Classical Mechanics: From Newtonian to Lagrangian Formulation
Prof. Debmalya Banerjee
Department of Physics
Indian Institute of Technology, Kharagpur

Lecture - 47
Lagrangian Formulation - 5

Hello and welcome back.

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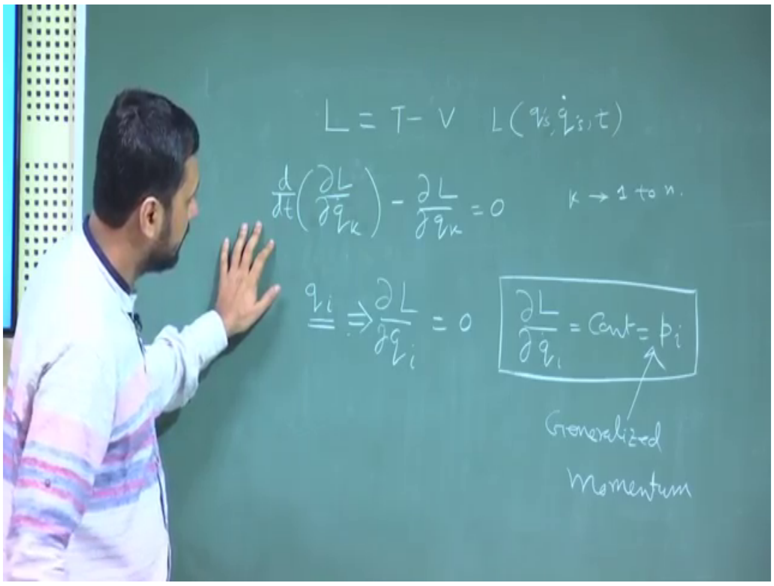

So, we in the last class we derived this Lagrangian which is T minus V for a system, and then we got series of equations which was of the form d dt of del L del q k dot minus del L del q k equal to 0. So this means; the Lagrangian L in general is a function of q q dot and time. I should write function of q's q dots and time. And this is a general form I mean this is 1 form of Lagrange's equation and there will be n number of such equations. So, the index k actually runs from 1 to n, so there will be n such equations n being the number of independent parameters required so that is 3 n minus k. That means, the number of degrees of freedom of the; I mean available degree of freedom of the system that is degree of freedom minus constraint. So we have such k number of equations.

Now, we at the end if you recall we I said something like, if Lagrangian does not include one of the coordinate explicitly that so we have a situation. For example, if I out of this different values different q q k's. If I take one particular q i which is not included in the Lagrangian explicitly; the example I gave was when we wrote the Lagrangian for the central orbit it did not include theta. So theta was included in terms of I mean theta dot was a part of the Lagrangian, but theta was not a explicitly included in to the system.

So, in that case what happens is if that particular parameter in this case q i is not explicitly involved in to the system; sorry I mean explicitly present in the Lagrangian then this gives you del L del q i is equal to 0, for this particular Lagrangian I mean sorry for this particular coordinate q i and this leads to the fact that del L del q i dot is equal to a constant. And this constant is called and this quantity del L del q i is corresponding generalized momentum. So far we have defined generalized coordinates we have defined generalized velocities we have defined generalized forces.

So, no harm in defining a new generalized parameter which is the generalized momentum which is equal to this so these terms are essentially a generalized momentum. So, now, it makes little more sense because if this is a first just a minute p I right. So, this is the derivative of the time derivative of the generalized momentum just like we have in Newtonian mechanics, Newtonian equation of motion and then we have something which is on the other side right.

So, exactly I mean something very similar to what we have in Newtonian mechanics where we have I mean rather p dot is equal to F an equation of this form right. So, it is somehow not very different and as I am saying it over and over again that this is just a another way of getting this type of equation which will sorry Lagrangian so far what we have learned is another way of getting same set of equation in a slightly different way. Now we can have situations where we have frictional forces present or rather dissipative forces present in the system and sometimes we can have a situation in which we do not know. We cannot really remove the frictional forces a priori especially it happens especially for dissipative forces.

So, now, we will learn how to tackle these 2 types of situation using this Lagrangian formulation right. So, the first one we will take up the relatively easy case I mean both are not very difficult, but we will just out of that two, I would say the first 1 is this case of dissipative force sorry case of what you call the viscous damping for example. So, in viscous damping what we have seen so far is the force is a function of velocity the viscous force.

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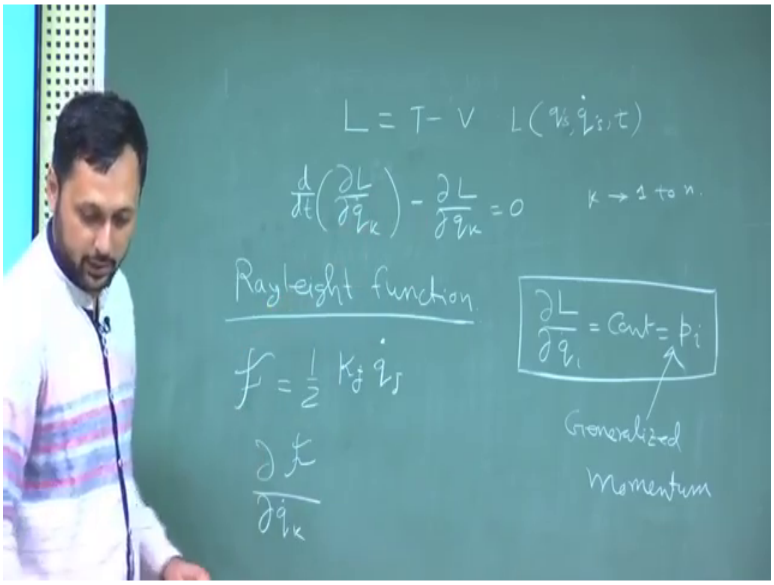

If you remember the viscous drag was written has a general form minus k v, so minus means it is working opposite to the velocity v and k is the co coefficient of the viscous drag and it is somehow related to the viscous forces present viscosity of the system and other forces present or other effects related to viscosity that we have seen. Now this thing this is a force so this is the viscous force or drag force we can write it as f d which is this.

In Lagrangian, if you see that Lagrangian has a dimension of energy now question is can we define an energy an define a function which has a dimension of a energy so that can be added to the Lagrangian the answer is yes we can for specially for forces which are or dissipative forces which are dependent on velocity and example would be a good example would be viscous force we can do that we can write something called a Rayleigh function.

Rayleigh function which has the general form it is written in using this curly f equal to half k or rather k j q dot j right. In this case it will be simply k times v q dot is equal to this is a q dot here. So, in the case of viscous drag it will be simply half k v square right. So, if we add this with the Lagrangian we can do that we can either add it to the Lagrangian or we can take derivative or a gradient of this function as a with respect to velocity which will have this form and we can put this on the right hand side of this equation. So my equation which was so keep this generalized momentum in mind we will come back to this in a moment here we are.

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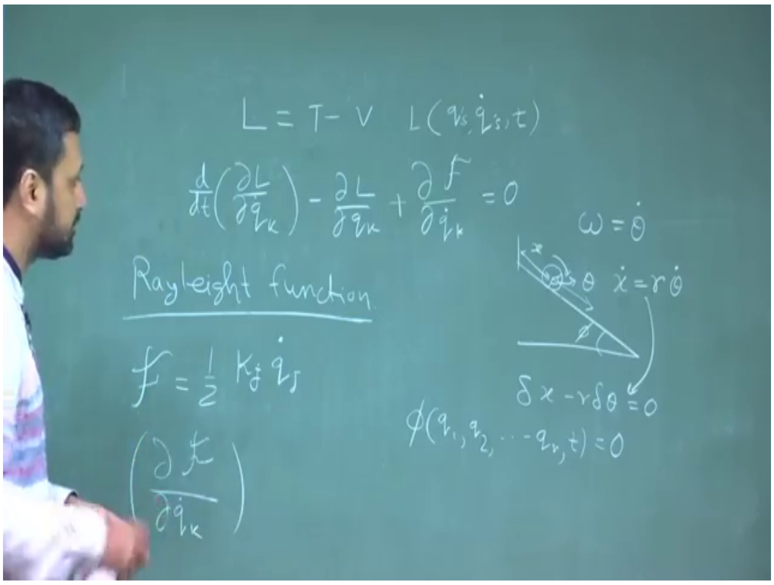

So, what we need to do is actually, so it should be we have to add this particular term with this equation so the total equation we will write as plus del f del q k dot is equal to 0 so I was bit confused whether I should put it put a plus sign or a minus sign here. Now so this is how we can do and if we do that for example, if we if we are trying to describe the motion of a system falling through a viscous medium like we did during our initial discussion of you know motion under resistive medium we can might as well use this Rayleigh function to get the final equation motion. And without surprise we will see that we will get back this 2 beta v term or right 2 beta v term which will give you a damping of the velocity or rather it will be a k v term simple minus k v term which will give you a damping of the velocity right so this is one.

The second one is sometimes we face situations where, even if we want we do not have the full knowledge of the constraint forces or sometimes the constraint forces are also associated with some dissipative force I mean certain dissipative forces can be written in to this particular form, but there are other types of dissipative forces which cannot be taken in to this particular form, but which are might be present in the system one example is the frictional force for example.

Frictional force let us take an example of ball or a cylinder is rolling on this platform so if there is no friction it just comes down, but because there is a frictional force present here it rolls. Now if this rolling takes place, how do we consider how do we include this frictional forces which are causing it to roll in to the equation. The answer is we can always find a relation which is somehow which will be something relation which is very similar to the constraint condition and we can include that in to the Lagrangian.

So, just to give you a brief overview of what talking about in this case see if it is rolling without slipping first of all how many parameters do you need in order to describe its motion it is an 1 d motion right. So, in 1 d motion in principle we can simply take you know some reference point and take the distance x of the center of mass and that should be sufficient, but here we have rolling and that is when there is no rolling in the system. But when we have rolling in the system what happens is we also need to define a angular velocity so we have to take some reference line.

Let us say this we take a line which joins the center of this cylinder with the point of contact and this is our reference line we took we have to take you know any point on the rim and we have to monitored this angle let us call it theta. So, I mean it could be this point it could be any other point, but it has to be a fixed point and this theta will give you the angular velocity omega in terms of omega equal to theta dot right.

So, when it is rolling it has an angular velocity and if the radius is a radius of this cylinder is a or r whichever you prefer let us call it r then we know that x equal to r theta dot this relation has to hold the linear velocity or rather x dot is equal to r theta dot. So, the linear velocity and the angular velocity will be connected in terms of a equation. Now this is a case of non holonomic constraint, why it is non holonomic? Because although, we can put it in terms of a differential equation again now if I start you know writing from here so I can this is d x dt and this is r d theta dt I can might as well get rid of this dt and write it in the differential I mean the virtual displacement form this will give you delta x minus r delta theta equal to 0 right.

Now this is some sort of an constraint condition whatever may be the motion this condition has to be full filled and this is this condition is there because of frictional forces. So, this is some kind of a constraint condition, but for a holonomic system the constraint equations are in proper form I mean this is a derivative form equation right but, in for holonomic system we have seen that constraint equations are given in terms of this type of equation. Please remember that this phi is a function of q 1 q 2 up to q n not that derivatives, but here we are getting a relation which is terms of the derivatives of the coordinates generalized coordinates x and theta. So, this is not a holonomic constraint it is a non holonomic constraint which can still be put in form of an equation.

If you recall during our discussion of the constraints I said there are certain classes of non holonomic constraints which can still be put in form of an equation this is 1 such example another example would be if I take disc which is rolling on a plane. Let us say I do not know may be so this is the this is the plane or let us say my palm is the plane and then disc is rotating I mean rolling on this.

So, what we need to do is we need to define its velocity v once again there will be a theta and there will be another angle phi which will be you know if my x axis goes like this and y axis goes like this. So, this is this angle there should be an angle which gives you the deviation of you know the disc plane corresponding to the axis. So, the situation is something like this I hope you have noted this already.

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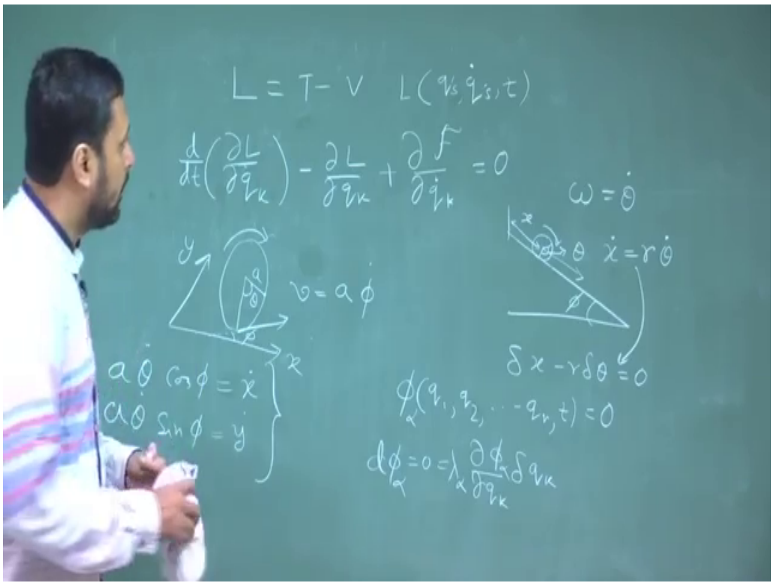

So, let us say this is my one axis this is my other axis in that plane and my disc is rolling, rolling in this particular direction so once again we have to take 1 theta there has to be a v velocity v, so and there has to be some angle phi which or phi or psi whichever which defines the direction of the disc the which basically defines the direction of the disc. So, this 3 if we can get v which is equal to let us call it x dot no vector needed so if we have x dot theta dot and phi then we know all about the system we will know it, but only after we solve the problem completely.

Here also we will know what are the values of x and theta and of course they are related, but this will come only after I mean we can get an equation with relation between r and theta. And basically we can get a relation in terms of proper values only after we solve the problem here also we can break down the velocity vectors, where velocity vector as if v cos theta which will be the x component of this let us say this is x and this is y and v sin theta will be the y component of it and I can might as well write this as and this v velocity v is nothing, but sorry that is what the mistake was.

So v is once again if this is your a, so this is your a right sorry again this will be my phi, phi, phi is the angle between the plane of the disc and the direct I mean the direction of the velocity v and the axis x axis so your v will be a phi dot right. So, if you replace it with a phi dot we get two relations I am sorry I just missed I mean mixed up between theta and phi so we have a phi dot sorry a theta dot once again sorry.

So, we have a theta dot cos phi equal to x dot and a theta dot sin phi equal to y dot so these two relations once again it cannot be taken in to a non derivative form unless and until you solve the problem completely just like we have here. So these are examples of non holonomic case systems and sometimes what happens is you cannot really even if you have an idea of the forces of constraint you cannot remove them a priory before you solve the problem it might happen.

So, the catch is we can always get to a situation which will lead to a I mean which will lead to an additional constraint equation which has to be taken in to account after we write the Lagrangian. So, it could be a differential form or even if we have some time we have a situation where we have a proper holonomic constraint which cannot be removed a priory for such situation we once again have to use the Lagrange’s undetermined multiplier.

So, the fundamentals are exactly same what we did well when we derived Lagrange’s equation of first kind. So, I am not repeating that the trick so basically from here we have to write d phi equal to 0 equal to sum over sorry d phi equal to 0 equal to del phi del q k delta q k and this is for the 1.

If there is a 1 constraint equation and if there are many constraint equations for each of them we have to write such equation each of them has to be multiplied with the Lagrange’s undetermined multiplier and then a sum has to run over alpha and each of them has to be added individually to this equations. So, the final form of equation will be we will take up examples do not worry about it we will take up examples.

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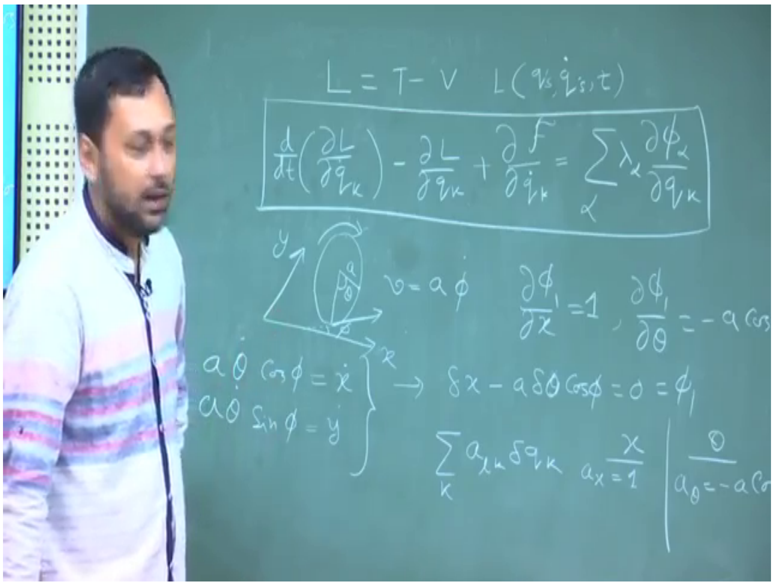

So, the final form of equation once again will be this equation for one particular parameter this will be the equation which has to be added up with a term which is sorry sum over alpha lambda alpha del phi alpha del q k right. Now in case of constraint where we have a situation like this what do we do? We break it in to for example, here there are 2 generalized coordinates; let us assume that phi is a fixed angle. So, we have 2 generalized coordinate theta and x and so we can bring it to the form that delta x minus a delta theta cos phi equal to 0. So, this will be the form of one constraint equations so there will be 2 constraint equations, but let us take look at it individually, ok

So, now what we can do is we can write any constraint equation in a general form that it will be sum over k a L k delta q k in this case we have 2 generalized coordinates x and theta. So, correspondingly for x and for theta we have a x is equal to 1 a theta is equal to minus a cos phi right and this is if you now try to correlate between this. And this it can be I mean we can I can tell you that del phi del x is equal to 1 and del phi del theta is equal to minus a cos phi.

Similarly, we can have a second equation so this is phi 1 and phi 1. Similarly, we can write an equation this is the first equations so this is equal to phi 1. And there is a second equation which will give you same set of parameters for phi 2 and we can add them up and get an equation. So, this is right now what we are looking at here is the most general form of Lagrange equation the first 2 terms are there for any a properly behaving holonomic systems.

The second third term is there because there is a viscous dissipation which can be written in form of a velocity dependent function called the Rayleigh function and if there are some additional constraints which cannot be taken in to account in terms of or additional dissipative force which cannot be taken in to account in terms of this Rayleigh function then we can write them in form of this part I mean in form of Lagrange's undetermined multiplier and include them. So, we have in this using this Lagrangian we can solve in principle we can solve any problems.

So, now what we are going to do is we will take up look up some examples and try to see how to use this Lagrangian we will take up cases where we have Rayleigh function in a system or dissipative system we will take up a case where we have to use this form formalism. So, we will do that, but before that before we start doing that I will just want to remind you about one our discussion on symmetric top. So, I will just erase this for now.

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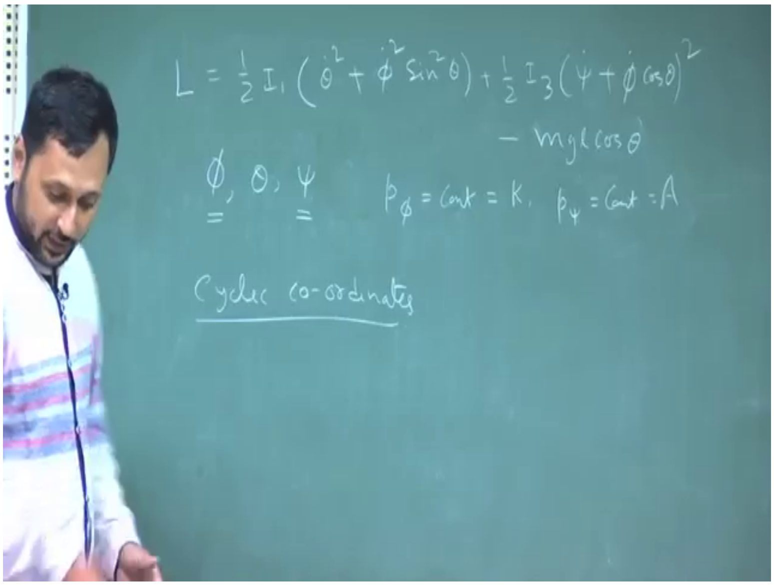

If you recall the kinetic energy and potential energy functions; I will just write it directly from this book whatever so t was given by half I 1 theta dot square plus phi dot square sin square theta plus half I 3 psi dot plus phi dot cos theta whole square that was our t for a symmetric term and our v was m g L minus m g L no sorry this is simply m g L or h we used L cos theta. So, where phi theta and psi these are the Euler angles right.

So, now if we form the Lagrangian what we have to do is we simply have to take write L here and we have to take minus m g L cos theta and this 3 Euler angles we can treat them as 3 generalized coordinates right. So, if you recall rigid body dynamics with 1 point fixed needs 3 degrees of freedom has 3 degrees of freedom I mean 3 degrees of freedom are taken away moment we fixed one point of it. So, this 3 coordinates can be phi theta and psi.

Now, if we examine this Lagrangian closely what do we see we see that out of this 3 phi and psi these two are cyclic coordinates, cyclic means which is not explicitly present in the Hamiltonian, theta we can see that there is sin square theta here there is cos theta here cos theta whole square there is cos theta here. So, theta at least in more than one places theta is present, but phi is it present here no phi is not present phi dot is there phi dot is here, similarly for psi psi dot is here, but there is no trace of psi. So, phi and psi they are cyclic coordinates.

So, in the beginning of the lecture and also in the previous class the theorem the what I taught you by going by that the momentum corresponding to phi and psi will be conserved right. So, we have p phi will be equal to 0 or rather p phi will be equal to constant let us call this constant k and there will be a p psi equal to a constant. Let us call it A.

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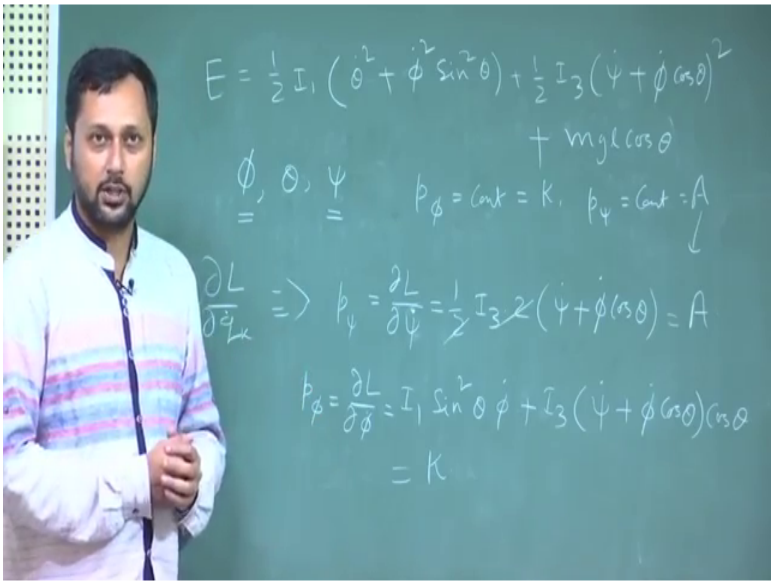

Now, if we try to calculate the momentum what do we see del L del q k dot is the momentum p k so going by this p phi will be equal to or let us start with p psi that will be slightly easier p psi will be del L del del psi dot which will be equal to only 1 term will contribute because rest of the terms does not have psi dot in it so it is a partial derivative. So, only this term will contribute it will be half I 3 2; will come it will be psi dot plus phi dot cos theta 2, 2 will cancel out which will have taken equal to A you wrote it the same expression of A which we have derived during our discussion.

And it came out the same constant came out from the third Euler equation, if you recall similarly so we immediately identify that we see we have 3 constants during the discussion of symmetric top one is the total energy which will be a summation of this and that then there was constant called A which was exactly that there is an I 3 extra here, but we can always do with one constant quantity I mean we can slightly manipulate 1 constant quantity, but at least this part is the same.

Now for the third constant k it was what was it, it was the z component from start looking from space set of axis it was the z component of the total angular momentum of the symmetric term. Our logic was because there is no component of torque along the z direction space z direction the z direction of I mean space z direction component of the total angular momentum will be constant.

So, we took a projection of this total angular momentum in the space z direction and found out there was a constant k now we will do this little trick here I mean little bit of calculation here p phi which p equal which will be equal to del L del phi dot now which one has phi dot this one and this one. So, it will be a slightly bigger calculation if you execute that let us do it at least 1 step we can do. So, this term will not contribute this term will contribute it will be I 1 sin square theta it will be 2 will cancel out phi dot plus other term this will contribute so it will be I 3 psi dot plus phi dot cos theta whole multiplied by a cos theta.

Now you please open your previous lectures video lectures or you if you have already taken notes please check the note you will find out this is exactly the expression of k we have derived in the class. So, the 2 conserved quantities which we had we had our own logic for taking it one came out from the third Euler equation the other one was a momentum total angular momentum component along the space z direction these two are nothing but 2 conjugate momentums of 2 Euler angles which will be constant because the Lagrangian is not explicit function of this Euler angles.

So basically I would say these are two momentum corresponding to 2 cyclic coordinates of the system so I just wanted to show this to give you an idea of what we have done previously with symmetric term can from here what we did essentially we wrote the total energy instead of L we wrote the total energy e we put a plus sign here right. And then we started manipulating this equation by applying this constraint so pure algebraic manipulation to bring that in the particular form. So, we can might as well have done this same thing using Lagrangian formulation that way we would have avoided all the Euler's you know we did not have to write any of the Euler's equation in order to do that all we need to do or we had to do was to write the energy expression in terms of Euler angles, anyway.

So, this is one thing I wanted to show you then rest will be for the next lecture. We will be solving mostly problems.

Till then thank you.