answer is a straight line which is not. So, if I just put a slope here it will take you it will take some time I mean some sometime let us a T 1. So, now, if this is the case if this if I put a straight line, I mean you know; sorry, inclined plane if it takes time T 1 it turns out, there it could be another plane of some another curve which can join this 2 and if it travels the particle travels along this curve; it takes a time T 2. So, it turns out that T 2 could be less than T 1, right. So, our job is to find this curve where along which the time will be shortest.

Now, in order to do that let us assume that you know to travel along this particular curve we take a small segment of length which is given as d S or d l let us say. So, travel this to travel for sorry for the mass to travel this small length d l with the velocity v the time taken is T is equal to or d T is equal to d l by v right length by velocity is your time. So, if we have to optimize. So, the total sorry the total time taken T is nothing, but integration of d t between 2 points; A and point B, right which is which can be translated to d l sorry d l by v.

Now, if the driving force of this motion is purely gravitation it is purely gravitation and let us assume; this point l has a vertical drop in a it has a vertical drop let us say d z or z sorry. So, basically what we need to is we need to find out what is the total vertical displacement of this particular particle and if this is z, then from nektons law, we can immediately get that v is equal to root 2 g set, right. So, this is something we are very familiar with this is the familiar expression and then what we do is. And of course, if this is z axis this is my x axis. So, d l can be written as or actually what we do is I am just changing, it to y because it will be easier to you know write nothing else. So, this is this and this will be simple y d x square plus d y square, right.

So, if I substitute with this here your time T will be integration A to B once again we have we can write this as d x root over 1 plus d y d x S whole square right; we can do that and once we substitute it here we get d x by d x root over 1 plus d y d x S whole square divided by 2 g y. So, we have to. So, for the path for the curve along which and this is for example, write now we have not taken any specific shape of the curve. So, d l is a is an infinite decimal length element.

And we do not say anything about the nature of the curve at this point, but now if the time taken has to be a minimum. That means, this integration I mean the delta T has to be equal to 0 which is what exactly what we expect in variation principle. That means, we have to calculate we have to find out the expression of this curve f which is simply given by sorry we have to find out the function f by solving this equation where f is or sorry we have to find out the equation of equation of the or curve which is of the form y is equal to y of x by taking this as my function f, right.

So, this is how this is this is where we end or this is where we come finally.

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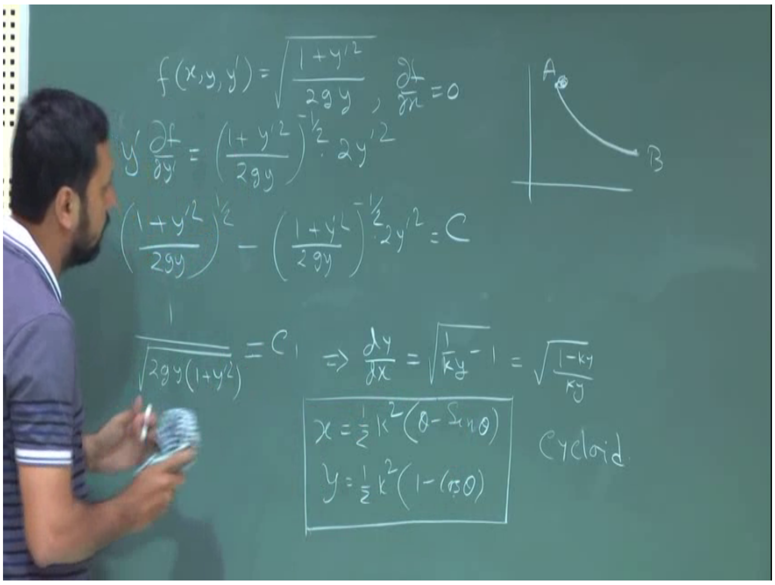

So, I will just remove this figure; I will draw it again if necessary. So, f of x y, y prime is this one which is given by 1 plus y prime square divided by 2 g y and now comes the Beltrami identity into the question; see f is not an explicit function of x so; that means, del f del x is equal to 0 and from Beltrami identity we know that we can write it in this particular form. We can write, we can use this identity actually. So, what we can do is we can calculate del f del y prime which will be here the expression y prime square by 2 g y whole to the power minus half times 2 y dashed, right.

So, if I have to compute y dashed times this. So, it will be 2 y dashed square and f minus this will be here 1 plus y dashed square 2 g y to the power half minus 1 plus y dash square by 2 g y to the power minus half multiplied by 2 y dash square equal to some constant C. Now if we start simplifying from here and I mean this will be just pure algebraic simplification, we can do that if we do that we will end up with the following expression I mean this is just a lengthy process, but not much to be done here. So, finally, we will end up in this particular expression root over 2 g y, it will be 1 plus y dash S square; it is basically 2 steps of algebraic calculation equal to some constant C 1.

So, we can do that and from here we can slightly modify this once again and we can write this final expression as or final differential equation as d y d x equal to root over 1 by some constant k y minus 1, right. So, if you wish we can write it as 1 minus k y; y k y right. So, this is the final differential equation, we have to solve in order to get y as a function of x. Now it turns out the solution is also I mean if solution is also very standard the solution is I mean it is not very difficult you just have to do a proper substitution you can do that we all know how to do this see if you do this the final expression will be half case square theta minus cos theta and y is equal to half k square 1 minus cos theta sorry this is sin theta not cos theta, right.

So, this is my final curve. So, if the partial travels in this particular along this particular curve which will be which we can draw. So, it is a equation of a cycloid this family of curve is called a cycloid; what is it and how is it I mean how does it look we will see in a moment. So, it is a curve. So, the thing is if I go back to my original picture I have point A and point B; if it follows a curve which the equation of which is given by this under gravity purely under gravity and object of mass m which is sliding down this curve will take minimum time to reach point p.

Now, sorry C y C l; sorry, my handwriting is not that great I know cycloid now how does this curve look like this is a curve as you can see there are periodic functions in you know both; both of them I mean you can take theta as a parameter what is theta I will come back in a moment theta you can take a parameter and you can see there are sin thetas and cos thetas in both the expression of x and y; now if you draw this in a more formal manner this curve will look something like this, right.

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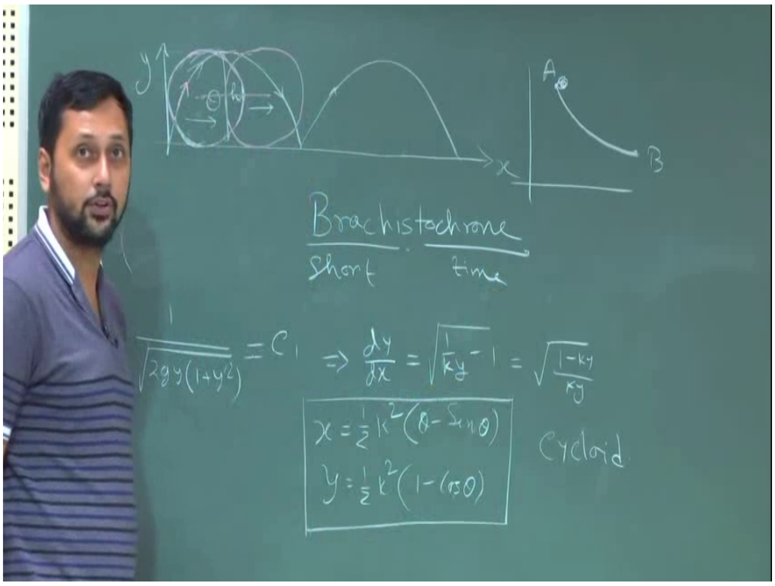

See, if you put theta equal to 0 the first term is 0 minus 0. So, its 0 and the second term is 1 minus cos theta at theta equal to 0 will be 1. So, at theta equal to 0; both of them are equal to 0. Similarly if you put theta equal to ninety degree you will see that this is pi by 2 minus 1 and this is 1 minus 0. So, it is has some value similarly at theta equal to pi you can find out its it will going it will go back to 0 I guess once again ya because 0 minus something minus no, it will go back to 0 at 2 pi only.

So, along this curve theta is also changing. So, if you take this is my x and this is my y theta is also changing in this particular direction and this curve will the; if a particle trace out this curve, it will look like this now what is the physical interpretation first of all it can it can be shown that this curve is the locus of a point on a wheel what I mean by this is let us assume that there is a wheel here of certain radius r right let us say this is this height will be your k I S, right I do not know. So, I am not sure. So, I will not write anything, but let us assume that there is a wheel with this radius let us call this h wheel of this radius. So, the wheel looks something like this or sorry this diameter not radius the diameter.

So, the wheel looks something like this and let us say the wheel is moving in this particular direction right now let us say this is the point which this is the position of one point on this wheel now as the wheel rotates what happens to this point see when you rotate it in a next position or in a later position your wheel; I will use another color may be that will be easy for you. So, this was my wheel at time T equal to 0 and now at a later position let us say here this is my wheel once again now what happens to the C this point this point as this center has shifted from here to here this point also will be shifting and it turns out this I mean locus of this point will always stay on this curve; so, at this point.

So, you can say without drawing anything that this particular point is here. So, it has moved along this curve. So, this is the concept of a cycloid a cycloid is essentially a rotational motion I mean it represents a rotational motion, but it is not a rotational motion in the conventional sense it is more of a you know I mean it is like one point on a wheel the locus of that particular point if we trace for the standard translation and rotation motion of a wheel that that point will trace out this particular curve.

So, I have prepared to a by the way this problem is called the Brachistochrone problem. So, Brachistochrone, I think the spelling is correct let me have a look brachisto- c h no just a minute the spelling is not right Chis Brachis; Brachistochrone. So, the word it is a Latin word Brachis means short and chrone is time you know the English word chronology which means the time bound documentation of certain thing. So, that chrone is that has the same origin from this word chrome.

So, this problem means the problem of short time now I have prepared slide for you sorry prepared a slide for you here.

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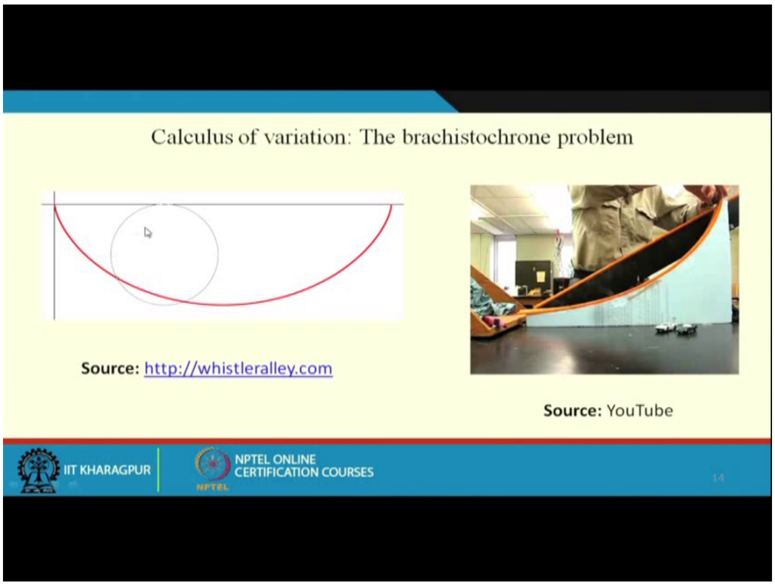

You see there is whatever I have discussed in the board just now it is there. So, it is a locus of this particular point which is stressed out in the space and this will draw this will I mean this particular locus of this particular point will trace out this curve. And in this in this link I have given an hyper link here. In this link there is a beautiful description of how you can get to the same equation using from the farmers principle drawing an optical analogy of the system.

Please go through it I will also describe try to describe it in very brief. And also there is a video if you search Brachistochrone in YouTube, you will find this video as within first 1 or 2 links and there is a beautiful video in this video they there are you see there are model toy cars present and there are 2 slopes or 2 two 2 channels which has been one is a straight channel and the other one is the I mean basically it is a Brachistochrone channel.

So, what they did in this video they what they have demonstrated is they are releasing this toy cars from the top of this exactly at the same time and it is very clear that this one the car which is travelling through this arc is reaching the ground at an earliest earlier time. So, this is a beautiful video if you look at this video you will be little more convinced that a it is actually not a straight line also I talked about an optimal analogy you can just a minute. So, whatever what did what do you have given in that particular website I will just try to describe very briefly without writing any equation.

So, a Brachistochrone curve can be seen; what happens is forms principle of least; least time says light travelling between any 2 point takes the shortest time.

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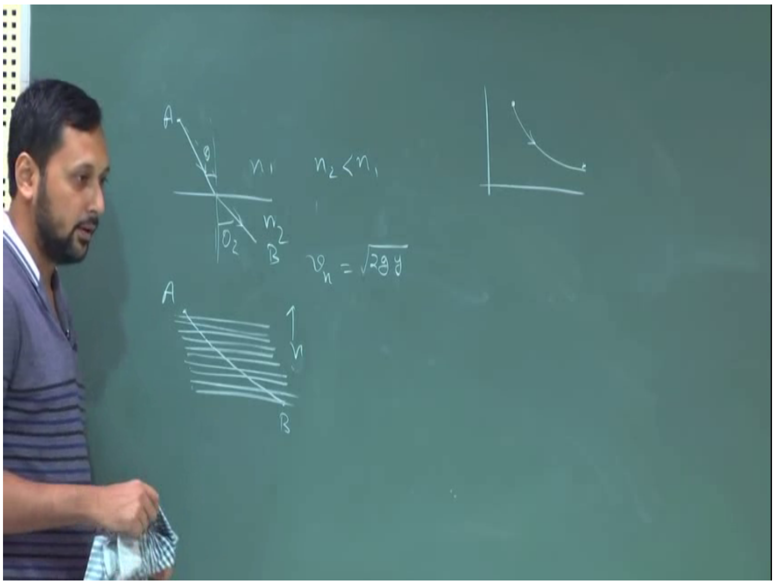

Now, think of it if light is travelling across an interface you have n 1 you have n 2 and n 2 is less than n 1. So, basically lighter medium sorry light is going from lighter medium to sorry denser medium to lighter medium, right and we all know according to Snell's law it will be an refraction phenomena. So, light will be moving away from this axis. So, if this is your theta 1 this is your theta 2 we know that Snell's law n 1 sin theta 1 equal to n 2 sin theta 2.

Now, what happens in Brachistochrone is this; we have a point here you have a point here and an object moving in a path which is like something like this now think of optical medium. So, what is what is the main idea or where is the analogy comes from here to here if I consider a point A here and point B here light according to Fermat principle of least time light takes the shortest time to reach from A to B I mean although it is not a straight line path; it is not a straight line path because there is a refraction happening.

So, even in this curve path light takes the least time now if I add many more levels let us say this is my point A this is my point B and what we do we add many more levels of materials I mean, it is an hypothetical experiment; for example, many more levels where the density is decreasing in this particular direction. So, as or whether I should I should put the arrow. So, as we go up the density is increasing and if as we down the density is decreasing. So, if a light raise starts from this particular position what will it look the path of it at each of the interfaces will it will deviate slightly more and more and more and more eventually after many deviations it will reach point b.

So, if even if it is not a straight line it will take the shortest time according to Fermat's principle, right and if I now draw an analogy between the speed of light in one medium v n with this law of a whatever we have derived from you know gravity, I mean whatever velocity expression. We have used in solving the problem classically that analogy is something we you can if you read that website you can you can read that link, you will be able to understand. And then in the continuous limit this equation of this curve or the differential equation which will produce this curve will be exactly give you thus exact same differential equation which gives the final Brachistochrone curve or final cycloid.

So, this is how an optical analogy. So, this is a very good example of how you can you can look into variation principle in a slightly different way you see essentially Fermat's principle of least time and variation standard variation principle as we have as we know as we have learnt; it is says the same thing it is it talks about optimization of certain things, right. So, you can always draw an optimization analogy if you can choose if you have a correct picture in mind please go through that website it is not exactly you know part of your syllabus strictly, but for me it is very important to important that I tell you all this and you read about it. So, that you have a better understanding of the subject.

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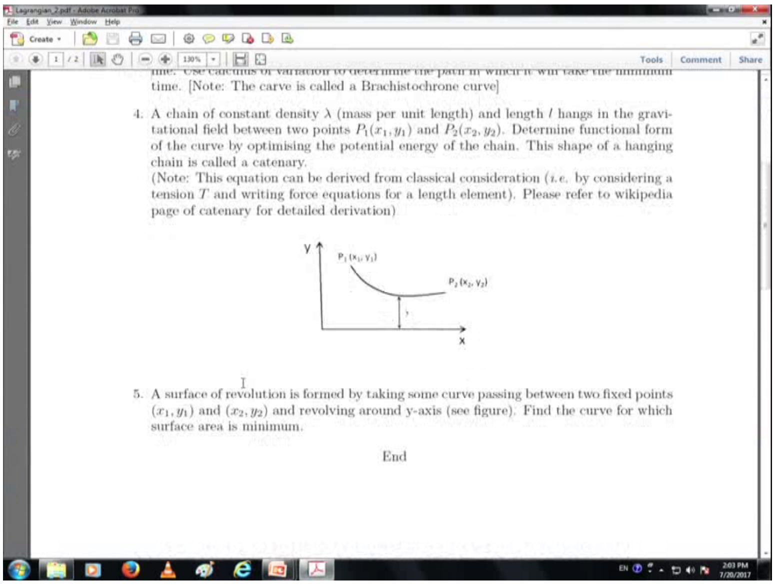

Now, let us go back to the problem sets problem number four is another very classical problem I would say; it is a problem of a catenary. So, we have a chain with the constant density and a length; some density lambda or you can take it sigma also because lambda will be using for something else just my basically it is my mistake; I should have used sigma instead. So, let us say the length is L and it is hanging in gravitational field. Now please remember that P 1 and P 2; this points; this 2 points are not necessarily in A; on the same horizontal plane it could anywhere any 2 points since space and. So, we have to find out what is the equation of this particular chain, right.

So, this is a shape which is called the catenary. So, we have a catenary here I mean we have a point P 1 we have a point P 2 and in uniform form chain is hanging.

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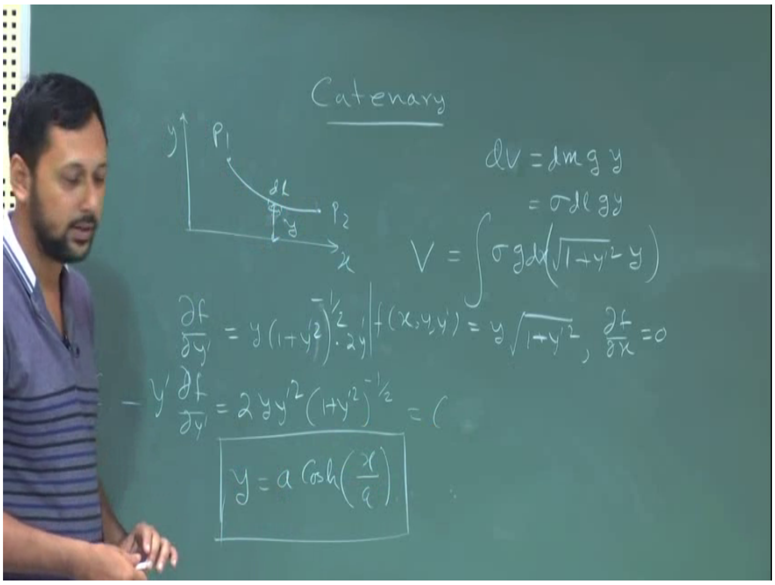

So, it will trace out some curve and the curve I think I have described it already, if it is an uniform chain if it is an uniform thread; if it is a rubber pipe, if it is a you know made of steel does not matter if the points are you know in the same line or if P 1 is up it is down or P 1 is down P 2 is up does not matter, it will still follow a same type of equation and we have to derive this equation.

Now, in order to do this; what we do is we have x and we have y; once again I take a length d l right, sorry, it is not very clear. So, this length is d l and the height at which it is from the ground level is given by y. Now what do we have to do here; what do we have to do; what we need to do is we need to minimize the potential energy of the system because we know from the fundamental laws of a physics that a system would like to go for a minimum potential energy configuration that is the nature the most stable configuration of system is the configuration at which it minimizes its potential energy keeping that in mind we have the potential energy v can write this for this particular length element is m or rather d m g y and d m will be sigma d l g y. I am just taking sigma because we will be using lambda for something else. So, just give me bear with me for a moment I will also in the final version I will correct it. So, I will have put sigma there.

So, this is your d v and your potential energy total potential energy of the system; v will be integration sigma g which is a constant d l; once again you can write d x by 1 plus y dashed square times y. So, this is your function f right. So, you have to once again write f x y y prime is equal to y root over 1 plus y dash square once again we see f is not a particular explicit function of x. So, del f del to x equal to 0 from here all we need to do we have to find out del f del y prime which in this case will be y 1 plus y prime sorry y prime square 2 power minus half into 2 y prime, right.

So, y prime del f del y prime is 2 y y prime square 1 plus y prime square to the power minus half right and we have to use the Beltrami identity once again which is this f minus is a constraint and once again there will be some algebra associated with it and the final equation which I will be getting is simply this let me give you the final equation just give me a second where is it how; oh sorry, I am sorry I did not have the final equation with me oh I think it is only way is to get this entire algebra done, but oh; you can find it out I guess it is not a problem. So, I really do not want to do this entire algebra now because it will be just a waste of time.

So, finally, what you will get I will give you the final answers that is more important. So, if you follow this procedure we will you will get essentially get a differential equation which will have an equation which will have a solution of the form y equal to a cos hyperbolic x by a. So, this will be the final answer if you set up the differential equation from here from this relation taking this side as a constant some C. So, you will get this where a is a constant which is which is; obviously, something which you can evaluate that is not a problem now this is.

So, this is an hyper bola the equation of an hyperbola with this curve will follow that is all good. Now, there is another if condition given if we look at the problem carefully there is another condition given here that is the length is given as l now how do we include that into the equation that is the question. Now if we include this if we want to include this into our equation what we need to do is I will just remove this bit this means if I specify the length total length.

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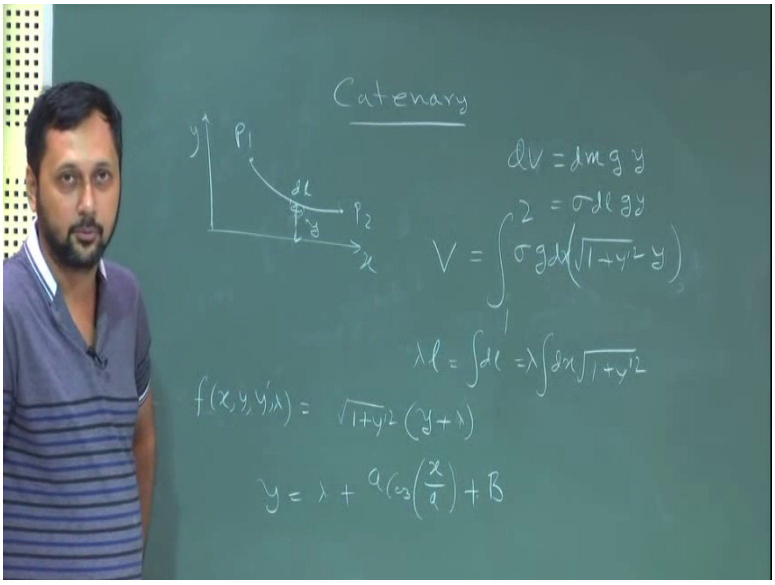

That means l is equal to integral d l between point 1 and 2 and we have this integral. So, that is that is something which is given, right.

Now, what we can do is we can use Lagrange’s multiplier like we did. So, this is a constraint condition actually. So, to include the constraint into this equation into the final function what we can do is we can multiply this relation with a constant lambda add this or subtract this from particular expression and we can get a function f x which is actually see d l will be nothing, but integration d x root over 1 plus y dash square right or I will just right it in here. So, that you can see; so, this is one equation and the second equation is l equal to integral d l which will be integral d x root over 1 plus y x square or sorry.

Now, if we if I multiply this with a constant lambda which is the Langrage’s undetermined multiplier try to construct a function which includes this our final functional form will be f of x y y prime and lambda will be given by. So, root over 1 plus y dash square will be the common factor in this 2 and we will have y plus lambda. So, this my expression for f and just like we did before we have to find a differential find out the set of equations differential equation which will include both lambda. Now if you do that, thank fully if we even if we do not do that this procedure his elaborate procedure nothing changes. Because the final expression what you get from by solving this equation is once again y is equal to some constant a plus sorry; this will this lambda will come here lambda plus a thus exact same expression cos x y A plus another additional term B which will be associated with your length total length l.

So, in just give right. So, including this function or including this constraint condition with or without his constraint condition your equation will be the same except in this will be little more complicated and you will have some more terms to it right, but the functional nature will be exactly the same.

So, I would say if possible I mean if you have to solve the catenary problem into for an understanding of your own just use the first procedure, but if you have to include length at some point then you have to follow this procedure. So, from here also, we can get an equation differential equation which is slightly complicated, but nothing difficult of course, if you can find it in almost every standard text book and may in some of the places; both the procedure has been done.

So, with this we are done with this problem last problem there is one more, but the time is little short. So, what I will do is I will just wait; I will keep it for the next lecture. And in next lecture we will be starting small oscillation.

Thank you.