**Classical Mechanics: From Newtonian to Lagrangian Formulation**
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**Lecture – 50**
**Lagrangian Formulation – 8**

So, we are back with discussion of variation principle, but before that I am very sorry, I missed small, I mean one small topic in before we go to variation principle.

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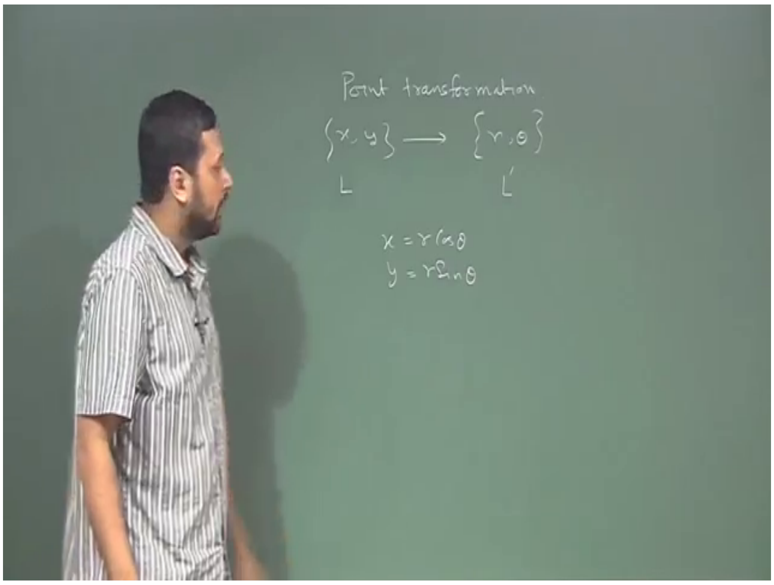


So, that topic we call it the point transformation it is a very simple concept so like for example, if we want to solve a dynamical problem in 2 d, let say we can either choose x y coordinate systems in 2 d plane or what we can do is we can take r theta coordinate system, right.

So, both in I mean depending on the geometry of the problem it might be advantageous to use one of this, but in principle we can use any of this in order to solve the problem in 2 dimension and we can also get differential equations which will be different of course, So, we have one set of differential equation we have one set of differential equation for this.

Similarly, the Lagrangian if I write Lagrangian l x y and yeah I mean or rather I will just call it l which will be a function of x y x prime x dot y dot and time and here let us call the Lagrangian l prime, so we say that it is what we have seen what we discuss here basically is that set of generalized coordinate the choice of generalized coordinate is not unique, but what is important to what we notice from here that x and y with their very strongly related to I mean they are related to r and theta by the set of linear transformation that x equal to r cos theta and y equal to r sin theta like this, so that means, although the Lagrangian's are different the equations of motion look different, but they are convertible from one in to the other by using this these type of relations right. This is the precisely the concept of point transformation this type of transformation of between generalized set of possible generalized coordinate is called a point transformation.

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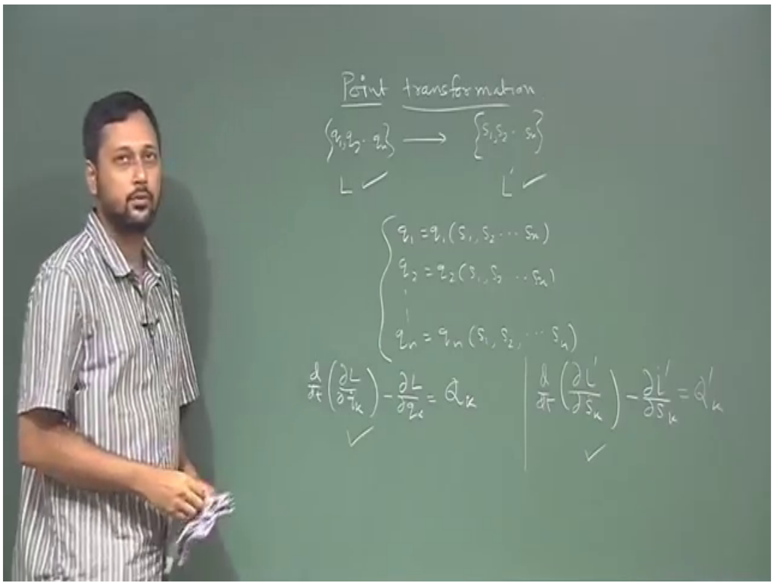


So, it is basically a transformation is between this set of coordinate to this set of coordinate. Now in general, if I write this set of coordinate using you know q 1, q 2, q n and this with s 1, s 2, s n then and if there is a set of transformation equation exist which will correlate these 2. So q 1 will be function of s 1, s 2, s n; q 2 will be a function of s 2, s n similarly q n will be a function of s 1, s 2, s n.

If such transformation equation exist between 2 numbers of 2 sets of generalized coordinate only criteria is 2 generalized coordinates has to be equal in number so if we have n number of coordinates here it has to have exact same number of coordinates here and they have to be related by this type of transformation equation.

So the concept of point transformation is if this type of transformation equation exist, then if l is a valid Lagrangian l prime will also be a valid Lagrangian little more formally if l gives you the equation of motion in the form d d t of del l del q k dot sorry minus del l del q k equal to q k q k being the generalized force I am just writing it in general form if this is a valid set of equation then this will or you should use prime here so if this is a valid set of equation this will also be a valid set of equation so this is exactly the precisely the concept of point transformation nothing new in it we already know that there cannot be an unique set of coordinate for a particular problem.

So, there is just a formal notation and actually it can be proved formally we can start from this set of equation and we can show if this is a valid I mean if this equations exist then also we can go to come to this equation, but this formal proof is more mathematical and conceptually it is it does not offer anything new so we are just keeping it so this is something called a point sorry; p o i n t; point transformation keep this in mind, we will solve may be a small problem on this.

So, with this we are moving totally in to discussion of variation calculus quantities why it is useful, but before we go in to all this details we I started off in the last lecture I did some small drawings and started doing something let us try that:-

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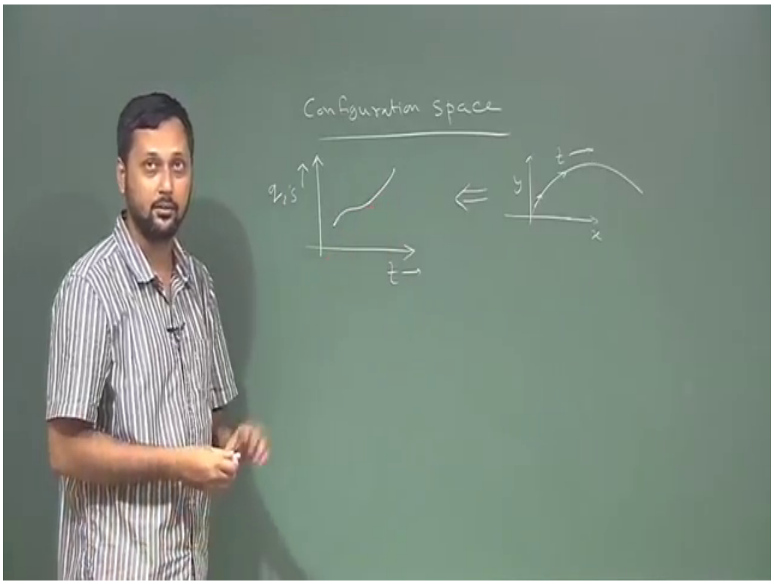


So, we define space called the configuration space for a system right so we are discussing variation principle now point transformation is over right. Now configuration space is 2 d space where we have time in the x axis and collection of all the coordinates q i's in the y axis, now this is something it might be difficult to grab as for you if this is might be something very new for you.

So let us assume that I am moving this object in space or I am throwing it, it will move right so it will go it will take some trajectory it is a parabolic trajectory it could be distorted due to air resistance, whatever it is I need to I need 2 parameters x and y right because it is a essentially yeah or you can call it x and z it is a 2 d problem.

Now, if I choose to draw the trajectory in real world if I have x here y, so we are familiar more familiar with trajectories which look something like this right so along the trajectory the time changes right, but if I represent it in a slightly different way where I stress the points if I stress the object whichever I throw; for example, this object as a function of time and see if it is x or y coordinate is changing.

If it is changing, then we move you know in a certain direction and if it is not changing we just stay and at the same point, now this is more of a hypothetical concept it is sometime not very easy to you know visualize this, but it is more of a hypothetical concept people use it a lot in theoretical mathematics and mechanics and for many other you know even in condensed matter physics , it is widely used the configuration space the thing is for this type of motion there will be an equivalent trajectory which will be traced out in the configuration space, right.

So, I do not know if you if I can draw a one to I can give you an one to one description of this picture and this picture, but all I can tell you because it is very difficult to plot variation of x and y together in to 1 axis, but I just took a general example it is not that it corresponds to directly corresponds to this, I am saying that we are familiar with trajectories where the coordinates are changing as a function of time. But here we are and we draw both x and y coordinate that is easy for the visualization, but we can do the same thing in principle at least you have to agree with me that in principle what we can do is we can plot time in x axis and any variation in coordinate in the y axis and we can represent this trajectory just like this, I really frankly speaking I do not know how a parabola looks like in a configuration it is space 2 d configuration space.

But, that a part if I do that there will be definitely some kind of a traceable path on it now let us assume that there are 2 end points what happens is we when we throw this object it starts at time t 1, let us say and it ends in drops on the ground; that means, they or somebody catches it in the mid air that is also possible, whatever it is it happens at a further time t 2, right. So, between t 1 and t 2 that entire trajectory is traced out right. Now we can think of a situation that due to some reason what reason we are not going in to the details right now.

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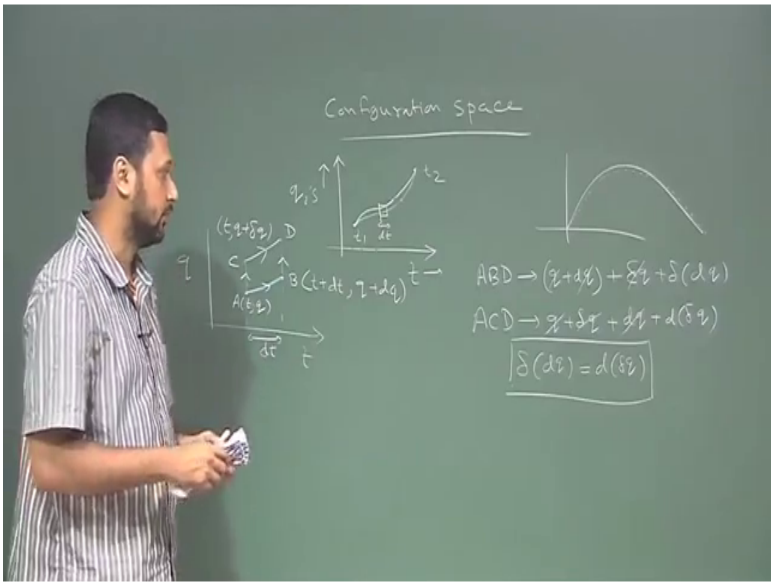


Due to some reason, the particle the which whichever I throw it took not exact the same exact tragic I mean this trajectory, but it took a trajectory which is slightly different from the one. Please remember please understand that I have not change the initial condition or anything or actually frankly speaking you can think of it this way that I have you know just starting point and end point is the same, but may be some initial condition some velocity something was different. So, that will be a trajectory which will be following this one closely which will be following this one closely, but not exactly the same right. So, it will once again as I said the starting and the end point is the same and only slightly the trajectory is different how and why could be due to some change in initial condition could be due to some other hypothetical reason, but let us assume that there is slight different in trajectories so it will be traced like this.

Now, if I zoom on this sorry if I zoom on this particular segment which took place over a time interval d t and try to draw it once again what do I see this is 1 segment and this is the segment in yeah. So, I will just you know elongate a bit something like this right and once again we have time axis here q axis here right q's. Let us assume for now that there is it is a one parameter system just for simplicity I want to demonstrate something. So, let us say there is only 1 parameter it is not that system, but some other system which also shows some kind of small deviation in trajectory between 2 trajectories.

Now what I started in the last class I am just drawing it again so I was wrong in the last class I said that we are going only advanced we are advancing only in time, but there is no change in q which is wrong because that will not give you a true trajectory; that means, the object is standing in the same place for a duration of time which is not a true trajectory it is a static case.

So, in dynamism the in dynamic case what happens if we go from A to B right, let us say a has a coordinate like we did in the last class of t and q right if that is the case then we have a point c we have a point d so b will have a coordinate t plus; so this is d t that is the real time displacement and q plus d q right. Now if see again it is a kind of hypothetical situation because we cannot repeat the same time point right I mean yeah the time interval might be the same. But in principle we if I throw an object and it goes to the if an object start from at time t 1 goes to time t 2, it cannot we cannot repeat it because we cannot reverse the time because time will always go forward we cannot come back to point time t 1 once again so it is kind of an hypothetical situation.

So, if I let us say hypothetical is picking the object could have taken one of these two trajectories right so this is the more correct statement the object could have taken one of these two trajectories now if that is the case then the displacement from A to C is actually a virtual displacement right because it is taking no time it is either could have been this or could have been that whatever it is between this 2 points there is a virtual shift and virtual shift means C will have a coordinate t and q plus delta q delta q being the virtual displacement right so we have coordinate for A B C.

Now what will be the coordinate of D I can go from A to D either like this or like this, so we can take A B D or we can go A C D if we go A B D the coordinate of D will be, if I take A B D coordinate of D will be again B to D is another virtual displacement so same time. So, forget about the time part time will be the same whether I go from this way or that way time will be t plus d t there is no doubt about it only the spatial part I am interested in A B D the q coordinate will be like this will be a virtual displacement of q plus delta q q plus d q so it will be q plus d q plus delta of q plus d q.

Similarly, if I take A C D see C to D is a real displacement in time right so it will be q plus delta q plus d of q plus delta q right so once we compare this and but D will have an unique coordinate right I mean the coordinate of D, whatever whichever path we take we should not it should not alter the coordinate of d. So, comparing these to this will be equal we will see that all the terms will cancel out if I open this bracket and perform there will be a d q here right so if I open it, it will be a delta q here plus delta of d q and there will be a d q here plus d of delta q right. So, if I take I mean I mean if we equate these two this, this and this will cancel remaining will be delta d q plus which will be equal to d delta q.

So, what we have established here is by if I taking 2 hypothetical trajectory of a system between 2 fixed points t 1 and t 2 that the operations d and delta which can be exchanged freely so they are commutative basically, so d of delta q is equal to delta d q this is the reason this is the result we have already used it we have if you recall if you go back to the derivation of the Lagrange's equation we have already used it. But now I thought I will give you a proper you know theory, I mean what you call elaborate I will elaborate on this point and give you a proof of that now, but this is what this is something that has come out as I would say as a you know side tock I would rather say, what I want to discuss actually so I demonstrate this for a single parameter system; it could might as well be extended for a multiple parameter system that is not the problem.

Now, what I am trying to demonstrate here is possibility of having very various paths between 2 fixed point t 1 and t 2 of a particular system. So, let us have a situation we have a system at time t 1 which has a particular orientation or particular set of coordinate which corresponds to a point fixed point so this is my point 1 in the configuration space.

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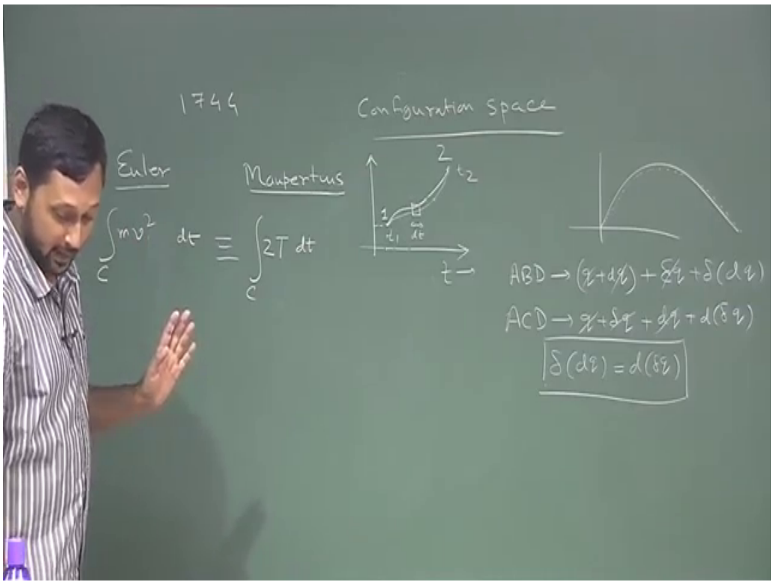


And I know after certain time gap it will at a time t 2 it will reach a point fixed point to you know between which has a time t 2 and a coordinate q 2 which is a point 2 now in between these 2 points which path the system takes that is the question I am asking here that is the question I am asking and not only me asking I mean it is not that I am asking it physicists has been asking this question for centuries.

So the first problem in a more physical mannered the problem is following I have a object in my hand I drop it and I catch it here what makes it travel in a straight line you might say through gravitational pool, but that is not not precisely the answer because whatever path it takes while coming from here to here if it takes to comes like this if it comes like that take any other path the initial and the final energy states are the same. Similarly, if I am hanging a chain you must have seen that this electric lines high tension lines between 2 poles they not straight there is a you know it due to gravity they have a particular curvature I mean particular shape and that shape is very well defined you hang a thread, you hang a solid chain, you hang a plastic pipe it should take the same shape I mean say it should follow the same family of curve I mean it have similar types of equation.

What makes it takes that shape it could have taken any other shape so there must be some parameter which is being optimized that was the question physicists were asking for centuries it started with Fermat we all know that Fermat principle, right.

What was the Fermat principle of least time that he was the first one to give some kind of an optimization principle he said light will travel between 2 points along a path which is which will take shortest time he never used the word straight line it happened to be that path is a straight line the because shortest distance between 2 points space is a straight line by the way have you ever seen we all know that right the shortest distance between 2 path is a straight line the 2 point in a in space straight line have you ever seen any formal proof of that no right, we will prove it.

There are ways of proving it using this variation principle right so then when it comes to I mean Fermat came up with this principle of least time back in 1600 year 1600 that time the you know classical mechanics was in it is early age before Newton so we did not have much idea of modern mechanics modern equations, but when that came in to picture then people started asking why a mechanical system takes up certain path for example, when I throw this chalk it will take a you know parabolic urge why a parabola?

Why not anything else? what makes it take the shortest I mean what makes it take that particular path so something must be I mean it is definitely it is not straight I mean it is not the shortest time path because shortest time path between any 2 point will be a straight line that is very intuitive of course, it might not be true all the time. So, why a parabola why not any other curve? So then in 1744, almost at the same time Euler the great Leonhard Euler and another French scientist Maupertuis; they came up with the principle which is called the which is the you know ancestor of the so called principle of least action.

What they said was actually Euler said, the mechanical system will follow a path along with which the quantity integration m v d s s being the length along the curve arc length v is the mass m is the mass v is the velocity where along. So, according to Euler that was in 1744, two independent minds one sitting in Germany, one sitting in France just a minute so I will so the pronunciation in my opinion will be Mauperti it is a French name Mauperti, but do not count me on this because they do not quote me on this because I am not an expert in French any way so, but and he said no, no, no, the path which will be taken is along which the integral the path integrals so this is a path integral along which the path integral c 2 times t d t will be optimized.

So, this they came up with these two suggestions, but then if I slightly manipulate it if I put d t this is nothing, but v so this will be v square and m v square is nothing, but 2 t right so from here so these are actually equivalent. So, this is nothing, but m v square d t which is 2 times t the t being the kinetic energy. So, they came up at the same time with the same principle, but even at that time in 1744, the formal approach of or you know Lagrangian calculus was not formed. Later, on Euler had a great role to play in the derivation of Lagrangian dynamics, but he somehow could not make a progress from here, what he derived was very true not very true, but at least for certain system it is it is it is right, but then we have to wait for another 100 years, then the Lagrangian dynamics was generate I mean was formulated and then people started working on that more and more.

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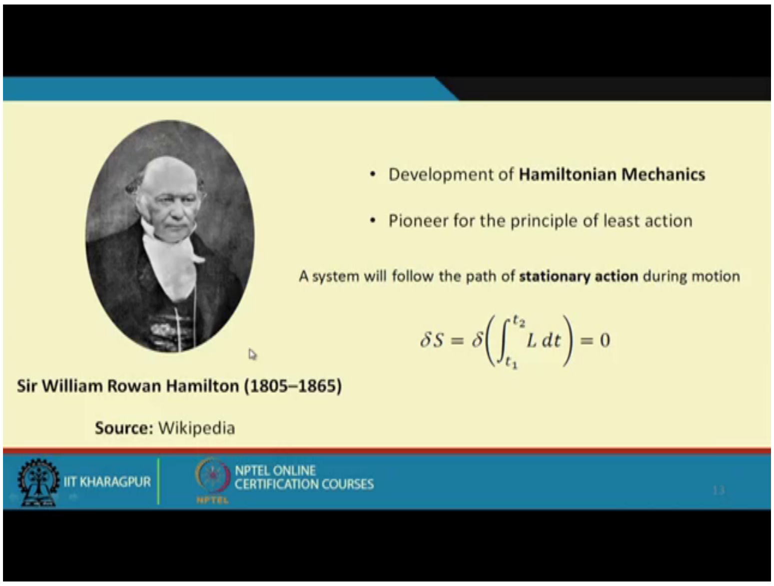


And then came this very famous Euler's I mean sorry very famous Irish scientist whose name is Sir William Ronald Rowan Hamilton that was he lived between 1805 and 1865 and he actually kept on building on the Lagrangian dynamics and developed something called the Hamiltonian mechanics, which unfortunately we could not we cannot you know discuss during this course which our course duration is will not allow us the discussion of Hamiltonian dynamics, but whoever taking an advanced classical mechanics course will start with Hamilton's principle.

Now he came up with a principle of stationery action or sometimes called the least action what he did he replaced this kinetic energy term ok with the Lagrangian of the system and between 2 time points t 1 and t 2 the system will take that path along which this particular integral l d t is stationery what is the meaning of stationary, stationary means the variation of that integral along a particular if a variation of that integral along a particular path is 0, the system will follow that particular path so that so what it did essentially was replaced this t with t minus v that is Lagrangian, right.

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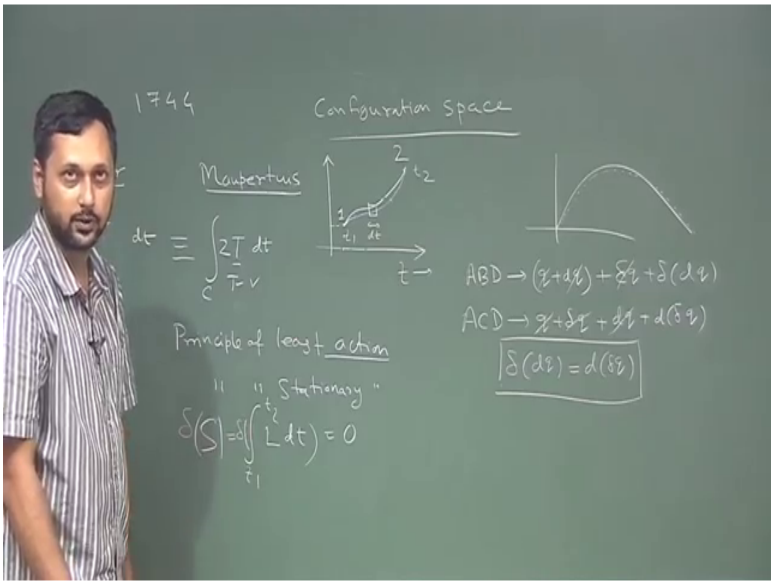


So, the principle of least action or principle of stationary action says action so this is a quantity action is this particular integral, integral t 1 to t 2 l d t a system will take a path along which variation of this action is equal to 0 and we see that this delta operation the variation operation is interchangeable with derivative it is very much like a derivative action so that means, they actually the system finds the path along which action is minimized so action is also a function after this integration also it remains the function of certain parameter right we will see that how to work with on that.

So, what we need to so what in the frame work of variation calculus what we deal with is we try to determine the variation of such integration and so first what we will do is we will take any function which is which might be a Lagrangian might be something else then we will see that in special case of Lagrangian this equation when we said this equal to 0 is actually give us back the set of Lagrange’s equation of motion; so let us try that keep this in mind I am going to remove it; now, but you I hope you already have your notes; so we will start it little more formally.

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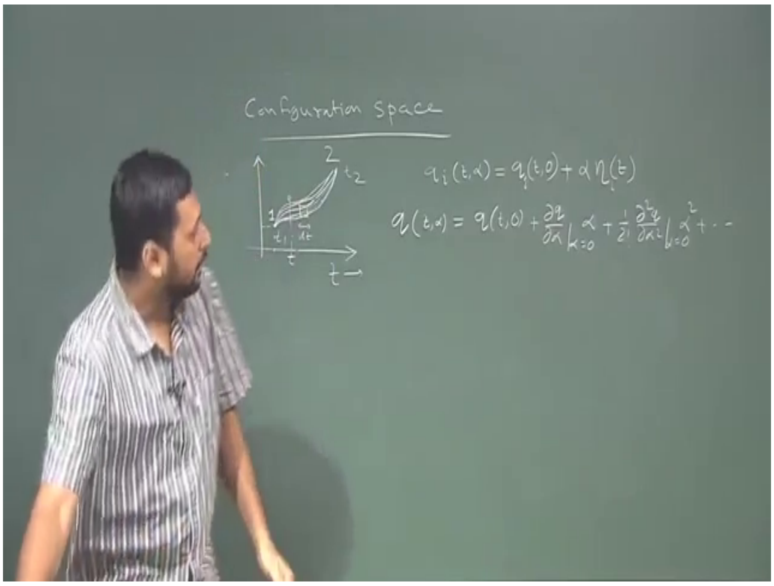


Now, in order to understand the variation in a more fundamental way what we are going to do is, we will take; we will say that there are many possible paths which are infinitesimally close from one another now out of that path system will choose one, but right now let us not go in to which path the system will choose that is more I mean let us concentrate on how to formalize this little bit.

So, let us assume what happens is each of these path, which are slightly different from each other, but the coordinates will also be slightly different right so what we can do is we can write any coordinate q i as or q i which is a function of t and we introduce a new parameter alpha which is also an infinitesimally small parameter and we can write the coordinate as 0 plus alpha eta t.

So, what we are doing is we are decoupling the time variation of a particular coordinate what is changing between 2 paths so, please understand that as we move from one path to the other along this line along the t; I mean along a path where t is fixed this are all virtual displacements. So, we have to somehow formalize the virtual displacement of delta q what how can we do that please remember in delta q there is no t evolution. So, we have to decouple the time variation and special variation of each and every coordinate in order to do that we what we do is we actually introduce a new parameter infinitesimally small parameter alpha and take a Taylor series expansion of q around alpha so q of t alpha if I take a Taylor series expansion at around alpha equal to 0 this will be q of t alpha equal to 0 plus del q del alpha at alpha equal to 0 d it will be alpha plus 1 by 2 factorial del 2 q del alpha square at alpha equal to 0 alpha square and it turns like that; right. So, this is the standard Taylor series expansion because alpha is infinitesimally small parameter.

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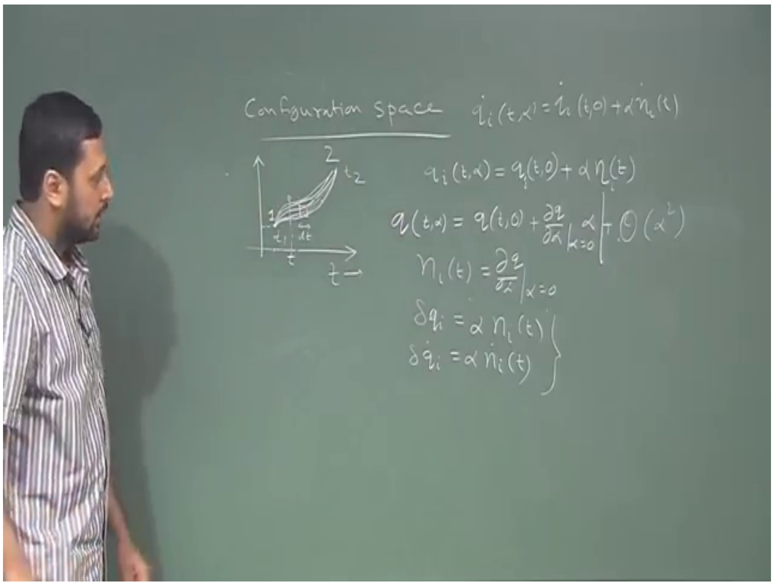


What we did we essentially truncated this here and we ignored any parameter we ignored any parameter which has a order alpha square so comparing these 2 equation, we see that eta I t is nothing but del q del alpha at alpha equal to 0 and what we can see is delta of q i will be nothing but delta of q i t 0 plus or sorry delta sorry delta of the or virtual displacement of q i will be nothing but alpha n i t so this is an infinitesimally small variation of the function for of a function which depends explicitly on time; right.

So, we can write it in this particular form if this is the case then the time for generalized velocities we also we can do the similar construction like, if I take a derivative of this time derivative of this we can write q i dot t alpha is equal to q i dot t 0 plus alpha eta e dot t we can do that and once again for generalized velocity also delta q i dot is equal to alpha n i dot.

So, once again what we did is we have decomposed t in to a q in to a time dependent part and another part which does not dependent on which and another part which is the function of a quantity which does not depend on time, but defines a special variation right and so that is why the virtual displacements can be written in this particular form.

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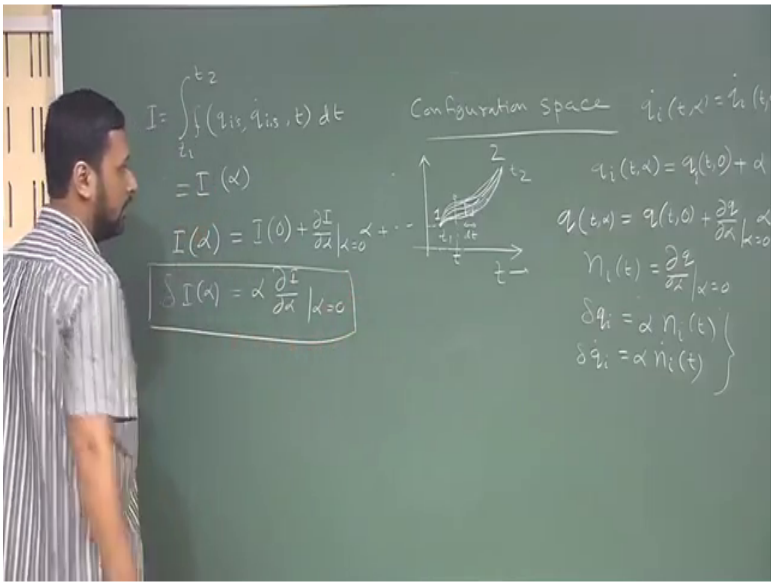


Now, we take any function f which is a function of q’s q i’s q i dots and time it could be a Lagrangian it could be any other function. So, let us take a general form and we have to we define and we have to let us say this integration exist which is between t 1 and t 2 d t right. Now what we have to do is we have to take a variation of this integral, now what happens is this integral if you see q’s are function of t and alpha q i dots are also function of t and alpha. So, if I integrate the time part what remains is this integration is actually a function the final after this integration what remains at the alphas because alphas are not integrable alphas does not depend on t so we have I of alpha.

So, what we can do just like we did in the case of coordinates and velocities we can also you know expand this integral in a Taylor series around alpha equal to 0 and we have del I del alpha alpha equal to 0 alpha and we can neglect the higher terms so the variation of this integral d of I alpha will be nothing, but alpha times del I del alpha alpha equal to 0 right. Just like we have variations in coordinates and velocities we have a variation in this integral and this is precisely what we are looking for right.

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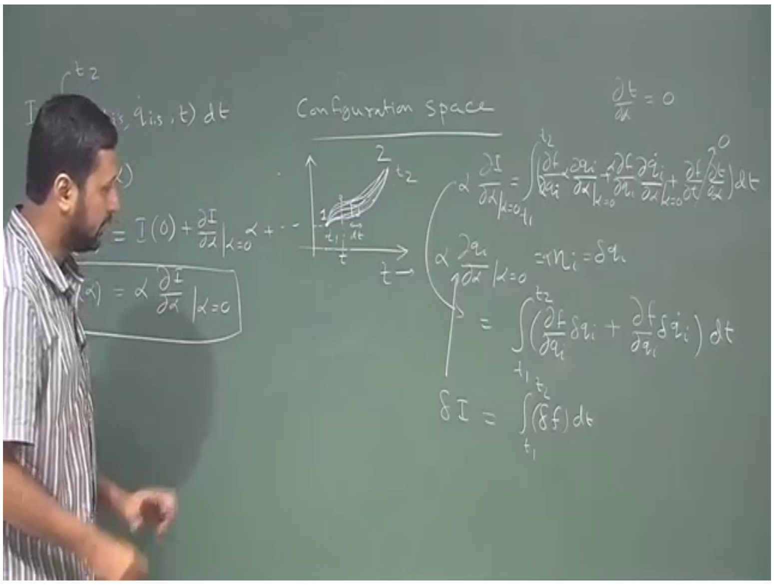

So, del I del alpha we have to evaluate in order to evaluate the variation in the integral we have to evaluate this term del I del alpha so del I del alpha going by this is nothing but t 1 to t 2 f of so it will be del f del q i del q i del alpha plus del f del q i dot del q i dot del alpha plus del f del t del t del alpha right intersection d t. Now the last term of this of this differential vanish because alpha and t alpha is not a function of time so del t del alpha or del alpha del t is equal to 0 right so the last term is actually a 0, 2 terms remains and in this 2 terms, what we can do is we can recall and so sorry this will be evaluated at alpha equal to 0 so all these derivatives at alpha equal to 0 alpha equal to 0, right.

Now if you recall d q i d alpha alpha equal to 0 is nothing but your eta I, right so that is what we did just now this quantity is your eta i and if I multiply this with alpha so we have a multiplication of alpha here we have a multiplication of alpha here sorry the we have not enough space but you understand right so we are just we have to multiply this whole thing with an alpha this is any way 0 so alpha of this will be alpha of del q i alpha multiplied by d q i del alpha is equal to alpha times eta I which is equal to delta q i right.

So, this integration takes the form integration t 1 to t 2 del f del q i by the way there is a summation over q i i parameter I mean this index because it is a repeated sum which is implicit in this delta q i and similarly we can do the same thing for the generalized velocity component and this will come out to be del f i del q dot delta q i dot right d t and this in short can be simply written as t 1 to t 2 delta f d t right and this is nothing but variation of i so this is equal to this. So what we have seen is if i is this integral then variation of i is the variation of this parameter and this is kind of intuitive because integrand is what defines a derivative I mean sorry I mean integrand is the main part of the integration so this is fine.

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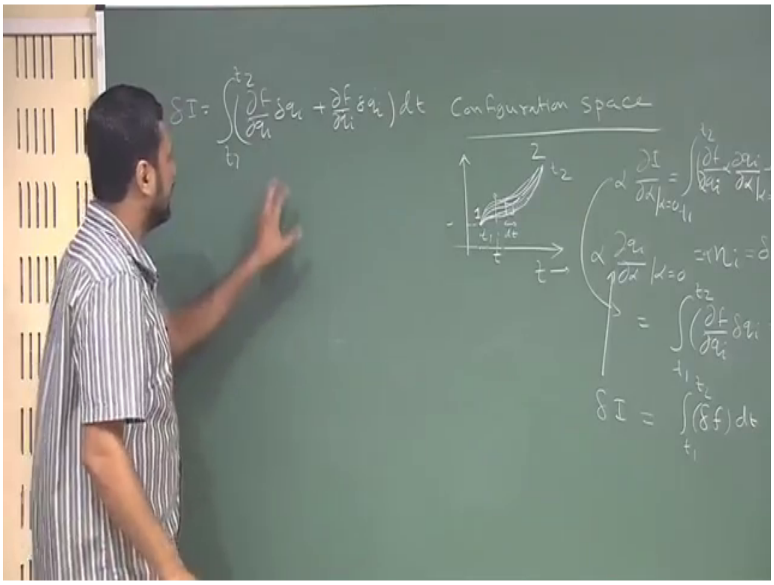


But then we can take it 2 step ahead from here what we can do is, we can what we can do is we can continue here from this particular form that delta i is equal to integration t 1 to t 2 del f del q i delta q i plus del f del q i dot delta q i dot times d t so these are there is a summation over q i and from here we can so there are 2 or 3 more steps which I will continue in the next lecture.

Thank you.