Classical Mechanics: From Newtonian to Lagrangian Formulation
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Lecture - 48
Lagrangian Formulation-6

So, in this lecture, we will be focusing mostly on problems and we will have a very brief discussion on, if we have if we have some time towards the end we will have a discussion on calculus of variation.

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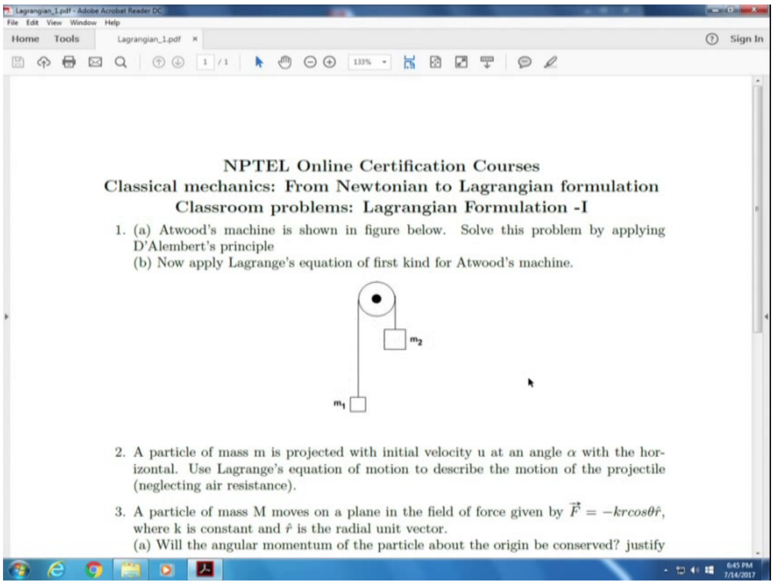

But for first let us start with the problem. So, I have as usual these small problems had prepared for the class we whichever we have covered. So, far in the class will be discussed in this problem set. So, will start with the simple problem of Atwood's machine and the question is we have to find out the equation of motion basically solved the problem means finding equation of motion first we have to use D Alembert's principle and then we have to use Lagrange's equation a first kind to solve this simple system. So, Atwood's machine is essentially, this 2 mass one pulley system which is very familiar to all of you and will look it up.

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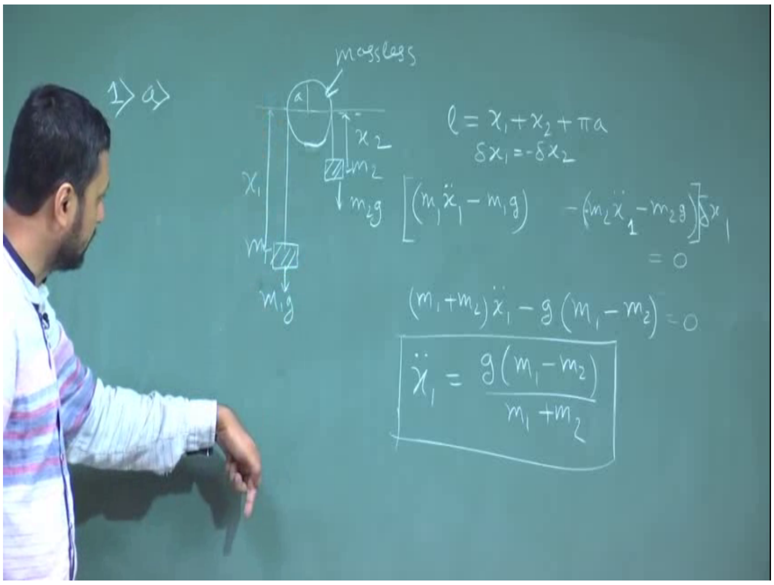

So, the first problems a we have the Atwood’s machine which is one mass m 1 the other mass m 2, right. Now first let us assume we can assume some radius a for this pulley. So, we can take the center there is a reference line at the center from where we measure all the heights this is the reference line. So, this is my displacement of the first mass, displacement of the second mass. What is the constraint here?

First of all let us identify the constraint the constraint is the length of this thread this is in at wood at wood machine it has been assumed, that there is no friction in the pulley we are very familiar with this frictionless pulley assumption right. So, this is one assumption the second assumption is the total length of this thread is constant. So, it is not a stretchable string. So, if we write down the constraint equation. So, L will be equal to x 1 plus x 2 plus pi times.

Now you can might as well just avoid writing pi time say because we will see towards the end that it will not contribute unless and until we are so also we are assuming that this is massless. If this is massless writing this pi a will not affect anything, if we consider that the system has mass then we have to take the angular momentum in to or that rotational energy of this in to account. Then this will have some effect, but anyway. Right now we are considering that there are only 2 masses in the system m 1 and m 2, and this is massless the pulley is massless string is also massless. So, in terms of D Alembert’s principle how do you write it D Alembert’s principle if you recall we have to get the forces the force on the first mass is m 1 g this is m 2 g. So, if we write it will be m x double x 1 double dot minus m 1 m 1, x 1 double dot minus m 1 g delta x 1 plus m 2 x 2 double dot minus m 2 g delta x 2 which will be equal to 0.

So, this is precisely our D Alembert's principle one equation single equation and as I said already and you also must have realized that this single equation is useless unless and until we use the constraint equation. So, basically it has it we cannot do much from here. Now from the constraint equation we can immediately just take a virtual displacement and say delta x 1 is equal to minus delta x 2 is not it definitely, it is it has to be I mean delta x 1 plus delta x 2 equal to 0 take a derivative of this. And it will get come out substitute it here. So, what do we need to do is we need to put a minus sign here, you can get rid of this delta x 1 put a minus sign put a bracket put this delta x 1 outside that is it.

So, you use this relation when you get this you start when you get it in one, ok there is a delta x 2 also which also has to be merged with delta x 1. So, this is the case if this is the case then the double dot will also be replaced has to be replaced no. So, you do that. So, you get m 1 plus m 2 x 1 double dot minus g (Refer Time: 05:39) take it m 1. So, it will be minus plus minus delta x 1 delta 0, right and now because there is only one coefficient of delta x 1. So, the whole thing can be equated to 0. So, we can get rid of this delta x 1 as well right. So, we can put this equal to 0, which gives you x 1 double dot is equal to g m 1 minus m 2 divided by m 1 plus m 2 that is it. And this is your very familiar form if you have a simple pulley air mass system this will be the acceleration.

So, if one acceleration is and this symbol means there is a minus sign between del, I mean derivative of x 1 and derivative of x 2. That essentially means if the velocity one mass is dominating then it will come down this way if mass 2 is dominating then it will come down this way. So, this is all included in this calculation in the final form. So, you just have to put the values of m 1 and m 2 and then you will get whichever way whether x 1 is positive or not. So, this is one way of solving it. The same equation we can solve by. So, this was by this; what you call at D Alembert's principle. Now the same equation has to be solved by Lagrange's undetermined sorry Lagrange's equation of first kind, which is like including this constraint condition explicitly and including this constraint means once again the constraint can be written as delta x 1 is or plus delta x 2 is equal to 0.

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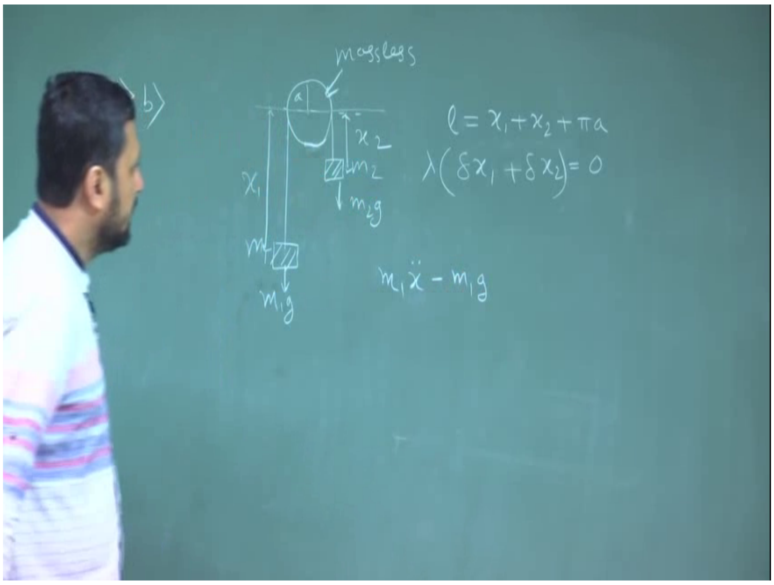

So, we have to multiply a lambda for each of this effort with this constraint condition. So, it will be lambda delta x 1 plus delta x 2, and then we have to separate I mean we can separate it out in to delta x I mean variation in x 1 and variation in x 2 form. So, if we do that we will get m 1 x double dot minus m 1 g. And it will be oh sorry you do not have to do this, you do not have to do this. So, your phi; so, your phi is x 1 plus x 2 plus pi a minus L equal to 0. So, this is your equation. So, what you need to do is. So, what you need to do is lambda del.

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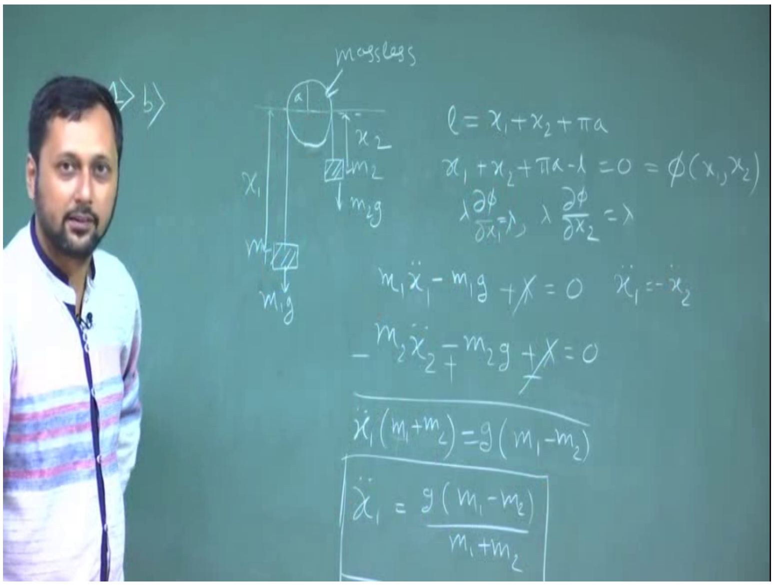

So, this essentially this is your phi of x 1 and x 2. So, it will be del phi del x 1 and lambda del phi del x 2. Now it so happened that these are all for both are first ordered. So, this is simply equal to lambda and this is also simply equal to lambda right. So, you get one and you have m 2 x 2 double dot minus m 2 g plus lambda equal to 0.

So, once again we got now we have 2 separate equations, in 2 parameters, but again we have to do some trick the trick is now we you know subtract one from the other. So, once we do that we get rid of this lambdas lambda cancel out. So, what we have we have and once again from the constraint condition, we see that x 1 double dot is equal to minus x 2 double dot because if we take a derivative of this it will be the just opposite. So, if we do that once again we get x 1 double dot is equal to m 1 plus m 2 and on this side of course, we will have m 1 or rather minus m 1 minus m 2. So, it put equal sign instead of writing it0. So, once again you get x 1 double dot is equal to g 1 minus m 2 by m 1 plus m 2. So, it is little simpler as expected compared to D Alembert’s principle; it is little more simpler we get 2 separate equations simple algebraic manipulation. And of course, we have to keep in mind here also we need to use that x 1 double dot is equal to minus x 2 double dot.

And then finally, we got this equation. So, I just took a very simple example of Atwood’s machine in order to demonstrate how this 2 techniques work, but please remember both this techniques are bit primitive in nature in a sense that they really need serious algebraic operations to go to the final equation also, we are we are good here because we have only one constraint condition have you did if we have more than one constraint condition it will be a not so easy job, right. So, this is the first problem we did, but again this is a good practice I mean Atwood’s machine is a very good model system to study very simple kinetics it is good to learn, that this techniques anyway I will I promise to give you a little more I mean a slightly very variation I mean slightly different problems in your assignments and I hope you will enjoy them.

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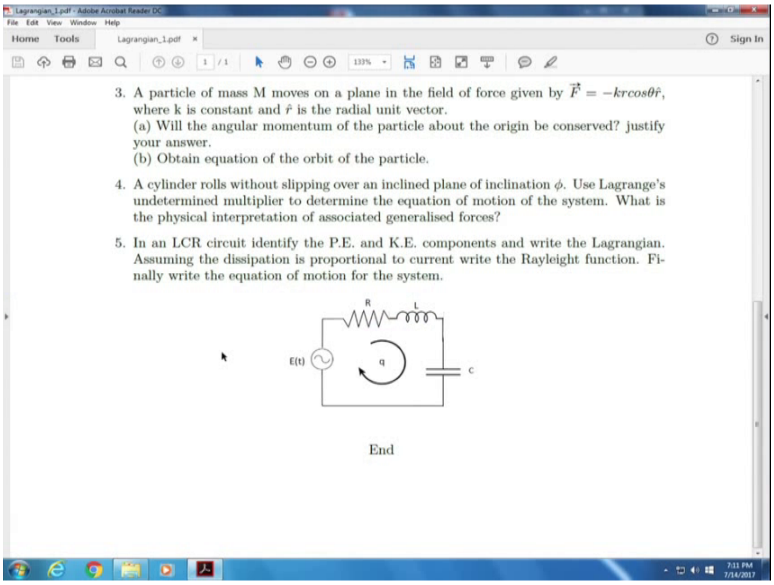

So, let us move to the next problem a particle of mass m is projected with initial velocity u at an angle alpha, with the horizontal use Lagrange's equation of motion to describe the motion of the projectile neglecting air resistance. So, essentially we are just we what we need to do is we need to simply write the Lagrangian of a system of a projectile and see if we can get to the equation of motion. So, it is very straightforward. Now we have finished Lagrange's equation of first kind we have finished Lagrange's equation, I mean D Alembert's principle and Lagrangian equation of first kind.

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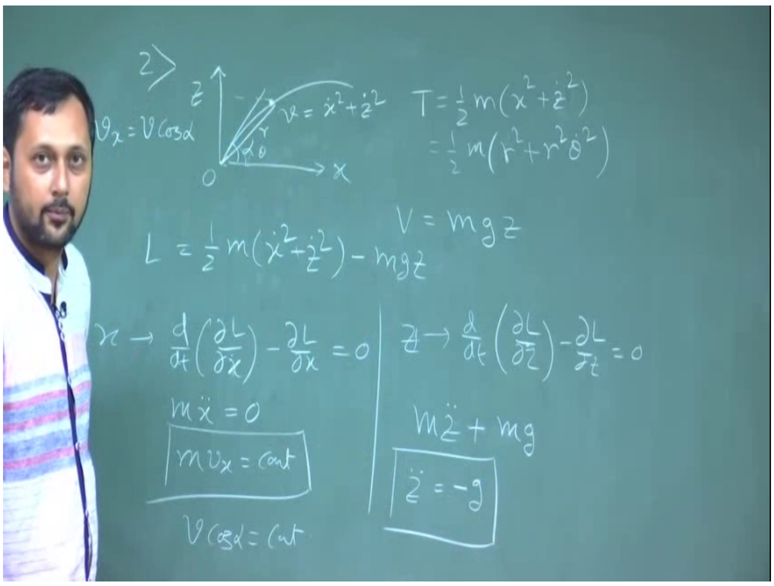

Now, it is the Euler Lagrange equation of Lagrangian equation of second kind. So, a projectile which is fired from ground. So, this is my x axis and this is my z axis which is fired from ground with an initial angle. So, maybe it is going yeah something like this. So, the initial angle was alpha and at any point the velocity v is given by x dot square plus y dot square or rather z, dot square.

How many how many parameters do you need generalized parameters, we need 2 generalized parameter because a free particle as I discussed in the last class. One of the previous lectures is a free particle is a needs 3 degrees of freedom the moment, we are projecting this particle as a projectile we are throwing this particle in the air as a projectile we are making it stay in this y z plane do we have to realize that. So, it is not a free particle anymore it has certain constraint. Now thankfully we do not have to take this constraint in to account because we are dealing with Lagrangian. Now, but we have to choose the generalized parameters. So, we need 2 generalized parameters which we have to choose wisely we can; might as well take r theta, we can take this as the origin o any arbitrary position this is r. And we can take this theta this is also a very valid choice of generalized coordinate I suggest we.

So, let us do it this way I will do it with x z and x and z or Cartesian, you will take it take a polar coordinate do it yourself it will be a good practice. So, kinetic energy t is given by half m v square which will be x dot square plus z dot z dot square. Now if I write the same thing in polar coordinate what will that be equal to half m r dot square plus r square theta dot square isn't it so we can write this we can do that I suggest you build up on this one right and v which is the potential energy is given by m g y right clear. So, the total Lagrangian L is half m x or sorry not y, z, x dot square plus z dot square minus m g z any cyclic coordinate yes x there is no x explicitly in this expression.

So, z is not a cyclic coordinate, but x is a cyclic coordinate typically kinetic energy will see that in most of most common case is kinetic energy is has only velocities. So, we will have, for example, here we have x dots and z dots very rarely we will see that kinetic energy term contains at least in this particular the going the next few lectures, which whichever example we will take you will hardly ever see that kinetic energy is a function of coordinate also. Of course, there will be cases when will be discussing small oscillation we will be seeing cases where kinetic energy can be potentially function of coordinate, but right.

Now let us not do it so this is one and we have. So, typically the coordinates not the velocities, but generalized coordinate come in the potential energy part now in this potential energy part you do not see a coordinate most likely it is a cyclic one. Of course, it is a close inspection I mean why I am saying this because sometimes the Lagrangian of a system could be. So, big that it is might it you might miss one or 2 terms in between. So, if you have to check for cyclic coordinate my suggestion would be check the potential energy put fast first.

So, equation of motion will be damn simple d d t of del l. So, there will be 2 equations one for x and 1. So, let us start with the x equation first. So, del L del x dot minus del L del x equal to 0. So, there is no dissipation forces present we have not included any resistance. Right now we have a 0 here. So, if we do that we have m x double dot is equal to 0 cyclic coordinate or rather a yeah m x double dot is equal to 0, which means m v x equal to a constant which is obvious because in projectile the x velocity has a constant value and from the starting point itself, we know that this will be if your initial velocity is v then v x will be v cos alpha isn't it so we immediately see that v cos alpha is equal to cos delta, right.

And then we have potential energy right sorry for the second equation y, it will be d d t of del L del sorry not y again, sorry, z z dot minus del L del z equal to 0. So, it will be m z double dot it will be equal to or rather minus m g. So, it will be a plus sign here because there is a minus here there is a minus here. So, it would be a plus sign here. So, which means z double dot is equal to minus g, which is absolutely fine this is how the equation of projectile looks like.

So, initial angle if you only if we want to solve this, we will need to use explicitly the values of the have an initial angles and all otherwise just to get to the equation this is this few one or 2 steps are good enough right now. So, second problem is solved let us move to third problem, third problem is a particle of mass m moves in on a plane in the field of force given by F is equal to minus k r cos theta r cap where k is a constant and r cap is a radial unit vector will the angular momentum of the particle about the origin, we conserved justify your answer and then we have to obtain the quantity equation of orbit for this particle.

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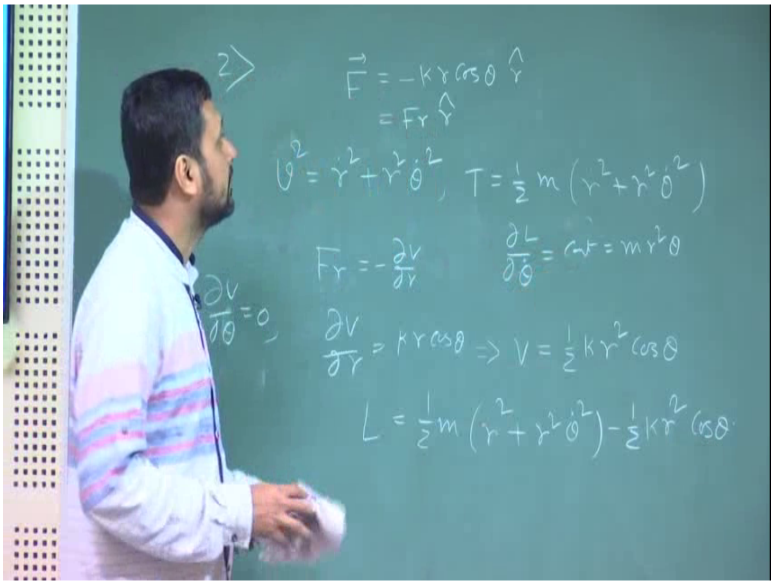

So, we have given we are given with a force field F which is minus k r cos theta r cap. So, we can write this as F r or minus F r. Right now we have to construct and it is a 2 d motion which is already specified it is a motion in a plane it is already specified in the problem. So, we can write velocity v or rather the magnitude of velocity v square to be equal to r dot square plus r square theta dot square, right. So, yeah. So, kinetic energy t is equal to half m r dot square plus r square, double dot square by the way in the previous problem; if we have a dissipation. For example, of viscous damping it will be a good practice if you can just add Rayleigh function with the Lagrangian and take the; I mean write the final equation that will be a good practice for you. Please do that here. Now we have to write v. What is v that is the question?

V is the potential energy and please remember that F is del v del r, F r is equal to minus del v del r. So, I will just write it here. So, F r is equal to minus del v del r which is given by if we you know just write the expression of f, which will give us or rather this is F r not minus. So, the minus sign will be included in here. So, we will get k r cos theta right. Now because it is a partial derivative the form of v can be predicted directly and please remember that total force is given in terms of only one component.

So, there is no so, we can assume that del v del theta is equal to 0, which is a valid assumption because there is no force in the theta direction right. So, if we do that if we integrate this, we will get half k r square cos theta right. So, v is equal to half k r square cos theta very good now; that means, the Lagrangian L will be half m r dot square plus r square theta dot square minus half k r square cos theta.

So, this is the lagrangian. Now we have to check for what was it, yeah will the angular momentum be conserved. So, the force is pretty much like a central force whatever force field is given it is also a radial force. So, in a radial force we have seen that in central orbit that for a for a force which has kind of similar nature. The angular momentum vector has been constant; will it be the case here the answer is no. Because if you if you recall angular momentum was the total angular momentum was that was conserved looking at the lagrangian. I mean from a Lagrangian perspective we what we have seen that because l. So, yeah because Lagrangian is does not contain theta explicitly. So, what we had was del l, del theta dot is equal to a constant which was exactly m r square theta dot for a central orbit.

But in this case the Lagrangian has a function of theta. So, in this case del l, del theta dot will not be a constant. So, that way this force field whichever is given although it is radial in nature. Because of this theta dependence over here it is not a true central force. Please keep that in mind that it I mean a force could be radial, it does not matter a force could be radial, but it is not as true central force unless and until your; I mean the total canonical angular can canonical momentum associate associated with the angle theta is a constant which is which is not the case here. So, we can now we have justified part a.

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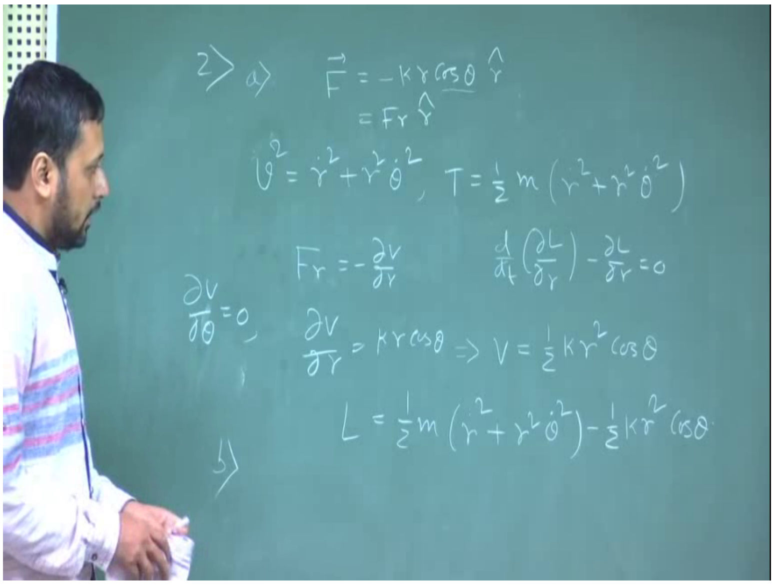

So, part a is over. Now for part b we have to find out the equation of motions equation of motion there will be one r equation which will be. So, you have to follow the same routine once again d d t of. So, d d t of del L del r dot minus del L del r is equal to 0. Once again there is no dissipation present here. So, it will be 0 and there is no another additional constraint forces. So, it is all included here and the second equation will be similar equation for theta. So, I am not doing it you can finish it yourself and it is it is pretty straightforward fine.

So, problem number 3, is sorry it was problem number 3 not 2 problems number 3 is over now 4. So, we have a problem 4, 1 problem 5 let us start with problem 5. Now in the during our discussion on generalized coordinate, we have say I have said many times that generalized coordinates need not have the dimensions of coordinate it need not be lengths need not be angels it could be anything I mean it could be any other quantity. So, this is an example where we are we are we have given with an LCR circuit, which is something which is very common for us and we identify. So, what we need to do is we need to write a Lagrangian for this system also we have to find out dissipation assuming that the dissipation is proportional to current we have to write down the Rayleigh function. First of all, before we do anything, we have to identify a generalized coordinate associated with the system.

Now, it turns out that charge electric charge which is moving through this circuit, can very well satisfy the criteria for a generalized coordinate. Now let us look at this problem. So, we are right.

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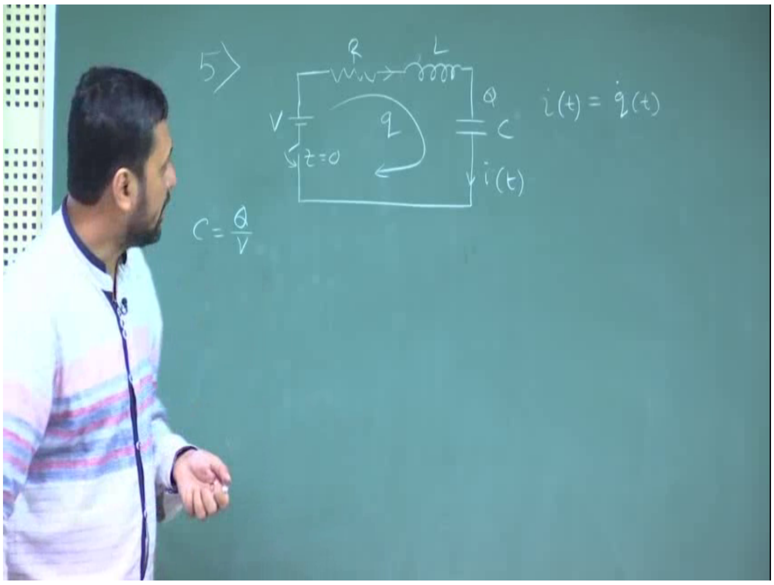

Now, we are doing problem 5, we have system of r and L and C present. And there is a current moving. So, the current in the system let us call it i which is a function of time. So, it is changing with time and there is a drive E which is also a function of time. This thing we have not discussed. So, far I will tell you now I will tell you how to have if we have a driving force on top of everything. For example, if we have a forced harmonic oscillation where there is a driving force constantly driving periodic driving force which is constantly driving a system how to take that in to account I will discuss this now.

So, now first thing is we have to identify the sources of kinetic energy, I mean the terms corresponding to kinetic energy and potential energy. Now if you look carefully we are probably we are all familiar with the basic operations of this component. What happens moment you close the circuit for, now let us assume that there is no driving here what we have maybe there is was a battery. And the there was a switch and the switch is closed at time t equal to 0. As and when the switch is closed there will be a charge flowing in this circuit. And that is why charge q has been taken as the generalized coordinate. Because it is the most fundamental quantity we can we can write I t as q dot. So, I the current serves the purpose of a sorry, here i is the generalized velocity in this case and q is the generalized coordinate.

Now, as the charge goes through this circuit what happens is the capacitor charges charged up, I mean it will accumulate a charge difference some finite charge different let us call it q. So, the; you we are all familiar with the relation that C is equal to q by v. So, whatever the voltage difference v here that according to that certain charge q will be accumulated, but that is the final step the steady state solution. Now in between what happens is this before you I mean before you reach a final value this q will also be a function of time. Now as the charge buildup there is a potential energy building up in to the system. So, capacitance capacitor is the source of potential energy.

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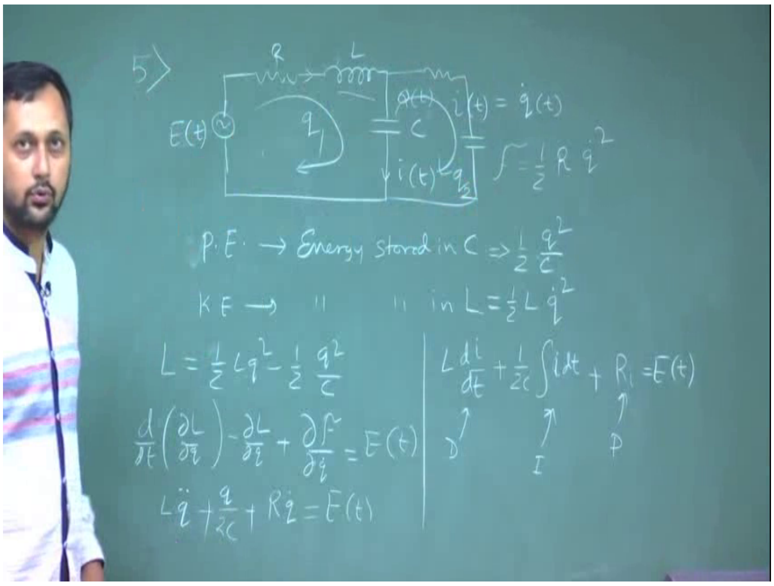

So, whatever energy is stored in the capacitor is potential, I mean is like a potential energy stored in C. Now on the other hand L what is the what is the effect of inductance in a circuit, where there is either no current or sorry where is a steady current, where there is a steady current flowing the answer is 0. There is no effect I can write out the different terms of I can you know the frequent I mean the fundamental frequency dependent terms for this each of these. And I will try to show you, but that is not strictly needed even if you do not want to go in to all this.

The basic concept is very simple an inductor will only participate, I mean inductor will acquire energy if and only if there is a charge flowing through it which is changing as a function of temperature or a current flowing through it which is functioning as a function of temperature right. If there is nothing flowing nothing will happen for a DC circuit a steady state, DC circuit and inductor is nothing what whatever register. It has register resistance it has only that has some effect otherwise it will not change, but in an s C circuit because the current is a time varying function the inductor has a finite effect.

So, because it is related to the dynamism of the system dynamism as in when the current is changing as a function of time only then this comes in to account. So, kinetic energy is the energy stored in L. And it so happened that energy stored in C is half CV square and this is half L I square. Where L is the sorry, i is the q. So, we what we can write is instead of I we can write q dot square half C. And for the first term sorry it is I have CV square yeah. So, now, what we can do is we can bring it in form of charge and we can write half. Q square by C right. So, we have a potential energy which is equivalent to half q square by C and a kinetic energy which is proportional to which is equal to half L q dot square. So the total Lagrangian L will be half L q dot square minus half q square. Let us see, but that is not all we have a Rayleigh function and there is a hint given that you have to take you know assuming that the dissipation is proportional to the current right a Rayleigh function.

So, the dissipation is proportional to current means it is like velocity dependent potential velocity dependent viscous damping. So, we might as well write the Rayleigh function as, I write it here F equal to half r i square. And i is nothing, but q dot. So, it is half r q dot square so for the so for the equation of motion. We have only one generalized coordinate and we have a Rayleigh function a lagrangian. So, for the equation of motion we have to do d t of del L del cube q dot, minus del L del q plus del F del q dot is equal to 0 right. For this particular system and if we do that we will find for the first 2 terms it will be L q double dot right minus 1 by 2 C q minus plus r q dot r q dot right. r i which will be equal to 0. Now this is when we have a battery in the system and the switch is closed at t equal to 0, but if we have a driving force in the system even after you know driving force is something or rather E driving potential this is something which cannot be included in in the Lagrangian at any form.

But what we can do is, see when we are writing this equation of motion we are bringing it down this is in the dimension of energy and these are in the dimension of forces. So, what we can do is to I mean we can if we have some force which cannot be considered in there as a force of constraint, which cannot be included in the external forces or rather sorry rather which is an external force which cannot be included in the force of constraints or dissipation forces. What we do is we put it on the right hand side of this equation.

So, your final equation is just like this. Now if we instead of writing q dot if we write I how does this equation look like this equation will look like, L i dot or rather I will just write it in long rotation notation just to make it look little familiar. So, it will be L d i d t minus oh sorry it will be already a minus here. So, it will be a plus sign here. So, it will be plus one by 2 C integral i d p plus r I equal to E of t. Now if any of you are familiar with PID control, this is the proportional term this is the integral term and this is the derivative term. So, this 3. So, basically what we have written is the PID equation of an LCR circuit here.

So, I hope it makes sense in order to also if we have more complicated circuits, if we have one more compartment here. For example, we have you know additional register and this we can assume that there is another. So, let us call it q 1 there will be another charge q 2 circulating in that loop. And we can might as well write an equation for that. Also you have to keep in mind that q 1 and q 2 are not entirely independent, but they are related by Kirchhoff's current law and voltage law.

We well we can we can find examples of such things in different literatures which I have mentioned also. You can find in the internet and you will definitely find this thing. So, I think today we will stop here. In the next class we will start with last problem that is problem number 4 of the set which is once again a problem which we have partially discussed in the class, but now we are running out of time currently. So, what I will do is we will start with problem number 4 in the next class. And then we will continue our discussion and will start a new topic called calculus of variation.

Thank you.