Classical Mechanics : From Newtonian to Lagrangian Formulation
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Lecture - 51
Lagrangian Formulation – 9

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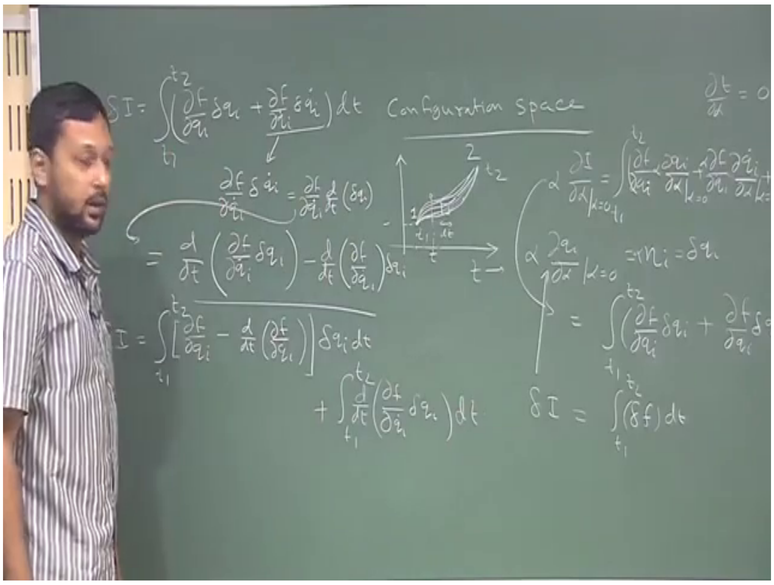

So, we derived up to this and from here what we need to do is so what we do is, we take this particular term which is del f del qi dot delta qi dot, and see this is a d there is a time derivative here. And what we have derived in the beginning of the first lecture that t and delta they can be derivative, standard derivative and this virtual displacement curly derivative they can be interchanged. So, if this is the case, what we can do is we can write this term as del f del q i dot d d t of delta q i. We can do that also this thing this whole thing can be written as d d t of del f right del q i dot delta q i, right minus d d t of del f del qi dot f del qi dot delta qi, right.

This we can do and if we so in place of this one, if I substitute this in to this integration what I get is delta i is equal to integration t 1 to t 2 del f del qi minus d d t of del f del qi dot, whole delta qi t t. Plus there is an additional term which will be t 1 to t 2 of so, basically I took this term and this term. So, this term will be there which will be given as del f del qi dot delta qi whole d d t and multiplied by d t right. Fine now this term is a total time derivative term. So, integration of this term will simply give you I will remove this part do not need this. So, if I just focus on this term. Now if I this is the total time derivative.

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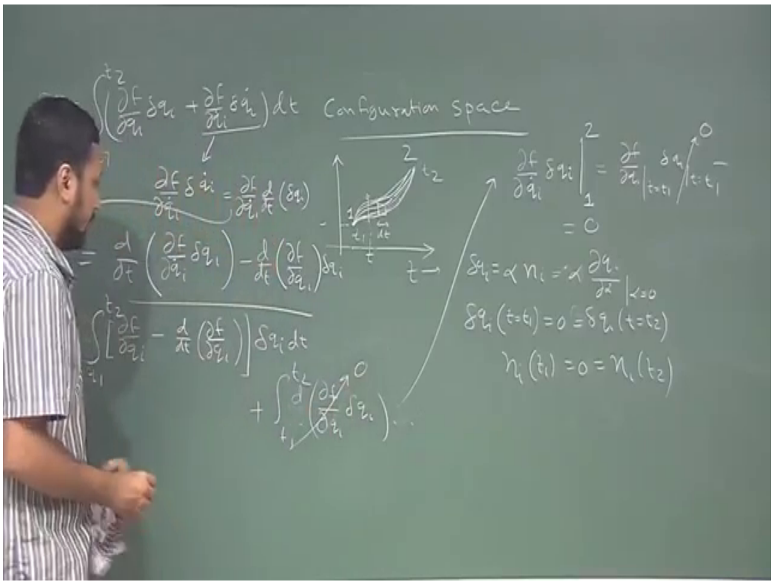

So, it will be d of so basically d t d t, we can get rid of so integration d of this term which will give you del f sorry qi dot delta qi between 1 to 2. Now first and the last points are fixed starting and end points are fixed points.

So, any oh that one thing I forgot to mention this parameter alpha is equal to 0 at the alpha in I is was equal to so delta of qi was equal to alpha n i right. So, n i where the or it was alpha del qi del alpha at alpha equal to 0. Now there is no variation of the starting and the end points. So; that means, this quantity at you know delta qi at t equal to t 1 is equal to 0 as equal to delta qi of t equal t 2. So; that means, eta i this is equivalent of saying eta i at t equal to t 1 and eta i at t 1 and eta i at t 2 is equal to 0 for all values of I. So, the variation of the parameters are not, I add there is no variation in the parameter at the starting and the end point well.

This is the case this will be del f del qi dot evaluated at t equal to t 1 and delta qi evaluated at t equal to t 1 minus same thing evaluated at t equal to t 2, but this is equal to 0. And the second term which will be delta qi evaluated at t equal to t 2 will also be equal to 0. And this is how the variation is defined I mean the integration is defined. We can have variation in between the fixed points if the fixed points are varying that it makes no sense. So, the fixed points are fixed. So, we have variation in between at the fixed point this will vanish now if this is 0. So, if this is the case this integration is unanimously equal to 0, right. So, if this is 0 then what do we have we can get rid of this one and we are left with this right.

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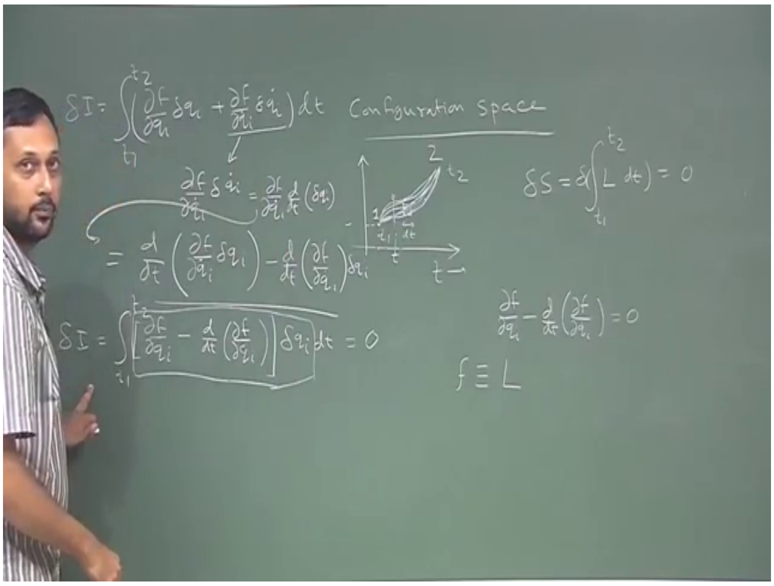

Now, if delta i is equal to 0, along as we see in the variation principle what is the variation principle. Once again the variation principle for a mechanical system is right. Now we are talking about mechanical system which turned out to be true for any other system that delta s, which is delta of L d t between fixed point t 1 and t 2 has to be equal to 0. So, if right now we are not talking about, we are we are not specifying the functional nature of f. So, we are not talking about principle of least action for say, but we are saying that delta of i, this particular integral will be equal to 0. Now if this is the case then we immediately see that it can an integration this is basically an integration and the integration can vanish if and only if, the integrand is equal to 0. And the integrand is actually this right please remember that there is a summation over all the q is ok.

So, if this is equal to 0. So, we have a situation which we have a very familiar situation of sum over I del f del qi minus d d t of del f del qi dot is multiplied by delta qi is equal to 0 and from there because stating that q is are independent. Sorry stating that q is are independent we can equate all the coefficients of each of delta qi term to equal to equal to 0 we have done that during Lagrangian, right. So, now, we have seen that it is not only for a mechanical system, but if we can define a proper function for proper functional form of f it could be you know Lagrangian, it could be total potential energy it could be the time with time traveled between these 2 paths you know 2 fixed points whichever parameter we want to optimize if delta i has to be equal to 0 delta i equal to 0. I mean the variation of that this; this particular integral if we said that variation of this particular integral equal to 0, we get the same set of familiar equation.

Which is this of course, there is a problem of negative sign, but that is not a problem because we have 0 anyway. So, we get back the same set of equations, similar looking equation if we set f equivalent to L. Then we get back the standard Lagrange's equation of motion starting from principle of least action. And so basically if we start from principle of least action taking this f is equal to l, we get back familiar form of Lagrange's equation right. So, Lagrange's equation although we got it previously we got it starting from d Alembert's principle this is lot more fundamental. Because here we do not have any prior knowledge I mean we for example, in what did was we started from Newtonians Newtonian equation we remove the forces of constraint.

Then we introduce generalized coordinate after. So, many steps we got this equation just replace f with l, but here what we are doing is we are starting from a principle, which is more fundamental in nature and we are getting back the same equation same set of equation without having any specific knowledge of the Lagrangian. Now this is this is called the principle of least action as I have already mentioned. The quantity s equal to L d t between t 1 and t 2 is called the action of a system and it is not only for mechanical system, but also it has been found it has been shown later on that these are also equally valid for quantum mechanical systems, equally valid for systems which are moving and very high velocity very high means at this towards the speed of light. So; that means, in general systems were general or special relativity relativistic theorems are applicable quantum field theories are applicable everywhere, this action integration is a very important parameter. So, a system will follow a certain path along which the action is 0 or variation of the action is 0 or rather say I would say action is stationary.

That is the most general statement we can make at this point. So, out of this; so, many path the system will choose one path along which this quantity this integration is the variation of this integration is equal to 0, right. And this will lead to an equation of this particular form now we will take up examples one by one and right now there is no formal I mean we do not want to carry forward from here we can do lot more things in terms of Hamilton Jacobi equations, and you know if we can go in to special relativity and all the this principle has way more applications that we could even think of, but right now that is not the objective we I wanted to introduce you the with the principle of variation calculus, we will now take up some specific examples and check for it is applicability of, and of course, we will take systems which are not only dynamic in nature.

We will also take up static systems all we need to do is we need to find out a suitable form of f. For a dynamical system, if we want to follow the dynamics of the system and find out which path it will take in the configuration space. We have to take the Lagrangian, but if you are not looking for the dynamics. For example, as I said I want to prove formally that the shortest distance between 2 points on a plane or in space is a straight line now it has. So, happens that shortest distance between 2 points can always be you know framed in to a plane, right.

If there is there are no other constraints I can connect any 2 points in this world by a plane, I mean we can we can make them sit on a single plane. So, when we are talking about shortest distance between 2 points we are talking about shortest distance between 2 points in a stay staying in a Nucledian plane, right strictly speaking. Anyway I will come back to that, but let us look in to the problem set I have prepared for you.

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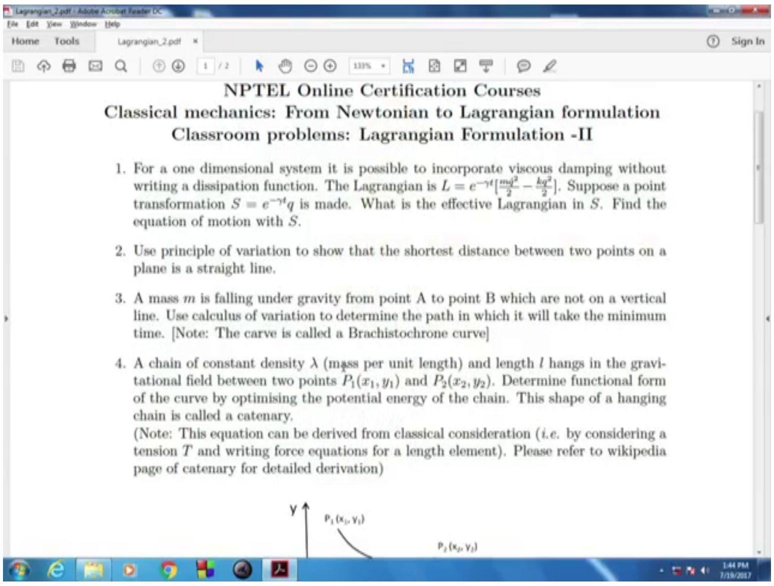

The first problem is a small problem on point transformation, which I will solve, but I mean yeah it is it is very easy to solve actually. So, the problem says. So, one dimension for a one dimensional system, it is possible to incorporate viscous damping. So, I am talking about simple harmonic oscillators by the way, I forgot to mention I should say one dimensional simple harmonic oscillators it is possible to incorporate viscous damping without writing their dissipation function Lagrangian is l, which is e to the and we what we do is we just add i e to the power minus gamma t term with the Lagrangian. And this will take care of the damping. Now we suppose there is a point transformation s is equal to e to the power minus gamma t of q what is it effective Lagrangian in s and we have to also have to find out the equation of motion.

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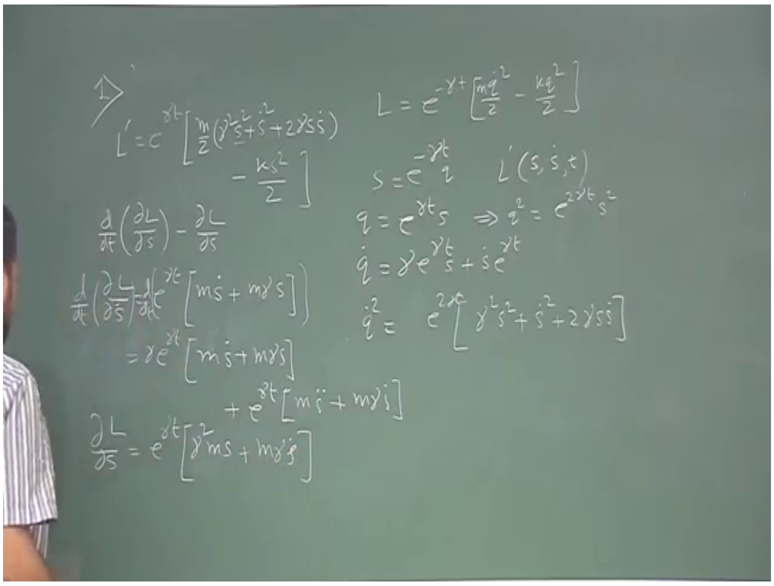

So, we start we will come back to this will come back to this very soon, but let us start with this simple problem. So, we have a Lagrangian L which is so it is problem number one of the problem set e to the power minus gamma t, q dot square by twice m of m cubed or square by 2 minus k v q square by 2. So, this part is standard one dimensional simple harmonic oscillator Hamiltonian. We add a damping term here what good practice would be to find out the equation of motion, for this Hamiltonian without altering anything right. Now what we do is, but right now what we are doing is there is a point transformation of s which is e to the power minus gamma r gamma t. So, you have to find out we have to find out L prime, which is a function of s s dot right. So, this is our objective. So, you start by writing this equation and taking gamma e to the power gamma t in this side. So, we just write e to the power gamma t s.

So, q dot will be e to the power. So, this is 1 and q dot will be gamma e to the power gamma t s plus s dot e to the power gamma t right. So, this is the derivative. Now we have to square both of them. So, the first term tells you that q square is simply e to the power 2 gamma t s s square. And q dot square will have 3 terms. So, it will be gamma square e to the power 2 gamma t. Actually e to the power 2 gamma t will be common from both all 3 terms. So, I am just writing it out. So, it will be gamma square s square plus s dot square plus 2 gamma s, s dot right. Now we can think of writing L prime which will be e to the power minus gamma t m by 2 q, q dot square which will be this.

So, you see we have e to the power 2 gamma t in both the square terms. So, what we can do is we can take out e to the power 2 gamma t, which if I operate this 2 it will be simply e to the power gamma t not minus right, e to the power gamma t. And inside we will have. So, we do not worry about the exponential term which we have already taken care we have modified it. So, inside we will have m by 2 gamma square s square plus s dot square plus 2 gamma s, s dot there is the first term minus k s square by 2 right. So, this is your L prime. So, you see L and L prime they have different forms in a point transformation, but both of them both of them are equally valid Lagrangian. So, you take what you can do is, as I said you calculate the equation of motion from here and we are going to calculate the equation of motion from this one.

So, the second part is we have to do d d t of del L del s dot minus del L del s, which is this is little longer we can do that, but I think you can also do it yourself. Because we have one s dot term here s dot term here. And please remember each of the term will be multiplied by and e to the power gamma t. So, finally, I will write the final answer or no. It is we can do it may be one more step, I can do del s dot which will be simply e to the power gamma t multiplied by m s dot. Sorry is it right here it will be m s dot plus this term 2 will cancel out m gamma s right. So, if I take d d t of this term this will be d d t of this whole thing here. So, the first we have to take out gamma e to the power gamma t m s dot plus m gamma s and we have to add e to the power gamma t m s double dot.

So, first we took derivative of this term and multiplied it here, second term is ready this is fixed and derivative of this term. So, since m s double dot plus m gamma s dot right.

And del L del s will be now see del l, del s will be from this term and this term right. So, we have once again e to the power gamma t m by 2 sorry 2 will cancel out. So, it will be m s gamma square gamma square m s plus 2 will cancel out. So, it will be m gamma s dot m gamma s dot right. So, if I you know subtract sorry if I subtract one from the other, just to get the final equation this will be equal to 0. You will definitely get the final form. So, I am leaving it to you it will be. So, what will happen is e to the power gamma t will you can take out from both the equations you get one gamma m s this gamma will go in yeah.

So, you can basically take gamma e to the power of gamma t common, and then you can simplify. So, you will get back the familiar equation of second harmonic oscillation with tempting. So, this is this was a little lengthy, but rather easy problem, I selected this problem. Specially I because I wanted to give you an idea of point transformation because theoretically sometimes it is very even it is not very convincing. So, always better to do an example. So, I hope I have convinced you that point transformation is something which is not very difficult may be it has a formal name, but this is something that we have done already we have already known for some time. Now let us come back to the principle of variation right problem number 2.

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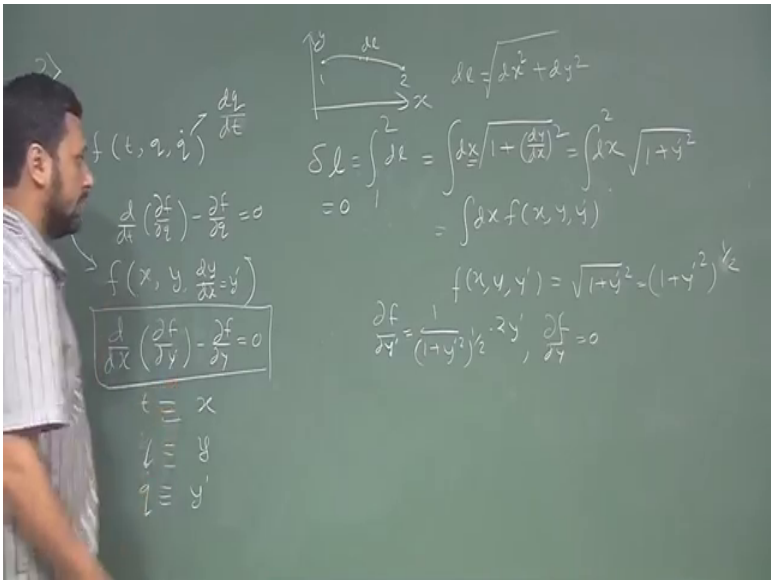

So, problem number 2 is the visible of variation, which we have to demonstrate the shortest distance between 2 point on a plane is a straight line. Now on a plane we should not write on a plane either we should just say you know 2 points or we should say 2 points on an Euclidean plane.

Because we can have we can go along the surface of some you know some sphere or some cylinder that is also plane right. So, when we say plane we call about we talk about Euclidean plane. So, what is the distance L between 2 points we have one point here, one point here right. So, if I should not write draw a straight line because we this is something we have to prove here. So, let us say this is a curve which joins this 2 right. So, what we do is we take a small length element d l on this curve. Now in a 2 d x y plane d l is equal to d x square plus d y square whole root right. So, L the total length will be integration of d l right. Of course, there will be limits you know one and 2, but that is not terribly important here. So, if I write this for d l what we can do is we can take out d x and we can write this as 1 plus d y d x whole square.

So, once we do that and oh by the way one thing, I should say that I have a function f of so far we have dealt with functions of t q and q dot let say. So, here t is the independent parameter q is a dependent parameter which happened to be a coordinate in this case and q dot is nothing, but d q sorry d q d t right. So, we can you know and we have proved that if I have a you know if I have to optimize the integral or rather optimum, we have to have a stationary integral. Then this should follow equation which is d d t of del f del q dot minus del f del q equal to 0. Of course, I can extend it to n number of parameters and put q y and q y dot that is not the point here. So, what I am trying to tell you is instead of this 3 if I have these 3 parameters.

If I have any function which will be a function of x y and d y d x, which we call y dash there also using the same you know same set of calculus mathematics which we have followed we can say that d d x for stationary integrals d d x of del f del y dashed minus del f del y will be equal to 0 right. So, this we can do. So, t will be equivalent of x which are independent parameters y or rather q, we will be equivalent to y and of course, q dot will be equivalent to y dash. So, we can extend our variation principle or whatever the principle of stationary variation to a case which is non mechanical in nature. These are for mechanical system and these are for any general system, where I have one independent parameter or one independent component one parameter one dependent component and the derivative of a dependent component. So, following the same logic here we see d x is our independent component. So, this is nothing, but integration d x between 1 to 2 root over 1 plus y dash square, right.

You note it. So, this is the case immediately we can identify this function now. So, where f of x y y prime is equal to root over 1 plus y dash square right. Now this function is not explicit depend dependent on x or y your y dashed, but that is not our problem this is our function. Now if this is the function it will follow this set of equation this equation right. So, what we want to do here we want to calculate the minimal distance; that means, either maximum or minimum right. Now we are talking about minimum is also minimum will be achieved when this integration the length. So, please do not mix it up with the Lagrangian; let me write d l small l that will be easier. So, this integration delta L will be equal to 0, right. Only then we will have the shortest length and for shortest length. So, if that this integral is equal to 0, the function f has to follow or yeah follow this particular equation I mean this particular equation will follow.

So, let us try this del f del y prime will be equal to so it will be 1 plus y s square 2 to the power 3 2 to the power half. So, it will be 1 by 1 plus y dash square to the power half times 2 y I dashed right. So, this is del f del y square y dot y dashed and del f del y is equal to 0. So, what we get is by applying this equation that d d t of this. So, this is equal to 0.

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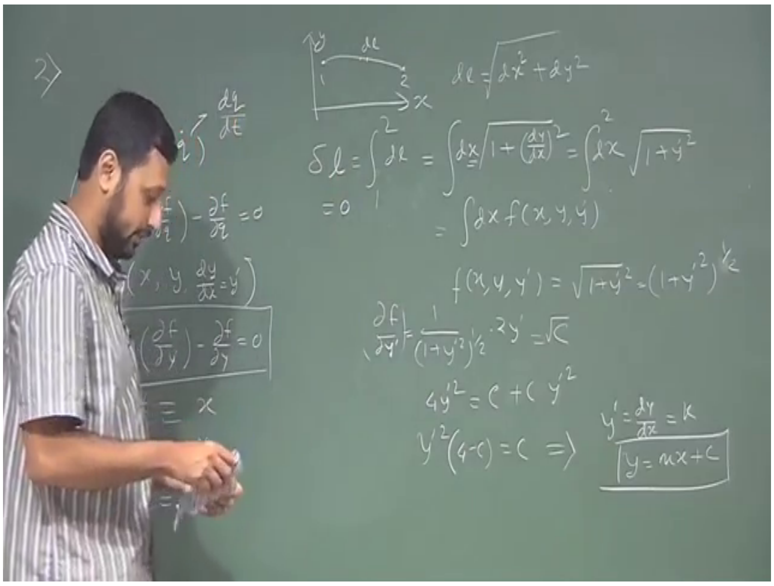

So, d t of d d x of this is equal to d d x of this whole thing will be equal to 0 right. This is equal to 0. So; that means, this is equal to a constant right. So, we can write this equal to some constant c, now if I simplify from here. So, what I have is if I take square from both of this. So, we get 4 y prime square is equal to C square plus C square y prime square right also. We can call C square and you I mean we can just absorb C and C square yeah.

So, if we want, I can put root C that is just for my convenience this is an arbitrary constant anywhere. So, if I change sides we get y dash square in to whole minus C is equal to some other constant. So, essentially what we get from here is y dashed which is d y d x is equal to some new constant k just by changing sides. So, it will be root over C by 4 minus C, which we can call k and that is it. So, if I integrate this we get y equal to familiar form of y equal to m x plus C. And this will be equation of a straight line right and now we have formally proved using variation calculus that the shortest distance on a plane between 2 points on a plane on a or an Euclidean plane is a straight line.

Please remember, this is the first formal proof of this statement which you have been heard you have been you have known since, you were in may be you know primary school third standard 4 standard that is why we started or maybe junior high school fifth or sixth standard. And now at this edge coming in to your; you know bachelors course or masters course you have finally, formally proved it. So, this is a beauty of variation calculus many things we just take it which we have taken for granted with this the beauty of advanced study many things which we are taking for granted at time, can now be proved more formally right. So, this is one, and I think we have just enough time for one more short calculation, let us see what do we have. So, we have another problem number 3, but I think in order to perform this problem, we need to learn an identity which is called the Beltrami identity.

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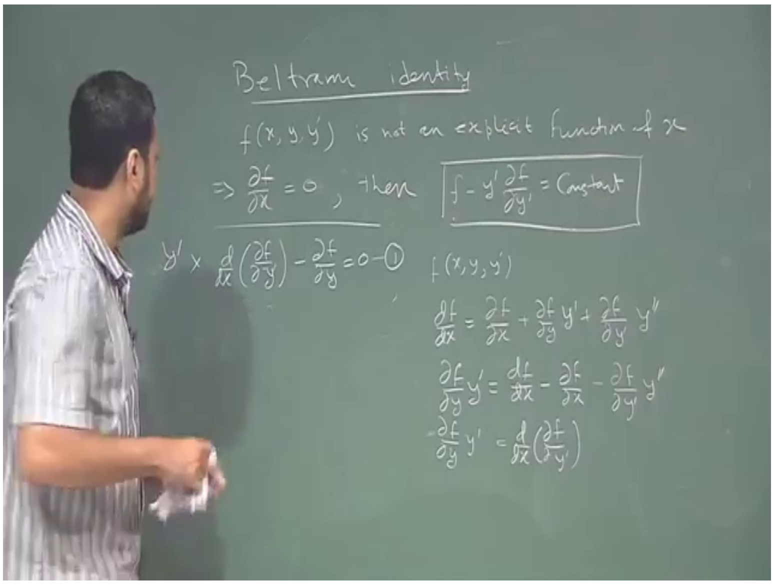

So, will end this class by proving Beltrami identity, what is this?

So, just give me a second right d x right. So, Beltrami identity says, if this function x y y prime is not an explicit function of x; that means, that means if del f del x is equal to 0. Then we have f minus y prime del f del y prime is equal to a constant. So, this is an identity, which of course, can be proved formula which we will do in a moment which is extremely helpful when we are discussing calculus of variation extremely helpful. So, we will end this class, by proving this identity which is essentially a 3 power 3 line proof so we to begin with we have this equation del f del x y dashed minus del f del y is equal to 0 please remember this f is not any arbitrary function, but f is the function for which this equation is valid.

So, this is the function which comes out as you know stationary as a function of stationery integral. So, it is not true for any arbitrary function. So, if this is the case and please remember that f is a function of x y y prime. So, d f d x is given as del f del x plus del f del y y prime plus del f or sorry yeah it is fine del f del y prime d u y prime d x right. Or we can call this the last term we can simply call sorry y double dashed right we can simply call that. Now this is let us call this equation 1. Now what we can do is we can change sides and write del f del y y prime is equal to d f d x minus del f del x minus del f del y prime y double prime. Now we what we can do is we can multiply this with y prime and write it down here. So, it will be minus del f del y y prime I am writing this term first changing sides it will be equal. So, it does not matter we do not need the minus will be d d x of del f del y prime, right.

So, comparing the right hand side what we can do is so it is slightly. So, we do not need this one anymore. So, slight mathematical jugglery we have to perform what we can do is we can you know take.

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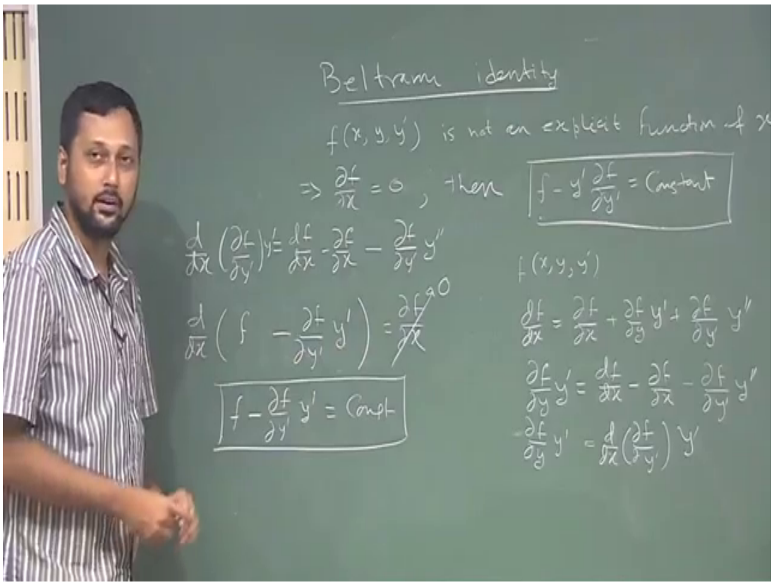

So, basically if I subtract one from the other then this side will be equal to 0. And so we can write d d x of del f del y prime is equal to t d del yeah. So, d f d x minus del f del x minus del f del y prime y double prime. And these one more step and what we can do is del f del y double prime right little bit modification and this can be written as. So, sorry I do not remember this by heart, but it is not very difficult anyway y double prime this f minus this sorry d d f of f right. Because I have this one and this one in hand I am sorry it will be y prime yeah. So, if I compare this 2 it can be equated with del f del x let me let me check this step once more. So, if I open this bracket. So, it is first term will be d f dx which is their second term will be d d x of d f del f del y prime times, oh there yeah that is what I was missing there will be y y prime dot here yeah.

So, now it all fits good. So, just manipulate this little bit and you can write it in this form now going back to this identity if this side is equal to 0. If this is equal to 0, then we immediately see this whole thing equal to 0. And we get f minus del f del y prime y prime is equal to a constant right. So, we proved it is a very easy proof 3 or 4 lines essentially. So, this is a this is called Beltrami identity please keep this in mind because we will be using this on and off. I mean whenever needed whenever we find of find a function f which is not an explicit function of x we can use this identity we can just do this for the problem, we have just now we have solved please remember that in that problem also f was not a function of x.

So, what I would suggest is instead of what instead of going what we doing what we did you can just apply this identity. And see if you get the same answer. Of course, you will get the same answer, that is not there is not a question, but this will be a good check for you. So, with this we end this lecture here. Next class we will continue with this problem set, we have few more problems to solve which will take more or less the full class. And definitely we will start with the introduction of our last topic which is small oscillation.

Thank you.