Classical Mechanics: From Newtonian to Lagrangian Formulation
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Lecture - 05
Systems with variable mass – 2

So we continue with the problem with vari problems on variable mass. So, let us look at problem two here.

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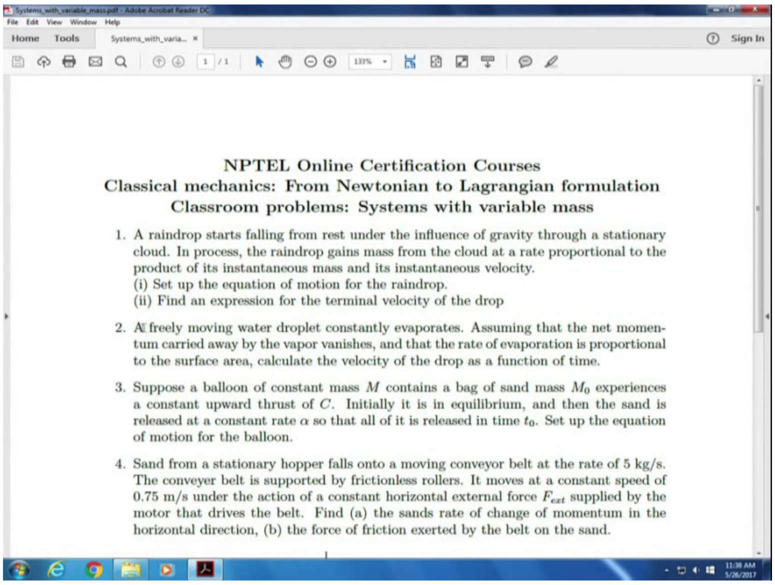
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Classical mechanics: From Newtonian to Lagrangian formulation
Classroom problems: Systems with variable mass

1. A raindrop starts falling from rest under the influence of gravity through a stationary cloud. In process, the raindrop gains mass from the cloud at a rate proportional to the product of its instantaneous mass and its instantaneous velocity.
(i) Set up the equation of motion for the raindrop.
(ii) Find an expression for the terminal velocity of the drop

2. A freely moving water droplet constantly evaporates. Assuming that the net momentum carried away by the vapor vanishes, and that the rate of evaporation is proportional to the surface area, calculate the velocity of the drop as a function of time.

3. Suppose a balloon of constant mass M contains a bag of sand mass M_0 experiences a constant upward thrust of C. Initially it is in equilibrium, and then the sand is released at a constant rate α so that all of it is released in time t_0. Set up the equation of motion for the balloon.

4. Sand from a stationary hopper falls onto a moving conveyor belt at the rate of 5 kg/s. The conveyer belt is supported by frictionless rollers. It moves at a constant speed of 0.75 m/s under the action of a constant horizontal external force F_{ext} supplied by the motor that drives the belt. Find (a) the sands rate of change of momentum in the horizontal direction, (b) the force of friction exerted by the belt on the sand.

So, the problem says a freely moving this one. A freely moving water droplet constantly evaporates. Assuming that the net momentum carried away by the vapour vanishes and that the rate of evaporation is proportional to surface area. Calculate the velocity of the drop as a function of time.

Now, in this particular problem the so far we have dealt with a situation where end drop is accumulating moisture from its surrounding that is to say when it is passing through a cloud. So, it is accumulating the moisture, but in this particular case the moisture is going away from the rain drop.

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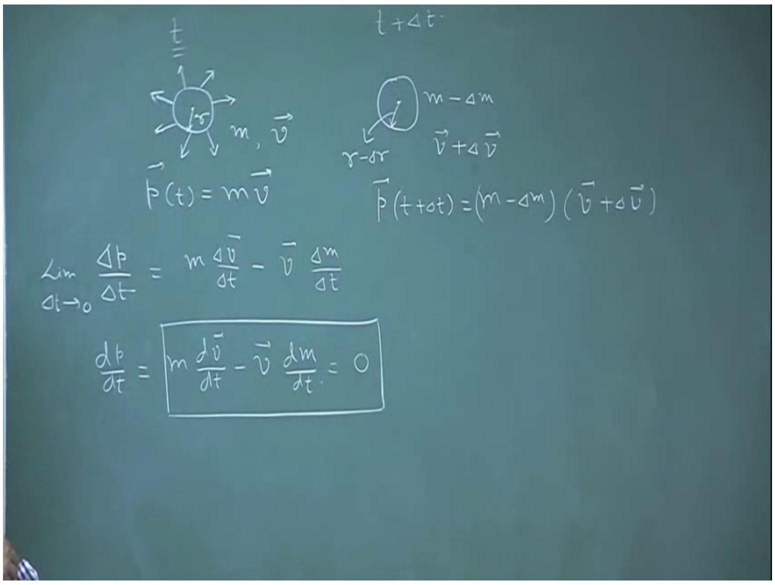

So, the situation is the following, we have a water droplet with which in which the water evaporates in all direction each of this water molecules that leaves this sphere surface essentially carries some part of momentum with it, but because the dispersion is even from all direction, the net momentum cancels out. So, the evaporation processes does not exerts any force on this particular droplet. So, we have at time t we have a mass m or let us mass m and let us say it has a velocity v, and assuming that there is a radius r at mass at time t plus delta t what happens is, it becomes m minus delta m. Let us say the velocity has changed from v to v plus delta v, we do not know if delta v is positive or negative as of now we will just keep it like this and the radius has changed from r to r minus delta r.

So, if we compute momentum it is m v, if we compute momentum at t plus delta t it is m minus delta m p plus delta p. So, if we compute delta p delta t limit delta t to goes to 0 this will be. So, this m v m v will cancel out. So, essentially we will have m delta v by delta t minus v delta m by delta t, which in differential notation simply m dv dt minus v dm dt which is equal to f external according to Newton's second law, but because in our case it is been mentioned that there is no external force on this particle, what we can do is we can simply put a 0 here.

Say essentially the equation will be working on is this great. Now what we can do is. So, you see as I said already during this during the discussion of system with variable mass there is no unique way of writing this differential equation, for each of this problems depending upon the nature of the problem we have to modify the differential equation, only this is we have to calculate the momentum change at a over a time delta t by taking momentum at p t and p t plus delta t and compute the difference.

So, we see that because of mass is evaporating from this rain drop it is or water droplet we are getting a different equation, generally we have a plus sign here when the rain drop is falling through cloud and its accumulating moisture we have a plus sign here we have a minus sign any way. So, let us try to integrate this and what we see straight forward is dv by v is equal to dm by m integration.

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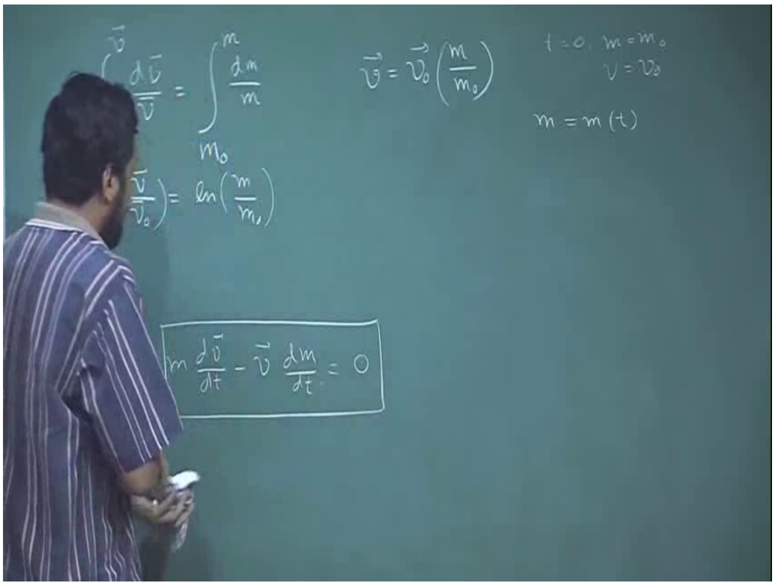

Let us assume that initially at time t equal to 0 we have a velocity v 0, and upper limit of the velocity is v just write it slightly below so that you can see properly. So, v 0 v dv by v is equal to integration. So, initially let us say the mass was m 0 and at time t the mass is m. So, we have dm by m. So, integration will essentially give us a relation which is v equal to v 0, please remember that v 0 is also vector quantity. So, we are left hand side of the integration is a vector I mean we are integrating vectors. So, keeping the direction in tact because we do not know which way it will going to change. So, we have m 0 m by m 0.

So, if we take this equation we will have a log logarithm and then we have on left hand side we have v by v 0, and right hand side we have m by m 0, comparing we get v is equal to v 0 times m by m 0. So, v 0 is initial velocity m 0 is the initial mass of the droplet at time t equal to 0 very good. Now question is how to get a functional form so essentially what we would like to do is we have to have m as m of t only then we can express v as a function of temperature I have yeah function of time not temperature sorry. So, in order to do that please recall that we have another relation given which is the rate of evaporation proportional to instantaneous surface area for this droplet.

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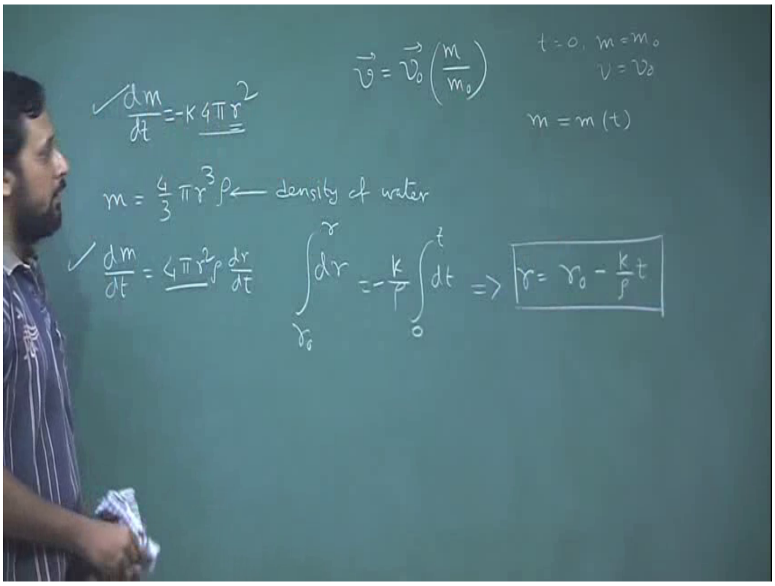

Now, if that is the case that essentially means rate of evaporation is dm d t, which is proportional to instantaneous surface area which is 4 phi r square this droplet r being the instantaneous radius. So, r essentially is r of t and we have a proportionality constant k. So, is please do not confuse it with constant for let us say resistive forces and all it is just a proportionality constant here. Now we also can write m is equal to four-third phi r cubed rho.

So, rho being the density of water fine now if we do that we can also get an expression for dm dt as 4 phi r square rho dr dt. Now if we compare this expression with this one we immediately see that comparing this 2 4 phi r square 4 phi r square cancels out. So, this whole term cancels out leaving behind, dr dt is equal to k by rho oh sorry there will be a minus sign here because as the water evaporates from the surface dm dt has to be negative quantity. So, we have to put a negative sign here. So, we have dr dt is equal to minus k by rho.

Now we integrate that we take dt to the other side and integrate at time t equal to 0 we have a radius r 0 at time we have radius r. So, this integration gives us relation which is essentially r equal to r 0 times r minus, so r minus r 0, r 0 minus k by rho. So, this is a relation which will have in hand. Now this is one relation and we already have another relation v is equal to v 0 times m by m 0. So, what we can immediately do is. So, at t equal to 0, r was equal to r 0.

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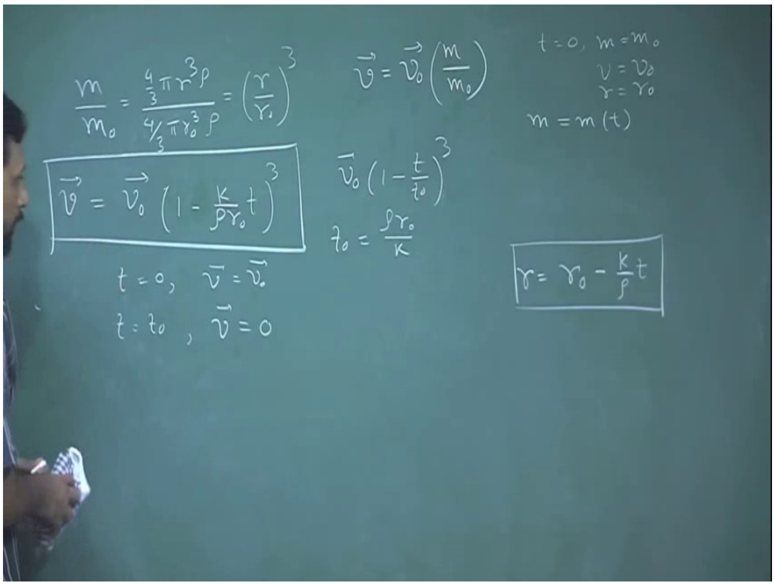

So, put t equal to 0 here we will get back r 0. So, what we can do is we can write m by m 0 as four-third phi r cubed rho divided by four-third phi r 0 cube rho which will be simply r by r 0 whole cubed and from this relation we straight away know r by r 0 whole cube is nothing but 1 plus sorry 1 minus k by sigma r 0 t whole cubed. So, we use this relation to final to get our final expression which is v is equal to v 0 times k 1 minus k by rho r 0 t whole cubed. So, this is the final answer.

Now if we examine this expression we put t equal to 0 we immediately get v equal to v 0 if we put t equal to let us say, we just write this as v 0 times 1 minus t by t 0 whole cube where t 0 is rho r 0 by k. So, if we have t is equal to t 0 then we immediately see that v becomes 0. So, essentially its starts with an initial velocity the droplet starts with an initial velocity 0 and at time t equal to t 0 when time reaches t 0 the velocity becomes 0 velocity of the droplet becomes 0 and also at time t equal to t 0, we can actually find the radius when it comes to a standstill ok.

Any way that is not terribly important just to give you some insight of this result, you can do it yourself if you wish. So, with this we move to the next problem which is this one. So, it is a problem where a balloon it says that there is a balloon with a constant mass m, and it has a bag full of sand m 0 in it. So, there is a constant up force upward thrust on the balloon which is given by c and initially it is in equilibrium. Now once the sand starts releasing at a constant rate v equal to constant rate alpha, then there is a change in velocity. So, the balloon essentially moves upwards. So, we have to we do not have to solve for the equation we have to set up the equation proportion for this particular system.

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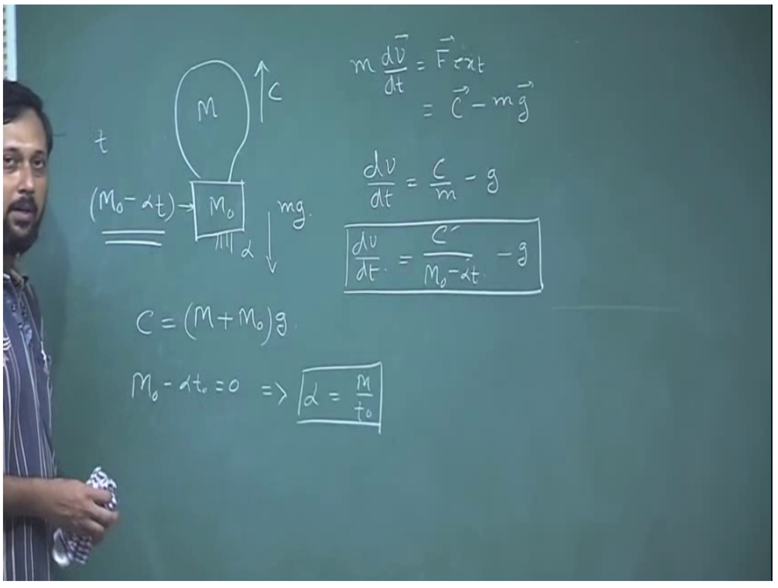

Now in order to do that, let us get into this picture once again. So, we have a balloon which is which is a fixed mass m capital M and there is a sand bag which is at time t equal to 0 this is the situation. Now as the time progresses at any arbitrary time t this sand bag starts releasing sand at a constant rate alpha. So, this mass is not a M 0 anymore, but this mass becomes M 0 minus alpha t fine. So, now, if we. So, essentially what we have to do is we have to write the equation of motion for this particular system at any time; now in order to do that.

Please recall that the balloon in this situation the equation of motion generally is given by M dv dt equal to f external now this F external in case of balloon. So, there is a upward thrust which is given by c in this case and there is a downward pull which is given by m g. So, your net external force is C minus m g because as the sand is getting released the balloon m g component is decreasing. So, the balloon is moving upwards. So, the net external force is pointing upwards so as the velocity.

Now because we are both in the same direction immediately we can get rid of this minus. So, vector sign and we can write this as dv dt equal to c by m minus g right. So, this is what we have in mind all we need to do is we need to replace to set up the equation of motion we need to replace c and m with some other quantity. So, essentially our c is also because c is already given we can essentially have to replace m. Now in order to do that we can replace m by m 0 t which is valid substitution, m 0 minus alpha t minus g we can. So, this is one form of equation which is already good.

Now, please remember we can have a slightly different version of the same equation because there is another condition given that initially the balloon is in equilibrium. So, at time t equal to 0 when the mass of this whole thing is M 0, not M 0 minus alpha t it is in equilibrium so; that means, c equal to M plus M 0 times g. So, we can replace this for c similarly at time t equal to t 0 the entire sand bag becomes empty. So, we immediately see that if we put t equal to t 0 here this whole term should vanish. So, we have m minus alpha t 0 equal to 0 which gives us alpha is equal to M by t 0. So, if we want to for certain purpose we might need to modify this equation modify this particular form of equation. So, we can use this 2 relation to get rid of c and alpha if we wish any way.

So, this is a the some manipulation some modification of this particular equation can be done, but as off now this is a very valid equation if we want we can put any of this function this expressions into it and we can modify this equation. So, I leave it you to as an exercise to modify and play with the equation and of course, you can put some values of for different parameters and see how what is the final equation looks like and this expression as we can see as it is very nice integral terms, the first term will give you a logarithm second term will be just will be proportional to g t on integration, ok.

So, let us move to the last problem of this particular section and the problem is this one, number 4 sand from a stationary hopper falls into a moving conveyor belt at the rate of 5 kgs per second, conveyor belt is supported by frictionless low frictionless rollers, it moves at a constant speed of 0.75 meters per seconds under the action of constant horizontal external force of external supplied by an motor.

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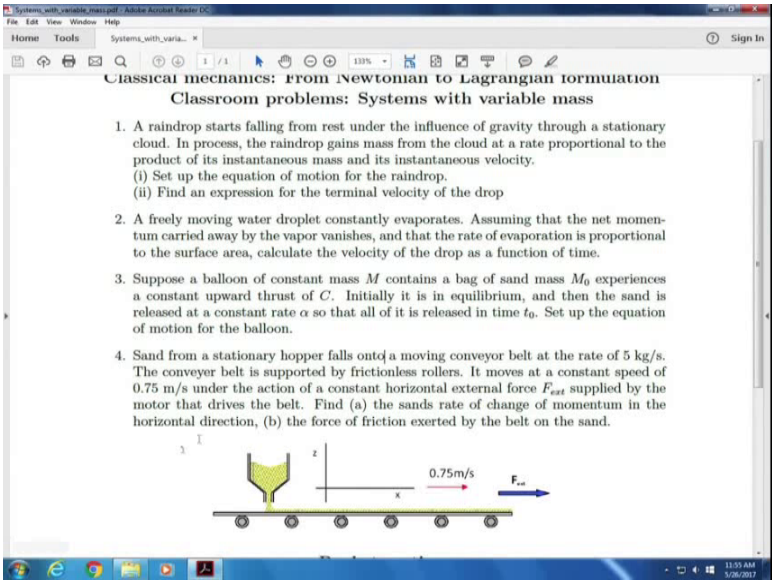

So, we have to find a the sands rate of change of momentum in the horizontal direction which is along the x direction and also we have to find b in b the force of fiction extended by the belt on the sand. Now in order to do that please recall that in this particular case equation of motion is given by v dm dt is equal to F external.

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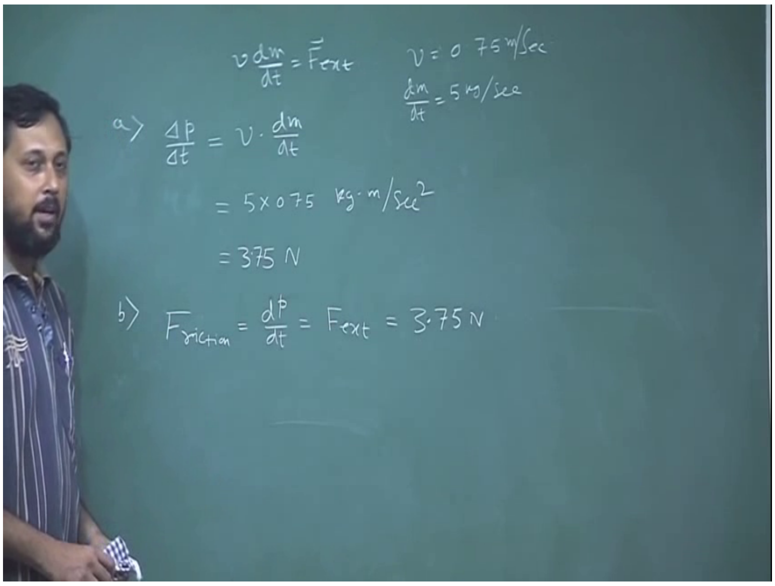

This was discussed in one of the previous classes. So, rate a we have to calculate the rate of change of momentum delta p by delta t, ok.

Rate of change of momentum and rate of change of momentum is simply given by delta p by delta t which is in this particular case is v times dm d t, and it is given that v is equal to 0.75 meters per second and dm dt is equal to 5 k g per second. So, we just put this numbers and we have 5 into 0.75 k g meter per seconds square which is nothing but Newton. So, essentially we have 5 3.75 Newton it should also be equal to the external force because as the conveyor belt is moving at a constant rate constant velocity. So, this has to be equal to the external force.

Now in part b we have to calculate the force of friction now please understand this as the sand from this hopper heats the conveyor belt the sand does not heat any more. So, if the frictional force this can happen if the sand is not moving this can happen if and only if the frictional force which is exerted by the conveyor belt on sand is exactly balanced by the force that is acting in the horizontal x direction. So that means, frictional force f friction will exactly be equal to dp dt in this case which will be exactly equal to F external and this all these numbers are the same 3.75 Newton.

So that is a pretty straight forward problem and I hope you understood all the 3 4 problems we have discussed in this section, now let us move to the next section which is the rocket motion. So, what we have here what is special about rocket motion is so far we have discussed systems where the mass loss or mass gain is typically a 0 relative velocity. So, for example, when a rain drop is falling through cloud it is gathering mass at and the gathered mass has no velocity on it. If sand is leaving the balloon the sand and the balloon at the point of point when the sand leaves at the time, that particular sand particle and the balloon has the same velocity. So, the relative velocity between mass and the big mass and the mass which is lost where has been saved, but rocket is different in that sense.

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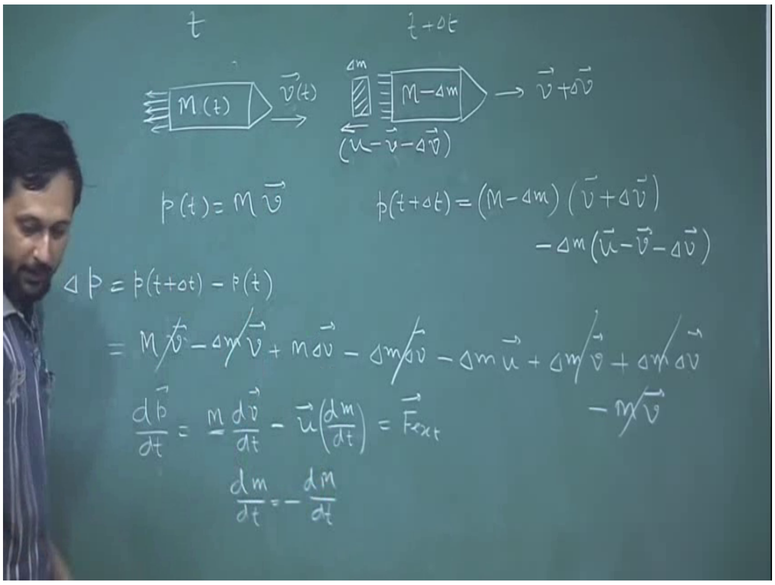

Because, what happens in the rocket is we have a full fuel tank which is filled with fuel and there is a combustion in engine which creates which burns this fuel creates some hot and yeah hot gas and if it exhausts the rocket in form of a jet, which has a very high relative velocity with respect to this rocket body. So, the situation here is slightly different. Now if we try to construct an equation of motion once again there is no standard rule for it we have to you know take this as a separate case and we try to calculate the momentum we try to once again try to construct the momentum for at time t and related time t plus delta t. So, let us see.

So, at time t we have a mass of rocket which is m t, and the rocket moves in this direction with the velocity v t. Rocket at time t t plus delta t let us assume that it has lost a mass or I will just use there is a reason why I will use capital M for this minus delta m and the velocity has gone from v to v plus delta v. Now what happens to this delta amount of mass the delta m of mass is exhausted. So, this is your delta m amount of mass that has been exhausted in form of a form of gas jet of gas and this leaves the rocket with a relative velocity u or with velocity u which is measured with respect to the frame of the rocket.

Please understand that frame of the rocket is an accelerating frame a rocket in general accelerates. So, when it accelerates it is not relativistic I mean it is not a inertial frame anymore. So, this velocity u which is measured which is given here is measured with respect to rocket frame. Now if we and the velocity v here which we which we are calculating I mean which we are writing here this is measured from a inertial frame; because please understand this the conservation of linear momentum and conservation of angular momentum we will see that angular momentum quite later on these are valid in an inertial frame.

This conserve 2 conservation laws are not valid in inertial frame say essentially when we are writing the expression for t of t and t of t plus delta t, we have to make sure that we are writing this expression with the velocities measured from an inertial frame. So, in this case p of t is simply m times v and p plus p at p plus delta t is m minus delta m times v plus delta v, but there should be an additional term here because this one it is moving in the opposite direction of v with the velocity u which is measured with respect to rocket and please remember that the rocket is moving with the velocity v in this direction.

So, the relative velocity at which the gas leaves the rocket body is given from an inertial frame as u minus delta v, you can also write u minus delta v minus u minus v minus delta v, but we will see that that will not affect the calculation that much for a simple reason that this will see in the moment ok.

We can also write this. So, and this is going in the opposite direction because this mass essentially there is an jet of excess gas which leaves the rocket body. Now if this is happening then this momentum has to subtracted from the initial momentum this total momentum and then we will have delta m times u minus v minus delta v. So, we compute delta p which is p t plus delta p minus p of t, if we go term by term we have m p minus delta m v plus m delta v, minus delta m delta v minus delta m u plus delta m v plus delta and delta v and then we have to subtract minus m v from it. So, we see that immediately this m v m v cancels out. So, this delta and delta v cancels out nicely and also yeah. So, this term delta m v delta m v cancels out leaving behind. So, delta p or rather I will simply write dp dt equal to M dv dt minus u d u dt which is equal to f external.

So, this is our equation we have got, but still there is a slight manipulation which we need to implement here, this dm dt is the rate at which exhaust gas has been going out of the rocket, but we want to write the final equation in terms of the rocket mass m.

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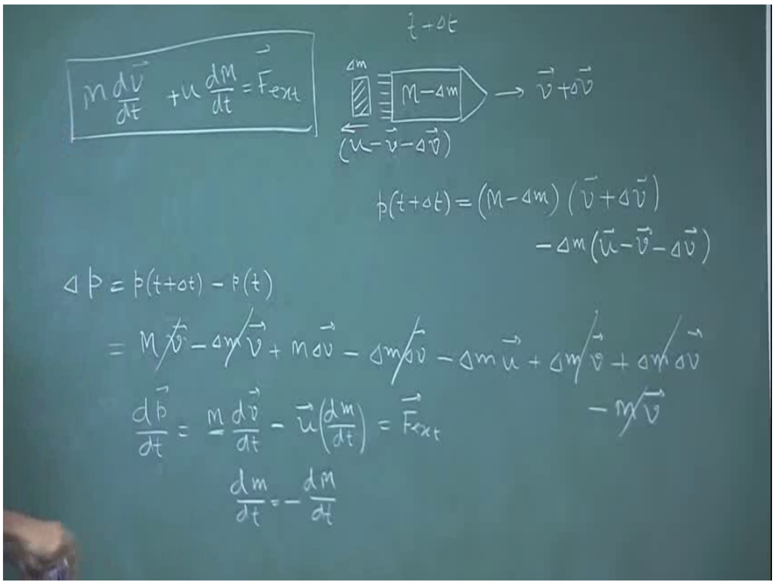

So, if we substitute dm dt is equal to minus dm d t, the final rocket equation becomes M dv dt equal to minus u dm dt or rather m plus u d b m s dt equal to F external. And here the m is the final is the rocket mass ok.