Classical Mechanics: From Newtonian to Lagrangian Formulation
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Lecture - 49
Lagrangian Formulation-7

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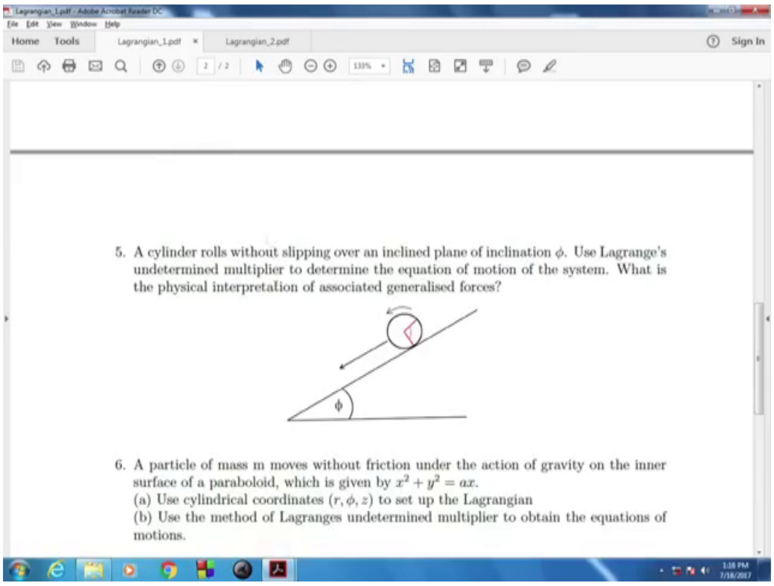

Hello and welcome back. So, we continue our discussion of this particular problems that. So, what I did actually here, so basically I moved this one problem which we will solve now in the last I mean end and then I have added two more problems specially on Lagrange's undetermined multiplier.

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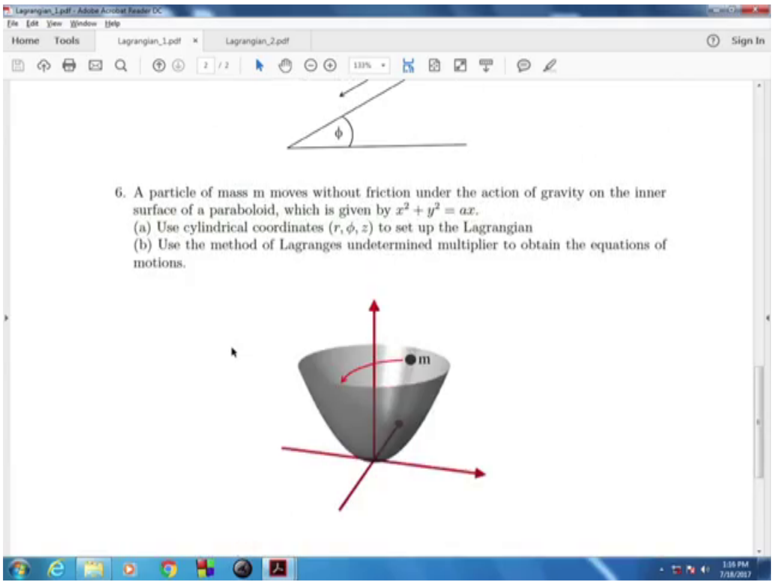

So, right now we have finished this four problems which are all there in the problem set we have to finish this two now and the last one I will leave it to you. So, let us start with problem number five which used to be problem number 4 in the last lecture whatever I have shown you, but now I have modified it. So, it is a problem when a cylinder is rolling without slipping from on an inclined plane. Now, what is the situation here?

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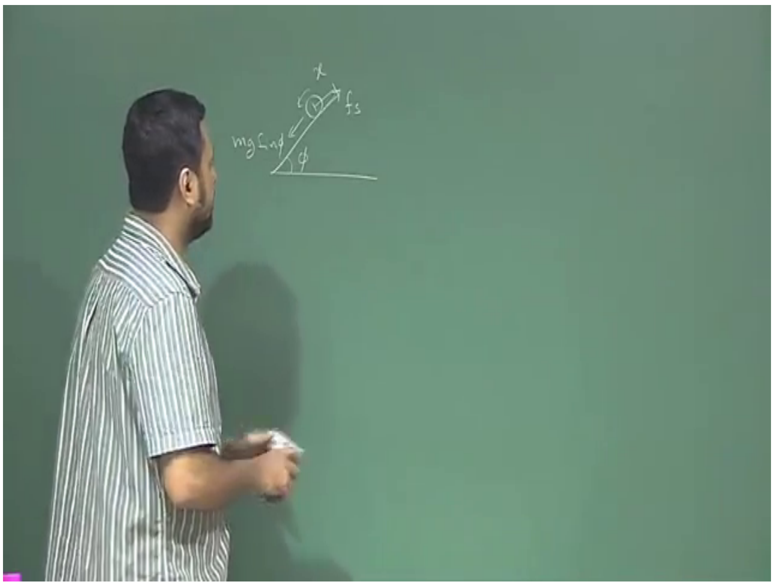

We have already discussed the constrain condition of this particular problem, but more importantly we have to understand what is happening physically, I have added a picture also you might have noticed. So, this is a inclined plane, this is a cylinder, this is rotating and coming down, so it is rolling without slipping. This happens if and only if there is a frictional force sufficient adequate frictional force, which is acting on the cylinder. So, typically if this is phi we know that the force. So, right now we are talking about the you know Newtonian picture of the thing. So, if the mass is m of this, so we have m g sin phi acting along this direction, and we have a frictional force which is given by f s which is acting in the opposite direction.

Without this frictional force in the absence of frictional force, it will just slide on it. So, there will be no practical difference between whether the mass is a cylinder or a block or a sphere or any other shape for example, it could be this bottle also I mean it does not really matter, because there is no friction it will all the masses which will be placed here it will come down. And only parameter we are will be interested in will be this displacement x from some origin, we have to set some origin here on the top we can also set it here, then it will be minus x nothing other than that. And we have to look at this one parameter, but then when, but now we are discussing a different situation in the in the current situation we have rolling without slipping.

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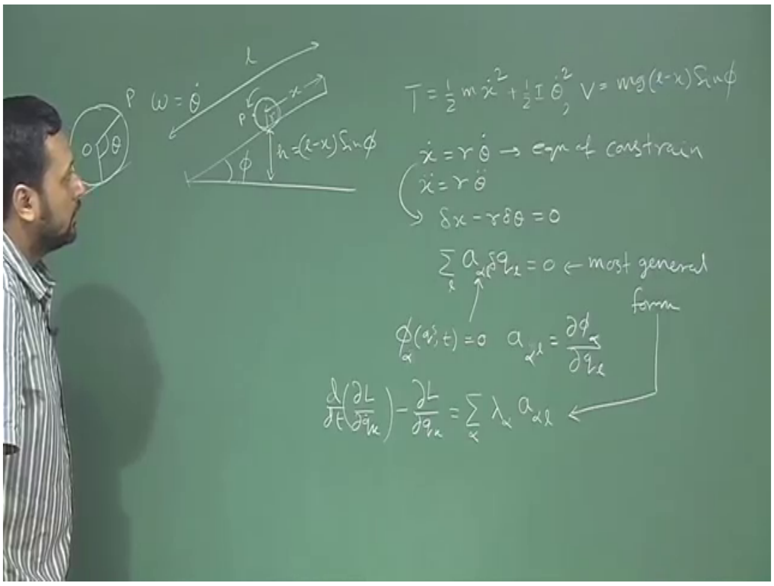

Now, when it rolls without slipping that is due to this frictional forces what we are going to do is we need to you know introduce a new parameter. So, this is my inclined plane some angle phi; this is my cylinder, so we are just looking at the cross section of the cylinder, so which has a radius r. So, one parameter again is this displacement x that is there, but to account for this rotation what happens when we are trying to write the Lagrangian of the system we have to also add a second parameter for this rotating term. So, the kinetic energy T will now be half m x dot square which is just the displacement plus half I omega square, where omega is the instantaneous speed of rotation; and I is the moment of inertia. So, we are familiar with this part. So, this is pure rotation, this is pure translation.

Now, in order to parameterize this rotation what we need to do is we have discussed it already, we have to take a point, any point p on the surface of this cylinder. So, I will just draw it little big here. So, let us say this is the cylinder, which is rolling on this plane. So, what we do is we you know take any point p or could be here could be here does not matter what we do is we measure the angle from the center of the cylinder let us call it o we measured this angle, but for any measurement of angle we need a reference line. So, our reference line is the point of contact which is not changing because it is you know rolling without slipping. So, the point of contact is we can take it as a stationery I mean frankly speaking it is not a stationery point each time different point on the cylinder is making contact here, but we can take this as a reference point, and we can measure this angle theta.

So, if this is the case then your omega will be equal to theta dot. So, what we can do is we can replace this one with theta dot square, but again because it is a system of one parameter it is essentially 1D problem. So, we need to have a relation between theta dot and theta and x and the relation is theta dot or rather x dot is equal to r theta dot. Also we can write we can I mean this r is a constant r is a radius, so it will be true for second derivative also, so we can write x double dot is equal to r theta double dot. So, this are all varied valid relation. So, now, we will I mean we can also write this, but we will take this as the constrain equation. So, this is will be the equation of constraint, and we can rectify this as we can rewrite this in differentiate I mean in virtual displacement form as delta x minus r delta theta is equal to 0.

So, we have in a general form of constrain equation is sum over lambda sum over l a lambda l q l equal to zero. So, this is the generalized form of a constrain equation, where for a holonomic system the constrain equations for strictly speaking for a holonomic system this are nothing but where the equation is given as phi of q’s and time equal to 0.

So, these are sorry this is delta sorry there is a delta q q l here. So, these are nothing but a k l will be or this will be or rather sorry lambda l. So, this is the lambdath constrain equation, it will be del phi lambda del q l, but this is in case of holonomic constrain, I am just giving you whatever we have done so far I am just giving you a recap of that.

So, and in this case the Lagrange's equation can be written as d d t of del l del q k dot minus del l del q k is equal to sum over lambda or yeah sum over just a minute yeah, so it will be sum over. So, basically we have to take this or oh I know by mistake I should write alpha here that makes little more sense yeah if I replace it with alpha right this is. So, lambda alpha yeah lambda alpha a alpha l, so sum over alpha right. Now this makes more and this we have discussed already this is the form. See for a constrain of holonomic in nature a l lambda ls can be represented by this, but in case if the constrain is not non holonomic also this is the general valid form most general form so we write this as the most general form of constrain equation. And eventually this is also the most general form of Lagrange equation with undetermined multiplier.

So, now, we are going to apply this equation in this particular case, also we have to write the potential energy equation. In this case it will be this you know the total length if from here to here, the length of the platform is l. So, it will be simply m g l minus x sin phi yes sin phi. So, the height will be so instantaneous height h will be l minus x sin phi yeah. So, this is now where this is how where we start.

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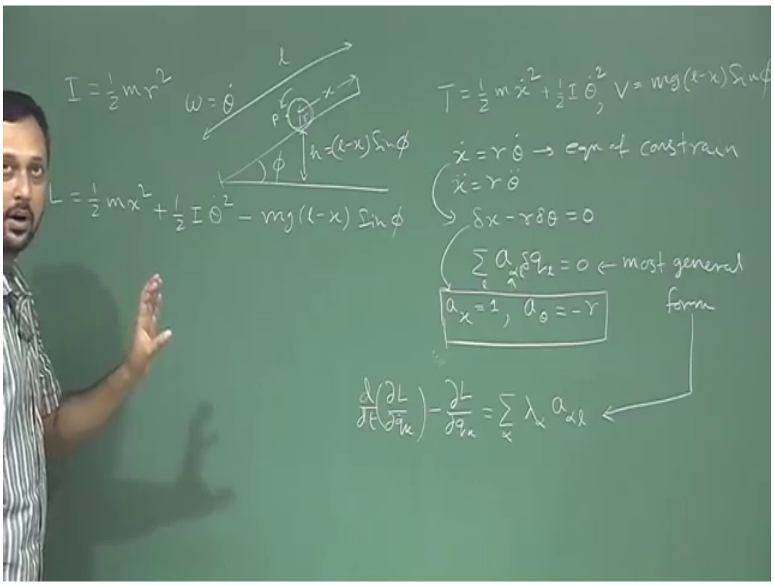

And then what we need to do is we need to follow the basically we need to construct the Lagrangian between, which will be t minus v which will be given by l is equal to half m x dot square plus half I theta dot square minus m g l minus x sin phi. Once again theta is a cyclic coordinate it does not appear explicitly in the Lagrangian, so only comes us theta dot. And also this I yeah I for a cylinder which is rolling around its axis, it will be half m r square that we have seen during our discussion of rigid dynamics. So, everything is there. So, all we need to do is we need to find out see there will be two differential equations to begin with one for x one for theta, but using this constrain relation we can get rid of one differential equation.

So, we will see in the moment that solving this problem actually we do not need to determine the Lagrangian multiplier alpha. So, we do not need this here because this is not a general holonomic problem, but what we can do is we can compare, so we have only one constrain equations, so there is only one alpha we need alpha equal to one all that time only one equation. So, there is no summation over alpha here because there is only one equation, but what we have here is we can calculate a x the parameter a corresponding to x which we see is equal to 1, and the parameter corresponding to r will be equal to minus r or rather theta sorry not r, theta.

So, we have the coefficients from this constrain condition from here we get this two values of a and r. So, now I will give you a demonstration on how to write the Lagrange equation. First, we have to write d dt of this equation for x. So, d dt of m x double dot simply speaking it will be m x dot square will be m x double dot minus, see for the second term we see if I eliminate the constant quantity m g l from this which will not contribute anyway, so it becomes plus m g x sin phi.

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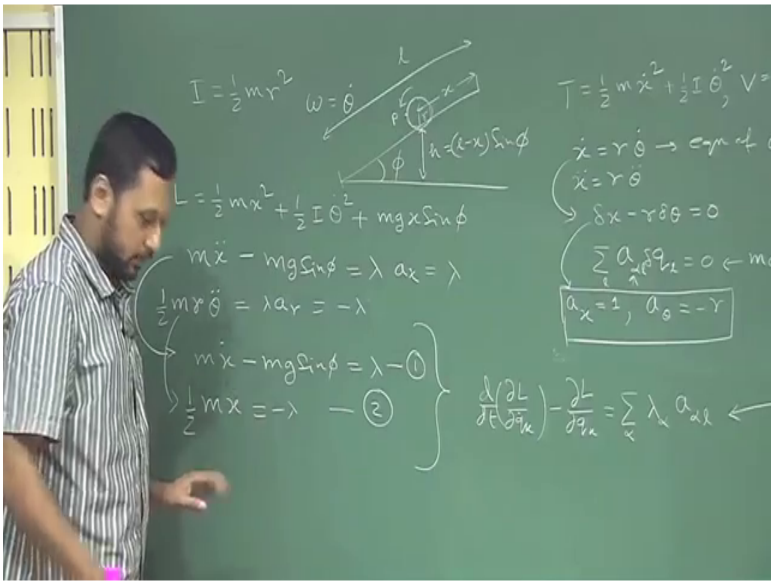

So, there will be 1 minus m g l sin phi, which is a constant term which will not just constant shift of overall you know energy, and we are just getting rid of this. So, if we do that the second term will be minus m g sin phi; and on the right hand side, we have to put lambda that is a Lagrangian undetermined multiplier times a x, a x is equal to one in this case. So, we just leave it like this. And for the second equation once again it will be 2. So, it will be I theta double dot, I theta double dot will be not equal to 0, but equal to lambda times. So, it will be basically lambda times a x; and this will be equal to lambda times a r. And we have seen that lambda times a x is equal to lambda, it will be equal to minus lambda r, clear.

Now, if you substitute for I, here it will be half m r square, so we cancel one r from both sides right. So, this gives you straight forward two relations, the first relation is m. So, from here m x double dot minus m g sin phi is equal to lambda; and the second relation from here is m. So, what we can do now is we can simply replace r theta double dot is equal to x dot; because at the end we need only one relation right I mean this is the equation of single parameter. So, m x r theta dot is equal to m x double dot half is equal to minus lambda. So, we have this two relations let us call them one and two and that is where we stand. So, one-way of just to solve the problem would be ok. So, I am getting rid of this I hope it is clear. So, one-way of solving the problem would be just eliminate lambda from this two equations which is very easy to do, but we can do is we can you know lambda is equal to minus of that.

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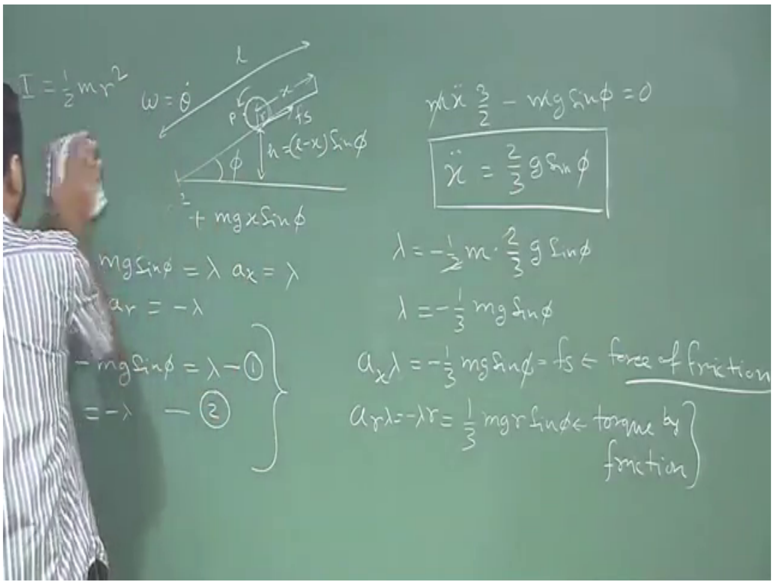

So, we can simply write m x double dot 3 by 2 minus m g sin phi equal to 0 using the second equation because we just put the value of lambda which will be equal to minus you know x double dot by 2 and change the side. So, it will be plus and m, m cancels, so we get x double dot is equal to two-third g sin phi. And this is our most familiar result just anyway let me verify because I have a tendency of you know doing things wrong oh great I have done it right. So, this is our very standard expression for I mean standard answer for this problem we all know that the acceleration in this will be two-third g sin phi.

Now you can ask me why we need Lagrange equation in that case I mean why we need Lagrange equation Lagrange multiplier, in that case we can just reduce everything in a single parameter equation, even the Lagrangian and solve. It in principle we can do that, but now we will see why we need that because now let us do one thing from the second relation lets determine the value of lambda. So, lambda will be minus half m x dot right x double dot just put two-third g sin phi, so that will give you lambda is equal to minus one-third m g sin phi. Now, the generalized equation what I wrote the right hand side, I have can be taken as generalized forces which will not be which was not possible to included to be included in the Lagrangian. In this case this generalized forces one of them has to be the force of friction.

So, we compute a x lambda which was right which was the right hand side which is the right hand side of this first equation a x lambda. Now, a x was one right, so a x lambda is exactly the same it would be one-third m g sin phi. And what is this, this is the force of friction which is acting in this particular direction. So, this is exactly equal to frictional force f s. So, this is force of friction sorry you cannot see force of friction. And then we have the second parameter which is we removed r from here cancelled r, but it will be lambda minus lambda r. So, when we write a r lambda which is minus lambda r, there is a minus sign already here. So, when I include this minus it is nothing but one-third m g r sin phi. What is this, this is the torque of this particular force which is causing this rotational motion. So, this is the torque by friction.

So, you see using Lagrange's multiplier when we determine lambda and put it back in to this equation in order to I mean relation in order to get the generalized forces we gain lot more insight. Then if we see just to get to this equation what we can do is we can reduce it to one parameter equation, and you know get one parameter problem and get the equation that is, but that is not the point here. We all know the solution how it works, but using Lagrange's multiplier what we can do is we can actually find out the forces of friction and torque which is causing this rotation to take place. So, it offers us something more than just getting in to the getting giving us the equation of motion right. So, this is what we have learned.

Now, we will go to the next problem. So, the next problem is particle moving inside particle of mass m moving inside the paraboloid. And the paraboloid equation is given as x square plus y square is equal to a x. Now, we have to first set up the Lagrangian using cylindrical coordinates systems, and then we have to use a method of Lagrangian undetermined multiplier to obtain the equation motion. So, here we are exactly frankly speaking we are not solving the problem here, but what we are going to do is we are going to use Lagrange's multiplier because once again it is a three parameter problem. We will in order to describe the motion completely we need three parameters x and y and z, but there is a constrain condition that the particle must move on the surface of this something a bowl like shape. So, there is an equation here, so it has to move on this surface, so that is the constrain condition.

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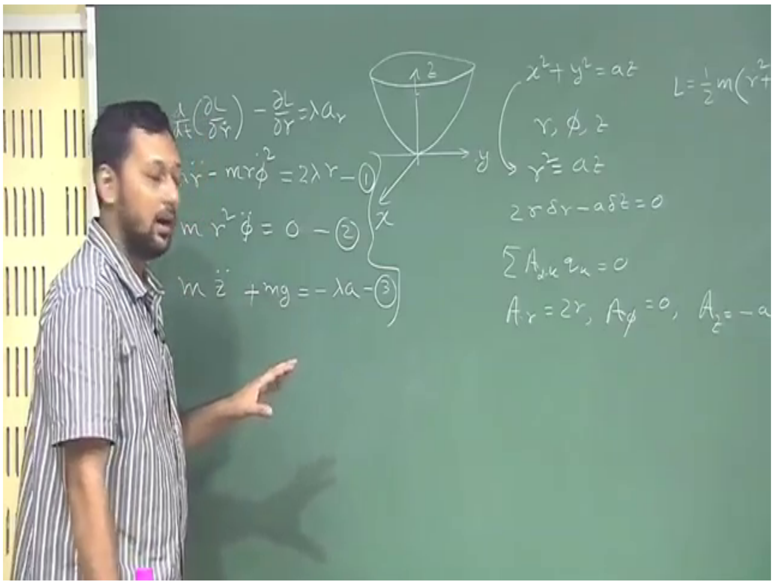

Now, how do we do that? First of all what is a paraboloid just in case if you do not know let us assume that we have x-axis here, and we have z-axis here. So, this is a parabola, we know how a parabola looks like it is a 2D object. Now, if I rotate this parabola it will form it will start around z-axis, it will form a surface right I mean surface of evolution of this parabola, this arc will be a surface and this surface is called a paraboloid which has the following shape forget about rotation now. So, this is our paraboloid. So, we have so in right handed coordinate system this is your x-axis, y-axis and z-axis.

So, first of all the equation of this paraboloid is x square plus y square is equal to a z. Now, we can when we are switching to polar coordinates system, cylindrical polar cylindrical polar is only the 3D extension of plane polar coordinates systems. So, the length elements are r I mean three parameters are r, phi and z. We can also write r theta z that is, but typically phi is more commonly used. Now, in this case x square plus y square will simply be equal to r square. So, the constrain condition can might as well be written as r square equal to a z. I am not writing the explicitly form of y how x and y and r related to r and phi it is exact same as we have already known in the plane polar coordinate system, only thing is there is an additional z which was not there. So, it will be r square equal to a z.

Now, if I reduce it now if I take you know I mean write it in virtual displacement from form. So, this is the constrain. So, we can write it as 2 r delta r minus a delta z is equal to zero in this form. So, in similar manner we see so they comparing with this a l k q k a alpha k q k equal to 0 comparing with this form we see that a r is equal to 2 r and a z is equal or a phi is equal to 0, there is no phi dependence into this equation. And a z is equal to minus a. To avoid confusion what we can do is we can let make it cap capital and A, A and A z very good. So, this is the constrains side.

Now, for the kinetic energy and potential energy kinetic energy t for this particle is simply half m. Now, for r theta or a you know plane polar coordinate system, we know that the general velocity expression is r dot square plus r square not theta dot we have to write phi dot square now it will be simply another z dot square term. And potential energy will simply be v is equal to minus m g z. So, we get the Lagrangian straight away. So, the Lagrangian will be (Refer Time: 24:52) you know just modify this a bit minus m g z. So, this is my Lagrangian.

So, the Lagrange's equations will be use this part of the board. So, it will be d d t of del l del sorry r dot minus del l del r will be equal to lambda a r. Once again there is only one equation of constrains. So, we need to use one lambda, there is no lambda I mean it is no not many to deal with. So, if we do that it will be m r double dot minus 2, so it will be m r phi dot square right equal to 2 r lambda or 2 lambda r. So, this will be equation one. Similarly, I will just write the equations just you know that to save time m I right yes good second. So, once again the phi is a cyclic coordinate, phi does not appear in Lagrangian explicitly. So, it will be m r square double dot or sorry not theta phi is equal to 0, this will be two. And the third equation will be m z double dot minus m g will be equal to minus lambda a right even in this m g no sorry this will be a plus sign here z double dot, how is it plus there is a yeah. So, there will be a plus sign here right m g is equal to minus lambda a right. So, this is my equation three.

So, right now we have to stop here because this is all which has been asked in this problem, I will probably will get more into your assignments, but this will be easy whatever you get once you have this equation derived, rest will be easy. So, we have to you know you can using this equations once again you can eliminate lambda you can calculate lambda equal to anything. Of course, you have to use the constrain equation this equation is there keep in mind. So, with this, we leave this problem, I mean we close this assign oh by the way there is a third problem or last problem, which I am not doing for you, you have to do it yourself.

Why I have add added this problem because please understand this Lagrangian multiplier although it started off as a tool in classical mechanics it is widely used. It is used in mathematics in its used in statistical calculus, it is used in economics you know. So, I will give you little few more examples although also you can use it in algebra for example.

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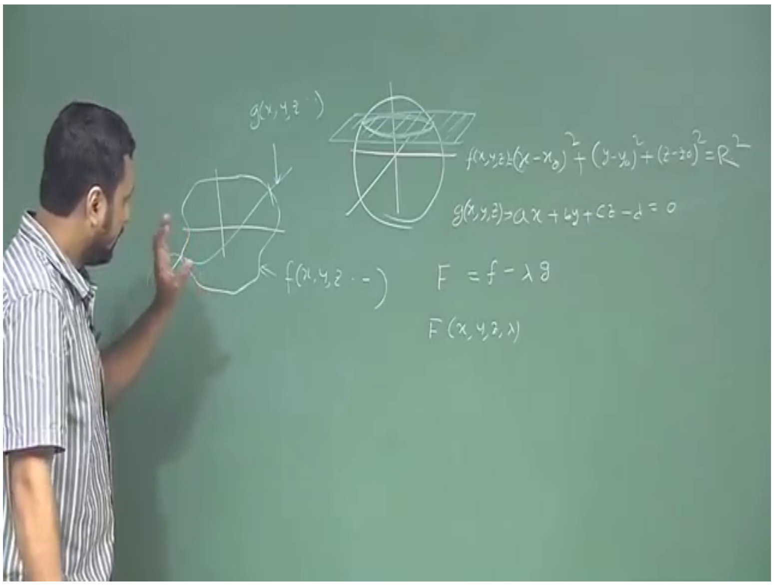

Let us assume that you know you have to find out the interception between a plane and a sphere or any 3D shape. Let us take it as a sphere. What is the situation we have in the in three-dimensional world as sphere is this right and a plane will be something like this. So, when they intercept, so the inter point of interception will be a circle. I hope you understand what I am trying to tell you the interception between a sphere and a plane has to be a circle. So, the problem is a geometric problem let say, but we can give it this following form we can write the general equation or rather I am not sure if it is you know the proper origin if it is 0. So, what will do is we can write x minus x 0 square plus y minus y 0 square plus z minus z 0 square is equal to s or r square. So, this is the equation of the sphere right and an equation of the plane is given by a x plus b y plus c z minus t is equal to 0. So, we can treat this function as f of x, y, z, and we can treat this function as g of x, y, z. So, we have to solve for solve this two equation simultaneously. Similar, a very similar thing is given in, so what is the way of doing it way is you have to include a Lagrange's multiplier into this problem.

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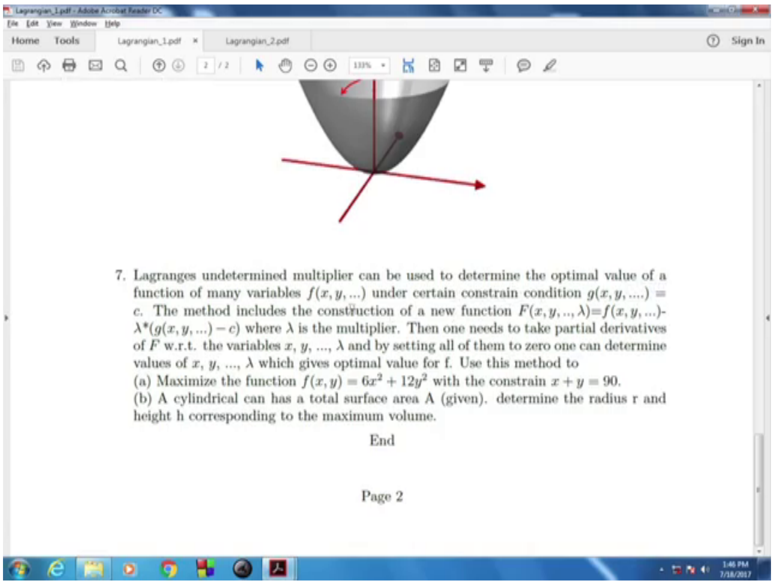

And the procedure actually is described here if you just read Lagrange's undetermined. So, the constrain condition in this particular the problem I have described in the geometric problems it is given as the pro the g whatever g is equal to c, this is the equation of the plane and variable the main equation is given as f x, y, z is the equation of the sphere. So, you have to construct a function capital F, which is this minus lambda times this by the way this star is the multiplication only. And then what you need to do is this f is a function of x, y, z and lambda as well. So, you have to take partial derivatives with respect to x and y and z and lambda. In order to gain optimal solution, if you want if you are looking for a optimal solution we have to optimize the b that big f function.

So, if we can optimize it with respect to all four parameters in this particular problem for example, we have only I mean we have three parameters x, y and z. The moment we construct this capital F which will be f minus lambda g we are actually including a new parameter. So, capital F will be a function of x, y, z and lambda. So, in order to get the interception which is this yellow circle to get to that particular solution in this particular case, we have to optimize the function capital F with respect to x and y and z and lambda both. And in order to optimize we have to put the partial derivatives equal to 0. So, this is a very standardized method I am pretty sure many of you are familiar with it, because nowadays this has taken place made its way in some of the school textbooks also. So, this is a technique called Lagrangian multiplier.

In general description, if I have a function space, I mean in you know in general space we have a function here, and we want to optimize the value of this function with respect to a constrain condition which is given by this. So, we have to take for example, this two interception points right I mean I am just writing it in 2D it could be a 3D contour also 3D contour and a plane. So, there could be the, so in order to get to this solution we have to simultaneously satisfy two conditions; one is this equation which is given by f of x, y, z this and another is this equation which is given by g of x, y, z. So, this is what we are we have to do mathematically speaking but if you think this is more complicated stick to this particular simple geometry description anyway.

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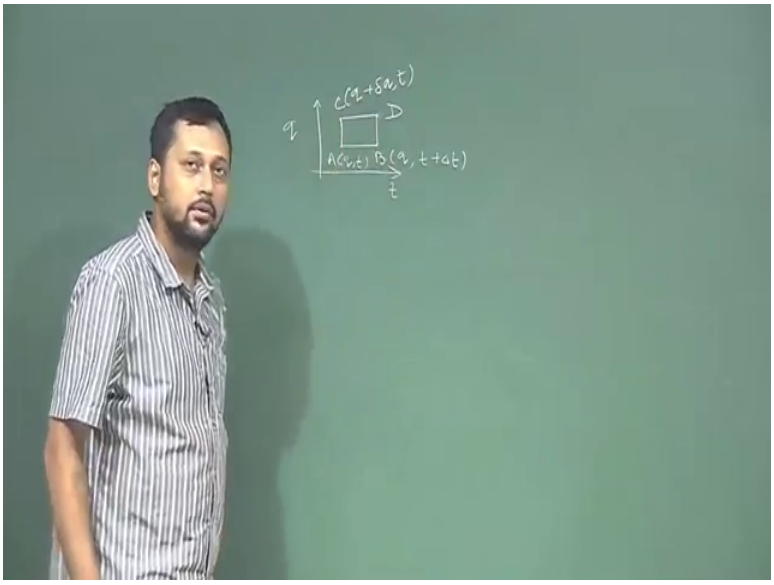

So, I am leaving this last problem up to you. And I will end this class by adding a I mean adding a very short description of a variation principle. Now, what is variation principle and how it is applicable, I will do that in the next I mean I will cover that in the next lecture, but before we go there very quickly let us assume that we have a q t q space. So, that means, in the x-axis, we have time which is the independent variable; and y-axis a system is moving if a system is moving it its coordinate change as a function of time we all know that.

Now, we have all the general I mean let say the system let us take a simple system which has only one generalized coordinate. Now, in order to you know track the trajectory I mean in order to draw that trajectory, what we can do is we can trace q as a function of t over a period of points. Now, let us take one starting point a, which is given by q t. Now, if I make a very small infinitesimally small displacement real displacement in time. So, this point b will have a coordinate of q t plus delta t. So, the idea is from going from a to b we have not travelled along the q-axis. So, the coordinate remains same we are just evolved little bit in time.

Similarly, if I, but now let us say if I want to represent a virtual displacement in the same diagram, let us say there is a virtual displacement please remember that virtual displacements during a virtual displacement there is no time elapse. So, a virtual displacement will be a vertical displacement, let us say this is point c and it is a coordinate to point c will be q plus delta q and t. So, in order to finish this diagram, I take the diagonally opposite point in this rectangle, which I call point d. What will be the coordinate of point d now, we have we know coordinate of point a, b and c. So, we will start from here in the next class, and we will build up what is called the calculus of variation.

Thank you.