Classical Mechanics: From Newtonian to Lagrangian Formulation

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Lecture - 46
Lagrangian Formulation-4

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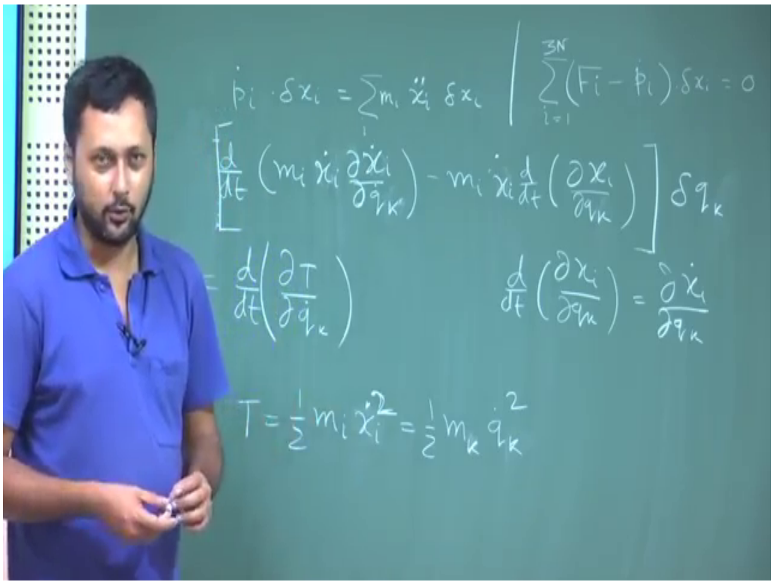

So, we are back with this equation here. So, we started off with D Alembert's principle which is here. And we took the first term; there are 2 terms in D Alembert's principle the first term and the second term. So, or rather one of this term. So, we took with this term there which is p i dot delta x i, please remember there is a summation convention explicitly, I mean it is which is not written explicitly, but on the repeated index is there. So, which is essentially p i dot; that means, say m i x i double dot.

I hope it is clear 2 dots which we could write in this particular form which is d d t of m i x i dot del x i del q j and also we have to understand that this is a double sum of i and j; am I right, yeah. So, it is a double sum of i and j yeah and this whole thing will be multiplied by del q j. So, this whole thing will be multiplied by del q j or q k, I mean whichever index you want to use we can use j or we can use k right. So, let us first get rid of this summations which are always already implied.

Because i is a repeated index j is also a repeated index if you want let us keep to just to maintain the clarity we can use k instead because in the last lecture also I remember we used k it does not matter because this is a dummy indices anyway. So, we can either use j or k or any or any other number, which is not, I now what then what we did from here we said that. So, this is my velocity and there was a result that we can equate this with this. So, we did that in the last class. So, this is the velocity term this is also a velocity term and we could write the first term as d d t of del t del q k dot where t is given by half m i v i square where vi is x i dot. So, we can also write x i dot square these are not generalized coordinates, but it is a sum over all the Cartesian coordinates of the system, and if we want to express in terms of generalized coordinates t will be simply half m i q k dot square, right.

So, q or m k sorry index has to match m k q k dot square. So, good thing is your kinetic energy is frame independent. So, whether you express it in terms of the Cartesian coordinates or you express it in terms of some generalized coordinate, it does not matter your t remains your t the total kinetic energy is invariant. So, the first term is might as well can be written in this particular form for the second term see this again it can be shown that this t d, d t can be taken inside and this whole thing can be written as. So, d d t of del x i del q k can be written as del x i dot del q k, I mean it is again once again it is not very straightforward, but it can be shown in 2 steps.

I am just leaving it to you because; so, that you can get a better grip on the subject if you can prove it yourself it is also there in the any standard textbook. So, you can have a look, but these are not very difficult. So, using this, what we can do is we can get rid of this d d t term here put this on x i, right.

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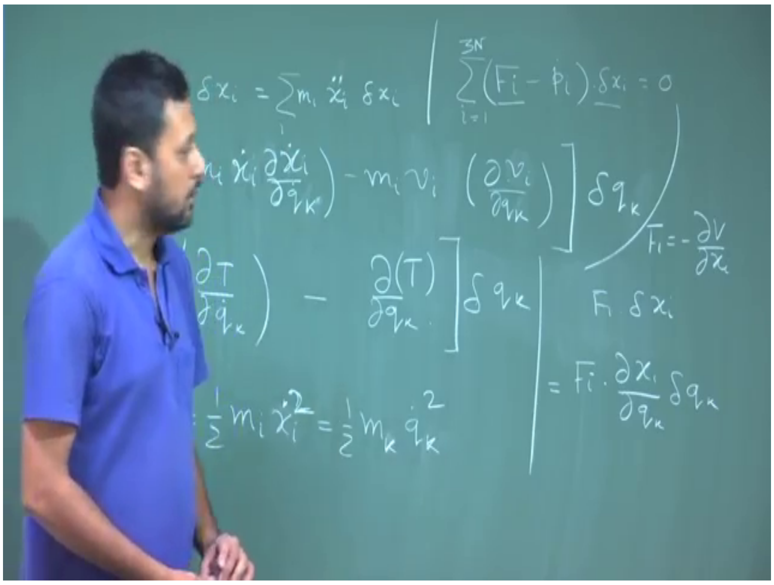

So, once again this is an x i dot this is x i dot. So, we replace it with vi here and vi here. And we identify this term as time derivative of or not time derivative, but derivative of t with respect to q k. Once again t is equal to half m this the same thing m i vi square or x i dot square right. So, this identification helps us in writing this equation the first term of D Alembert’s principle, I mean first term in this particular equation as this and this whole thing will be multiplied by delta q k, right.

So, if we go back to D Alembert’s principle we have 2 terms that is or rather actually it is a combination of many terms each term has 2 called 2 aspects of it. So, what we did is we separated out all the p i dot sums and we have shown that it has a general form assuming all the summation is going on. Now there is no summation over x i anymore all the x is are included all these sums are included in this t and kinetic energy is frame independent. So, we can might as well express in terms of q k does not matter.

Kinetic energy is a constant I mean sorry not a constant, but it is a it is a quantity that we can measure in any frame and we can express this express it in any coordinate system we want. So, most important thing is this whole thing is reduced to this particular form where the only variables are the generalized coordinates and the whole thing is multiplied by a delta q k now for the other term this term. So, we have sum over i F i dot delta x i right these might as well be written as just like we did in the previous case. Once again first let me start by removing this summation symbol here. If i dot delta x i implies summation. So, we can do Fi what we can do is we can write Fi delta x i delta q k del x i del q k delta q k. So, putting it in terms of generalized coordinate and Fi. If you recall there is if there is a potential function, I mean let us say we are talking about conservative forces there could be 2 types of force in general there could be a conservative force there could be a dissipative force, right.

So, let us assume that for now we are not I think, we have discussed it already we are assuming that there is no dissipative force acting on the system as of now. How to handle that situation will come slowly and slowly we will see right? Now let us assume that this Fi is are conservative in nature if this is conservative then each, I mean every force can be written in terms of some potential function, right. So, each of the force components can be written in written as a function of time derivative of some generalized potential component right. So, if we substitute for it just give me a second or rather I would use grad that will be better probably is it.

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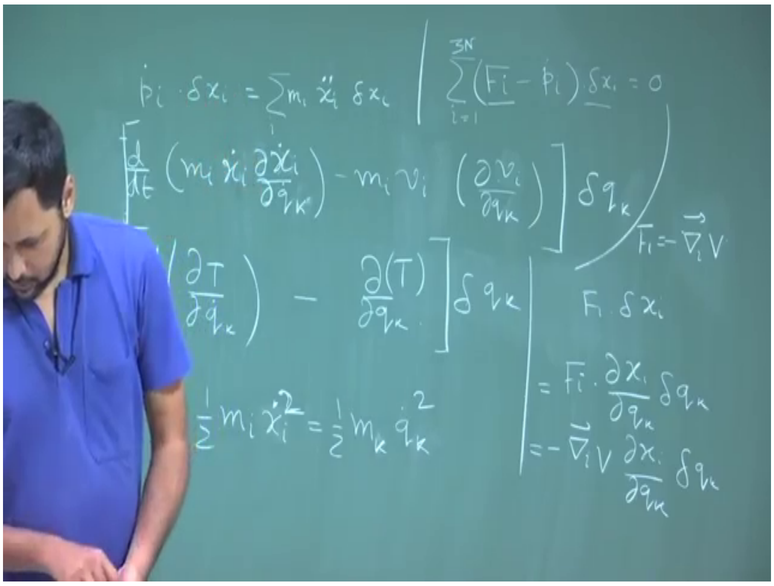

To use grad also actually grad knew the same thing just give me a second yeah better to you scribed. So, it will be grad i v yeah. So, that is the most general. So, if I mean you know if I. So, what you call if I bring it in to components I get the same. So, if I put it here. So, it will be minus grad I v del x i del q k delta q k now.

Or actually there is another way of doing it, let us do it this way better because it will be bit confusing as the calculation says there are many steps which we can avoid. So, what we can do is we can define you can define this term as a generalized force q k and then we multiply this as with delta q k.

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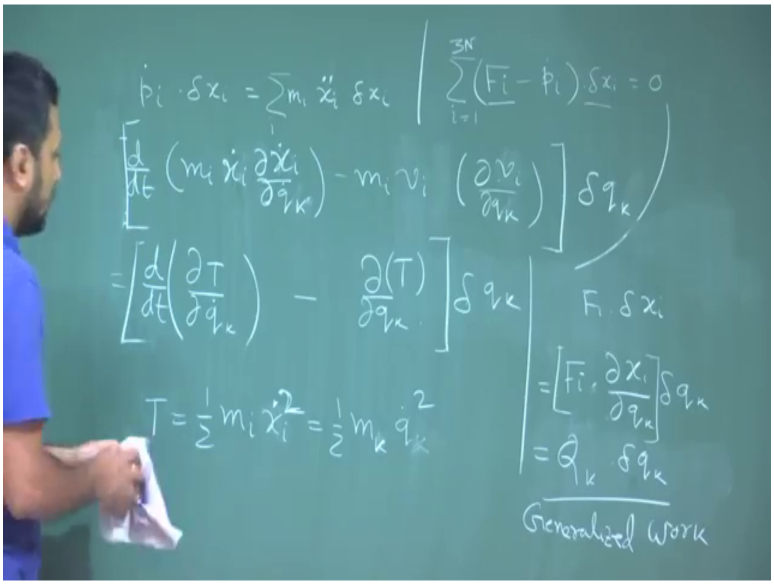

So, we can have a generalized force term which is which has this particular form now. This is a definition of a generalized force. So, what we did here we have a force component measured in Cartesian coordinate. And then we have a basically a transformation term i it can be shown that this is actually a jack and one of the elements of the Jacobean transformation matrix with which we are not going in to details, but. So, essentially we are taking this force component and converting it to a component which will be a one equivalent component in the generalized force or generalized coordinate dimensions.

So, this is this quantity is called the generalized force term now. It is not only force please understand. Now this is the force multiplied by a displacement although a virtual displacement it is a displacement. So, this whole quantity is called generalized work. So, this equation all the terms has the dimension of work. So, here now if you recall what is what did I tell you that generalized coordinates need not necessarily have the dimension of coordinates similarly generalized velocities need not necessarily have the dimension of velocities and generalized force, which is this term need not necessarily has a dimension of a force, but the criteria is the generalized work has to have the dimension of work in real dimension in real life what is what is the dimension of work it is force times. So, Newton times meter. For example, So, generalized force has to have this dimension proper dimension of force. So, that this equation; I mean please remember it is a transformation of the same force equation we wrote in a standard Cartesian coordinate system.

Now, if this is my generalized force, what I can do is I can add this term in to this equation and we can write this as right instead of closing the bracket here. We can add this term which will be with the opposite sign because there is a sign difference between these 2 terms.

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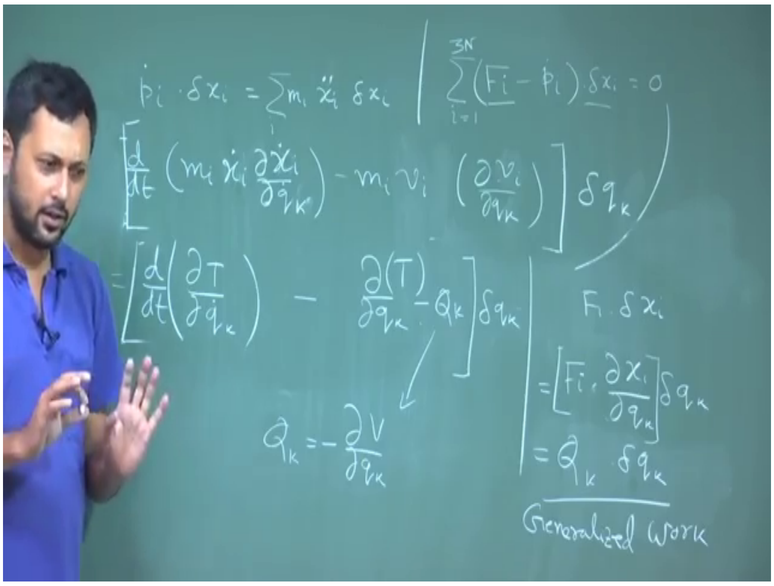

So, it will be minus q k delta q k. So, now, this comprises my full, I mean transformed form of D Alembert's principle transformed in to generalized coordinate space. Now this q k because it is a for generalized force term, we can ask of course, define a potential function for it the thing which I intended to do in the real space and then transform that is not required I mean we can do it that way, but this transformation will be little more complicated what we can do is, now we can start writing this q k as minus del v del q k. So, instead of defining this potential I mean force in the Cartesian coordinate system and transforming the whole thing I am writing a you know the force as a function of I mean as a differentiation of potential in the generalized space itself that is very well justified. Because finally, we will see the; we will be working in the generalized coordinate framework only.

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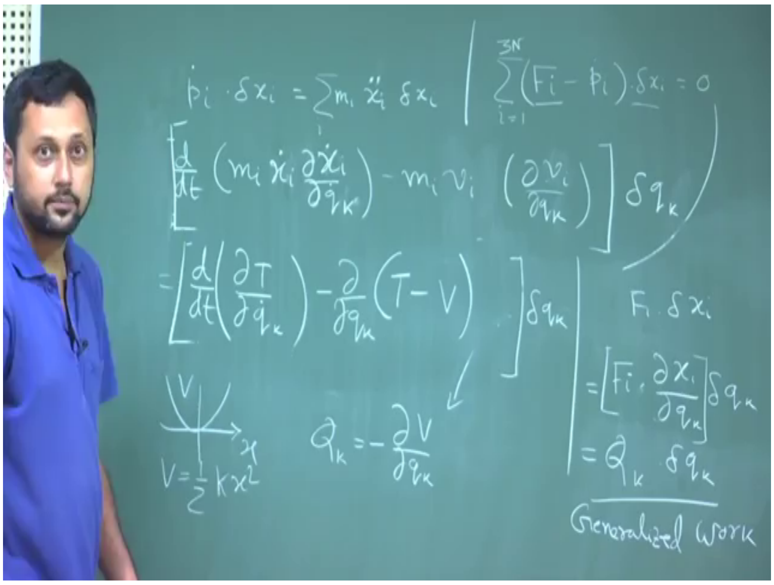

So, it is if we define it like this, now if we do that and we slightly rearrange this then my equation looks like minus del del q k t minus v. So, it will come with the opposite sign that sign will be yeah if we put a negative sign it will be positive. So, it is fine, right excellent.

Now, a potential if you if you realize, a potential typically is a surface which we can we can define a surface of the potential in the coordinate system. For a simple harmonic coordinate system what is the potential; how does the potential look like the potential looks something like this. So, we have displacement in this direction and energy is in our potential energy v in this direction. So, it looks like this similarly we can define a 2 d potential we can define a 3 d potential or we can also define for a system of complicated particles we can define a multi dimensional potential.

Now, whatever may be the definition I mean sometimes a multi dimensional potential we cannot just draw it on the blackboard it is impossible to visualize, but important thing is v in this case what was the form of v it is half k x square so; that means, v most importantly the potential whatever we wrote.

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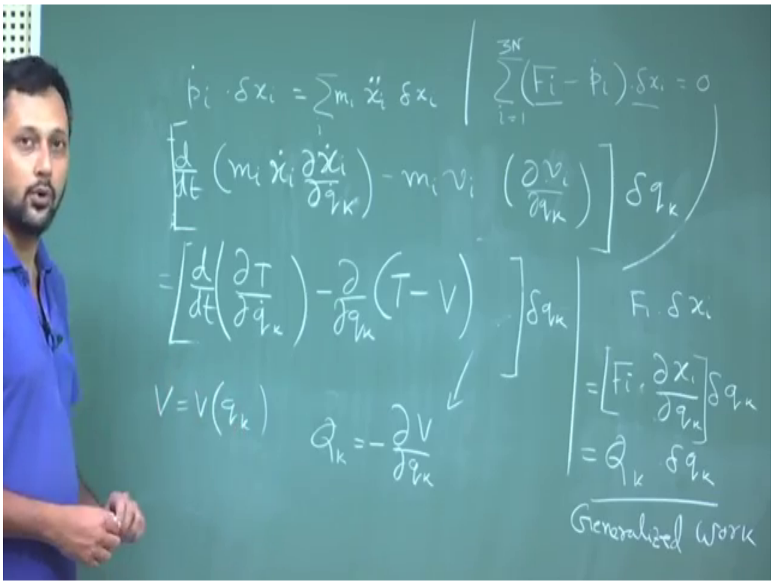

Whatever we can think of is a function of the coordinates only q k; it is not explicit function of time and also this is these are not functions of velocities does not matter. How fast the particle moves it put it the potential will should not change is it not simple harmonic oscillator, if it moves faster or slower, it is potential function does not change in certain cases for example, in particle movement in any electromagnetic field there is an effective potential which can be a function of velocity, but these are these are special cases which will be not discussing here, but in general for the systems we will be considering the mechanical systems where v is typically a function of q k only or basically I should not write q k, but in general.

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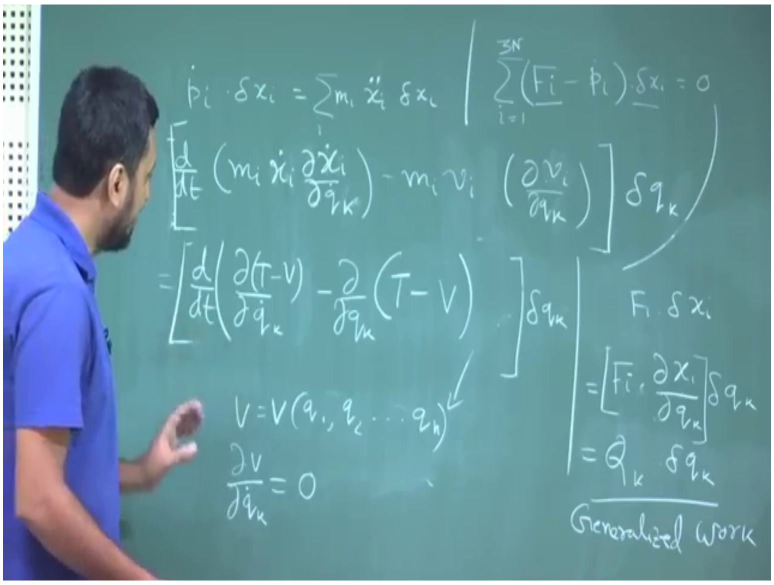

We can write v is a function of q 1, q 2, q n, I mean all the generalized coordinates no generalized velocity involved here if this is the case then del v del q k dot will be uniform unanimously equal to 0 for all case is it not because it is not a function of velocity anyway. So, the partial derivative will vanish. So, without losing any generality what we can do is we can use this property and instead of t here we can also write t minus v it is totally justified because del v del q k dot will be equal to 0. Anyway does not matter, but if you if we do that we see what we get here if we do that we are not losing anything, but we are gaining because in a sense we can define a quantity, new quantity which is I do not need this anymore I can go up from here.

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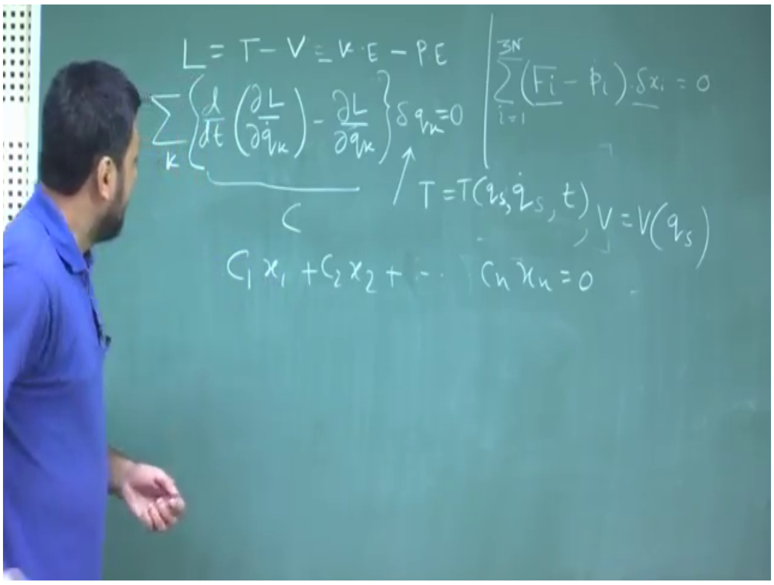

So, we can define a new quantity l which is t minus v and the equation what we wrote here in terms of Cartesian coordinates which is the tip which is the standard equation for I mean which is the D Alembert's principle now has a particular form I am writing the summation explicitly sum over k d d t of del l del, q k dot minus del l del q k delta q k equal to0. So, this is the form this is the form of D Alembert's principle in the generalized coordinate system what we get what we see here. So, l is equal to t minus v; that means, kinetic energy minus potential energy. So, t is your; so, this is equivalent to a variety kinetic energy minus potential energy.

Of course, kinetic energy and potential energy has to be defined in terms of generalized coordinates. So, we have a situation where t is a function of q qs q dots or, I would write an s assuming I mean saying that it is function of all q q dots and time right. I mean of course, kinetic energy might vary as a function of time and v is typically is function of q only right. So, this is the equation now the beauty is here we have reached a situation where all the functions all the you know in in this form if I write c 1 and c 2 and c 3 I mean all the coefficient vanish only if x 1, x 2, x 3. They are independent of each other.

Now we have reached a situation where these are my cs and these are my xs all the xs are independent, because that is the definition of generalized coordinate generalized coordinate means the coordinate system which is I mean all the in a it is a definite, I mean it is a set of coordinates which are all independent of each other. Now if this is the case we can; might as well equate each and every coefficient of each and every coefficient equal to 0 that means.

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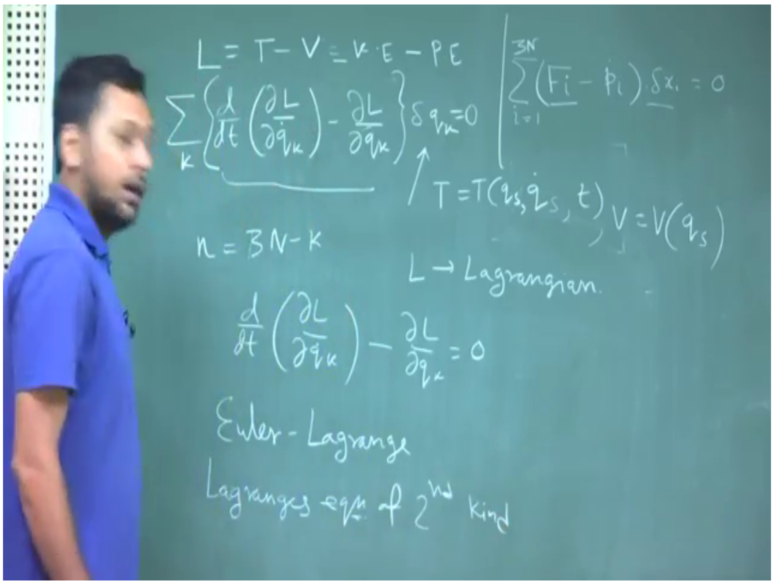

We can get n number of equations n is equal to if you recall 3 n minus k this many number of equations which has the form d d t of del l del q k dot minus del l del q k equal to 0.

So we have n number of such equations for our systems, and I will show you in a moment that these equations are exactly equivalent to the equations we get for a system using Newtonian dynamics, Newtonian kinetic kinetics, but look at the advantage now we do not have to worry about constraints we do not have to worry about the nature of forces.

All we need to know is if a system is moving if a system is moving under any particular type of in a particular type of potential, all we need to know is a potential defined in a set of first of all we have to look at the constraints how many or rather we need to know how many independent coordinates we need of course, that that is something that is a prerequisite of the system, but once we know that, if we can define velocity I mean potential and kinetic energy in terms of generalized coordinates and generalized velocities., Then that is it that is all we need we can construct the Lagrangian.

So, this one is called the Lagrangian of the system and readily we can get n number of such equations n is equal to the number of independent coordinates. So, this equation this set of equations are called Euler Lagrange equation or Lagrange's equation of second kind. So, this is where we end starting from or I should not say end, but this is where we come starting from D Alembert's principle. So, first we took D Alembert's I mean first we removed all the constraint forces got the D Alembert's equation, which is a long equation single equation.

From there we came to Lagrange equation of first kind where we involved the constraint forces I constraint condition also in the situation in the system and we got n number of independent equations or rather I would say sorry 3 n number of equations out of which n numbers are independent. And then finally, we have removed all the contribution from the constraints and we have got n we small n number which is equal to 3 n minus k number of independent equations which we which is sufficient to describe the system totally. Of course, we will see in a moment that this will also give you exact same second order differential equation as in the Newton case.

So, we have to solve it we have to we cannot avoid integrations, we have to integrate it twice in order to get the full description of the problem. Now there is another formalism which is called the Hamilton formalism, there is an advanced version of it Hamilton's Jacobi equations, which are essentially first order differential equations, but this will be too much for one course. So, we are not covering Hamiltonian I am pretty sure many of you will take advanced courses on classical mechanics where the Hamiltonian will be covered in great details. So, we are not going in to Hamiltonian will be staying in to Lagrangian and we will build up from here.

So, let us start by taking examples of systems sorry, if I have taken too long to come to Lagrangian, but as a student when I stride I mean see I can in the first class itself I can define there is a quantity Lagrangian which is nothing, but t minus v. And we can get an equation of this form and we can start solving problems that will not help in understanding why and how we have come this far as a student I always had this problem of I mean during my student's life, I never had access to so many textbooks.

Nowadays, we have lots of internet I mean content in the internet we never had that. So, I had a tough time understanding the concepts which will lead to this particular equation.

So, if I have taken too long that is totally for your good anyway let us take examples, the let us start with a free particle yeah that will be good start.

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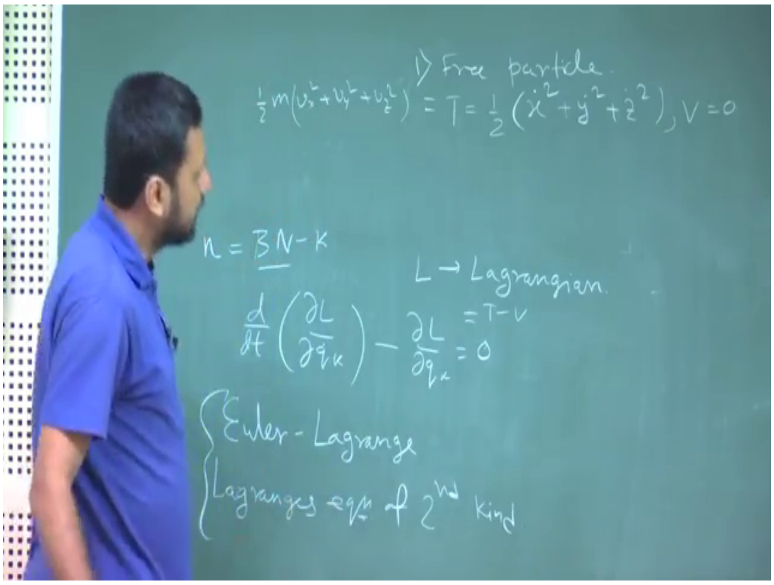

So, for a free particle we have what do we have, we have t which is given by half first of all how many degrees of freedom a free particle has the answer is 3. We have already discussed it many times that we need 3 independent coordinates in order to describe the motion of a free particle. So, this 3 independent coordinate we can might as well choose x y z we can choose r theta phi or we can choose any coordinate.

We can think of the please, but please keep in mind in Lagrangian also you have to integrate it just like in Newtonian, it is just another way of getting your Newtonian equal equations of motion it will not give you a new physics in terms of the final equations. So, it is always wise that you choose simplest possible coordinate system keeping the symmetry of the problem in mind just like we did in the Newtonian case. In order to get integral equations, I can take any arbitrary coordinate system any curvilinear coordinate system to describe the motion of a free particle, but that will not help me integrating it I will get the equations I mean we do not have to worry about the constraint and all will get the equations.

So, if we take free particle it will be best if we take xs; x, y, z coordinate system. So, your kinetic energy is x dot square plus y dot square plus z square z dot square I mean frankly kinetic energy is nothing, but half m v square and v is v x square plus v y square plus v z square isn't it which is equal to this and your vi mean v as in the velocity and your potential energy v is equal to 0 for a free particle, right. So, if I start writing my start by writing my Lagrangian.

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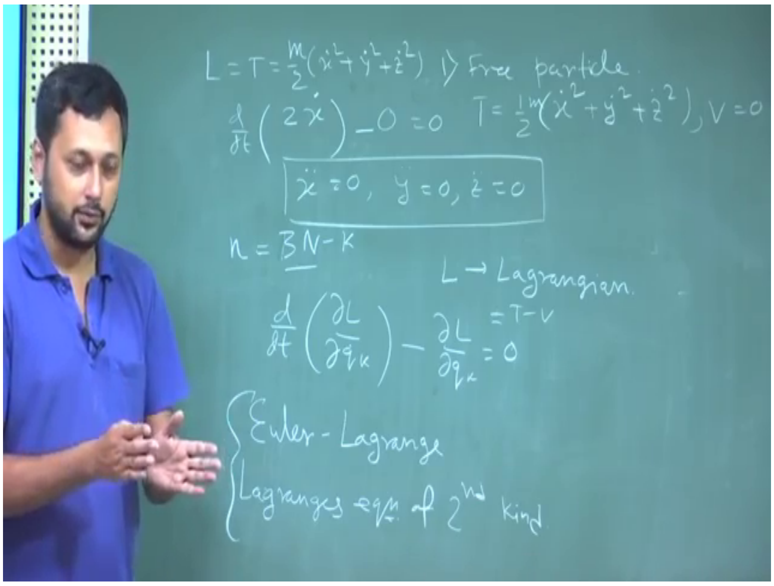

So, my Lagrangian for system which is t minus v will be exactly equal to t equal to this half x 2, there is an m let us say free particle of mass m x dot square plus y dot square plus z dot square. So, your Lagrange equation will be d d t of del l del q k dot. So, for each of this q ks we have to write one equation. So, for there if there are 3 independent parameters, we need to have 3 equations now in this case 3 equations will be identical that is not a; that that is not the issue, but we still have to write 3 equations to describe the system totally. So, the equation will be m by 2 see. So, m by 2 will be common from all. So, we can just get rid of it.

Del l del q k dot, I have to take a derivative with respect to generalized velocity the first generalized velocity I encountered is x dot second is y dot third is z dot. So, when I take derivative it will be 2 x dot right d d t of this minus, del l del q k it is there is no q dependence yeah I mean they in this Lagrangian nowhere this x and y and z comes. So, del l del q k for all the q ks q ks are what right now. The q ks are x and y and z. So, del l del x equal to 0 equal to del l del y equal to del l del z. So, we have a 0 equal to 0. So, my first equation is nothing, but x double dot equal to 0 and just from the symmetry of the problem my second and third equation will be y double dot equal to 0 and z double dot equal to 0 and this is exactly what we have for a free particle all the accelerations are 0 free means that is not no force is acting on them.

So, the free particle has a constant velocity if we integrate them we have v x v x equal to 0, v x equal to constant v y equal to constant v z equal to constant. What are the values depending on the initial condition that is it? So, all I am trying to tell you that it is the same equation it is just another different way of looking in to it, but this is very important though you might ask I mean if it is not giving us any new physics while learning it the answer is it is a different approach to physics first of all, we have do not have to think in terms of constraints anymore.

And secondly, and there are other advantages. We will see later slowly and slowly. Secondly, this is a gateway to advanced physics physical constructions like Hamiltonian is one thing we will learn a little bit of principle of variation integrals. So, these are all pathways which will lead to I mean, this is actually the gateway which will lead you to higher advanced physical situation I mean physical formulations. So, it is important that we understand Lagrangian mechanism by heart.

I hope you have this equation noted down, somewhere if not please do that because you will be needing it lot of time put it in front of your eye as much as you can because this is something that.

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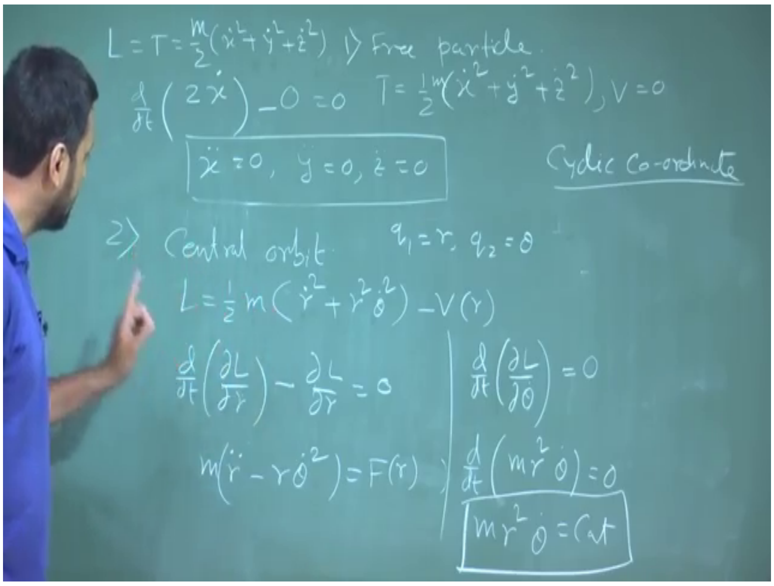

You have to memorize by heart second equation second case central orbit. Now in a central orbit your kinetic energy t once again we have to keep in mind that we have to take the; we have to consider the symmetry of the problem. Now in central orbit typically what happens is we have we have seen that without writing a single equation we can prove that the angular momentum l is constant so; that means, it is a planar motion and in planar motion it is always preferred that we take r theta. Instead of x y I mean we can also take x y, but because central forces always give rise to orbits for orbital planar motion it is always better if we take r n theta.

So, we straight away start writing half m v square will be what v r square, which is r dot square plus r square theta dot square right and v is equal to v r this is the potential function of the central force in question. So, your Lagrangian l. So, l will be simply just not write it again right. So, we already know the answer to this, but still we; I want to show you how to get to the equation of motion. So, the first equation will be d d t of del l del r dot yeah because r dot what are the coordinates we have our generalized coordinates are q 1 equal to r q 2 equal to theta right.

So, first equation it will be del l del r dot minus del l del r is equal to 0. Now what is del l del r del l del r see we have one r here and this is a function of r and the first what is function of r dot only the first term. So, first term will give you d d t of half. So, half will cancel out it will be m r dot. So, actually I can already take this perform the second derivative operation. So, I can write directly m r double dot no point wasting here step for that minus del l del r the first term will give you m r theta dots square it is a partial derivative. So, theta will be theta dot term will be unchanged minus v r sorry del l del v r will be there which is nothing, but F of r minus sign just give me a second. Sorry yeah.

So, there is a minus sign here also. So, it will be minus which will be equal to 0 so; that means, I can take this on that side and I can put the equal sign here, which will give you m r double dot. So, m goes out we can m we can take m common. So, it will be r double dot minus r theta dot square equal to F r and this is the standard equation for central orbit, we all know for the second equation this is for r for the second equation theta del l del theta will be equal to 0. Because there is nothing in this is a function of theta. So, second equation will be simply d d t of del l del theta dot which will be equal to 0 because del l del theta is anyway 0.

There is no nothing is function of v theta here and this will give you d d t of what is it m r square theta dot right equal to 0 does it sound familiar, I mean does it look familiar definitely it is this is nothing, but your angular momentum magnitude. So, the second equation gives you m r square theta dot equal to constant and this is exactly what we know from our discussion or from our for yeah lectures on central forces, that this is one equation and this is the other equation. So, we could reproduce the 2 equations of central orbit starting from here. So, I will I will stop the class here just by defining a single quantity which is called the cyclic coordinate just give you an example.

I will give you an example theta in this problem is the cyclic coordinate because theta does not occur in the Lagrangian explicitly the generalized coordinates which does not occur in a Lagrangian explicitly are called the cyclic coordinate which is in this case theta. So, we will see more examples of Lagrangian; how to solve problems with Lagrangian, we will take little more complicated problems we will have we have to add terms in the Lagrangian for frictional forces, we have to add terms for additional constraint forces if we have any. So, we will do that in the next lecture.

Thank you.