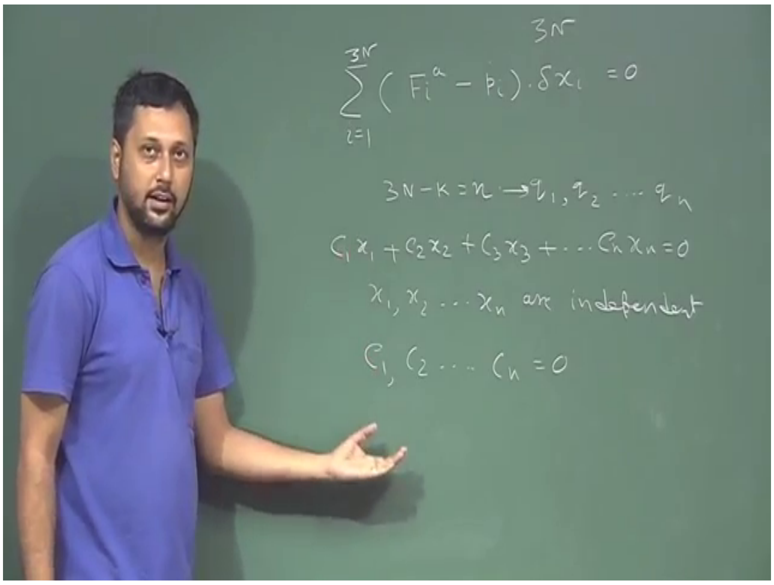
**Classical Mechanics: From Newtonian to Lagrangian Formulation**
**Prof. Debmalya Banerjee**
**Department of Physics**
**Indian Institute of Technology, Kharagpur**

**Lecture – 45**
**Lagrangian Formulation – 3**

So, we have seen generalized coordinate helps us a lot in terms of reducing the number of independent variables to exactly match or rather reducing the number of coordinate to match exactly the number of independent variable required and. So, what we can do is we will try to express the D Alembert's principle which is given as m d or rather m x i double dot minus or rather sorry my mistake.

(Refer Slide Time: 00:41)



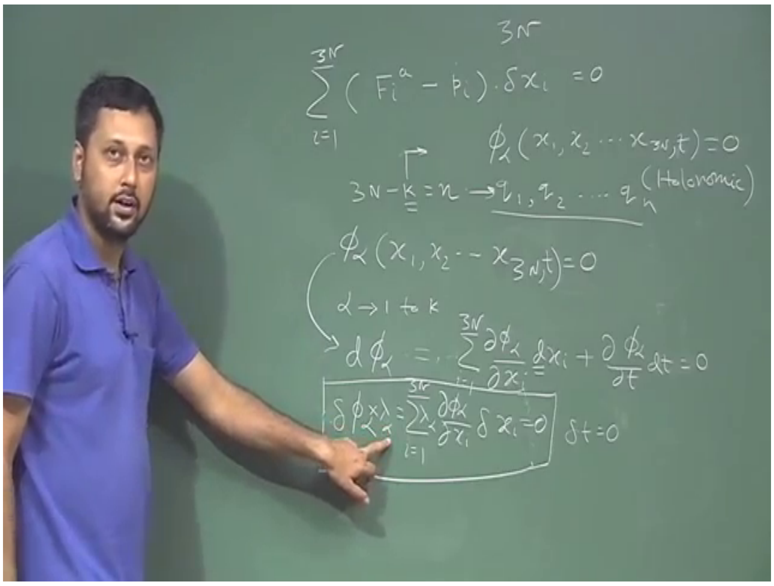

Which is given by F i applied minus p i dot into delta x i equal to 0 and of course, there is a summation over i.

So, we would like to write the, please remember that this x i's are not generalized coordinates, these are the standard Cartesian coordinates, could be Cartesian coordinate could be any of the standard coordinates, but there total, there are total of 3 n of this guys. So, this summation goes from 1 to 3 N right. Now, what we need to do is we need to reduce this number of independent, I mean we will try to write this equation in terms of 3 N minus k which is equal to n, number of independent variables which are given as q1, q2, qn the so called generalized coordinate.

So, my, our next goal would be to reduce this D Alembert's principle in terms of this equation, I mean this quantity. So, that please remember the advantage what will gain if we can write in terms of this generalized coordinate, each of this coefficients will be individual equal to 0 if you recall we discuss this we have c1, x1 plus C2, x2 plus c3, x3 all the way up to Cn, xn equal to 0.

If and only if x1, x2, xn are independent we can write c1, C2, cn individually equal to 0. So, this is something that is from the linear algebra and I am pretty sure you are familiar with this concept, if not you can always check your linear algebra textbooks and class notes and you will definitely find this there. Now, what I am trying to tell you is if we can write this equations in terms of we can convert this D Alembert's principles in terms of this independent coordinates then we can individually equate each of this coefficients to 0, but before that it turned out there is another way of doing it. So, we have 3 N number of this guys.

(Refer Slide Time: 03:36)




And we have k n number of constraint equations of the form phi alpha is equal to phi alpha of x1, x2, x or rather x3 for any holonomic constraint, could it be rheonomic or could it be steronomic does not matter. The constraint conditions can be written in form of an equation that we have seen, that is the definition of a holonomic constraint. Also there is an explicit, I mean in general there is a time dependence depending on whether we are dealing with steronomic constraint or rheonomic constraint this time dependence could be explicit or implicit, but any way this is the most general form for a system with holonomic constraint. So, this is for holonomic and so this is sorry it is a slight mistake here. So, the constraint equation is actually this equal to 0. So, this is the general form, if you recall we have equations of the form x square plus z square equal to l square when we were solving simple pendulum, by the way if you recall in the last lecture we could not locate a minus sign the origin of a minus sign.

That was because we took z equal to l or z nearly equal to l towards the end it has to be z nearly equal to minus l because we have, we are working in right hand coordinate system. So, this is x if this is x, z has to be like this I, what I did was I took the z along this direction which is wrong. So, this is actually minus z. So, we have to take z equal to minus l which will solve the minus sign problem, anyway apart from that the equation of constrain is x square plus z square minus equal to l square. So, what we can do is we can take it in this side and we can write it as x square plus z square minus l square equal to 0. So, this in principle is our constraint equation of the form t equal to 0 there is no time dependence here, but in principle this quantity is x and z, they can vary as a function of time. So, you we will write put it in this way.

Now, so the general constraint equation has this particular form I will just write it once again. So, phi alpha, alpha runs from alpha runs from 1 to k, k being the number of constraint conditions. So, it will be x1, x2, x3 equal to 0. So, we are if you recall in the last class itself we said we are not writing you know r1, r2 components of r1, r2 and r3 independent rn independently separately. So, instead of writing x1, x2, x sorry instead of writing x1, x2, x3 or x1, y1, z1, x2, y2, z2 we are writing x1, x2, x3 up to x3 n. So, this is something that we have adopted.

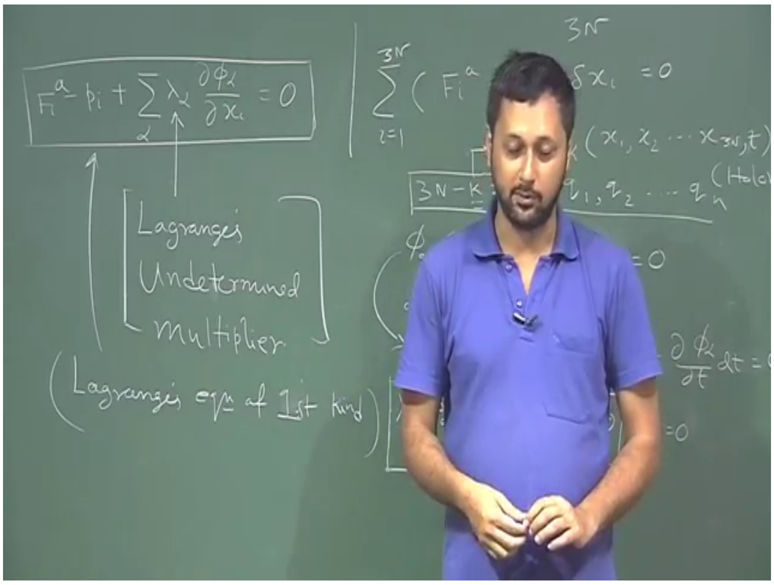
Adopted as in we instead of in writing x y and z we are just writing one index x. So, if we have alpha of such equations it is obvious that d of phi alpha will be equal to 0 because this side is 0 any way. Now d of phi alpha in its most general form has this, this nature is del phi alpha del x i where i runs from 1 to 3N, d x i plus del phi alpha del t which is once again so it is 0. So, put the 0 on this side right, what I did here is simply the we have taken the partial derivatives of this relation there is a t here.

So, it will be the partial derivative of each individual coordinate and a partial derivative due to type. Where if I convert this relation in terms of virtual displacement of this constraint condition phi; that means, instead of writing d phi if I simply put d delta, I mean delta as in curly delta phi alpha which is a symbol for virtual displacement, what happens is the first term stays as it is except we have, what change we have is this d is also replaced by delta x i right. So, this and this term oh and there is a d t sorry Del like Del d d t equal to 0 right. So, what happens to this term this term goes to 0 as delta t for any virtual displacement is equal to 0 unanimously right. So, this is the constraint equation or this is a this is a straight forward outcome of the constraint equation is this equal to 0.

If you I hope you understand this partial derivative I have taken, if it is a function of let us say x and z. So, if phi is equal to x z and t. So, we have to write d phi equal to del phi del x d x plus del phi del z d z plus del phi del d t. So, we just if use that all the coordinates are put in to this sum and times were there which we set equal to 0, now what we can do is there are alpha, I mean there are k number of such equations right. So, what you can do is do is we can multiply each of this relation each of this k number of relations with some constant alpha or rather some constant lambda we can multiply this with some lambda alpha which will be equal to a lambda alpha here right. So, this is some arbitrary constant and this will also be, once again will be equal to 0. So, what we are doing is we are simply writing this as lambda alpha delta phi alpha, which is also equal to 0 because multiplying anything with 0 is 0.

Now, what we can do now because this is a null quantity as in it is the total the total value of this quantity is 0 we can always go back and add this number of all the equations in this form, i in to this D Alembert’s principle. So, what I am trying to tell you is we can rewrite the D Alembert’s principle as F i a minus p i plus sum over alpha, lambda alpha, del phi alpha, del x i whole in to delta x i equal to 0.

(Refer Slide Time: 11:31)




So, that could be an alternative form of D Alembert's principle without losing any generality, what I am trying to tell you is this is valid and this is also valid because of this relation this are essentially this summation is 0 ok.

But then please remember that there are k such summations and the (Refer Time: 12:18) of this equation was we have 3 N number of parameters in D Alembert's principle, but actual independent parameter is n is equal to number of independent parameter is equal to 3 N minus k. See if I can somehow I mean get rid of k number of equations from or k number of relations from this relation by the way I am just switching to Einstein summation convention as i is a repeated index, the summation is implemented over i right. So, I am not writing it explicitly, I mean it is very difficult to write summation every time. So, will just follow the summation convention; so, this means there is a summation over i.

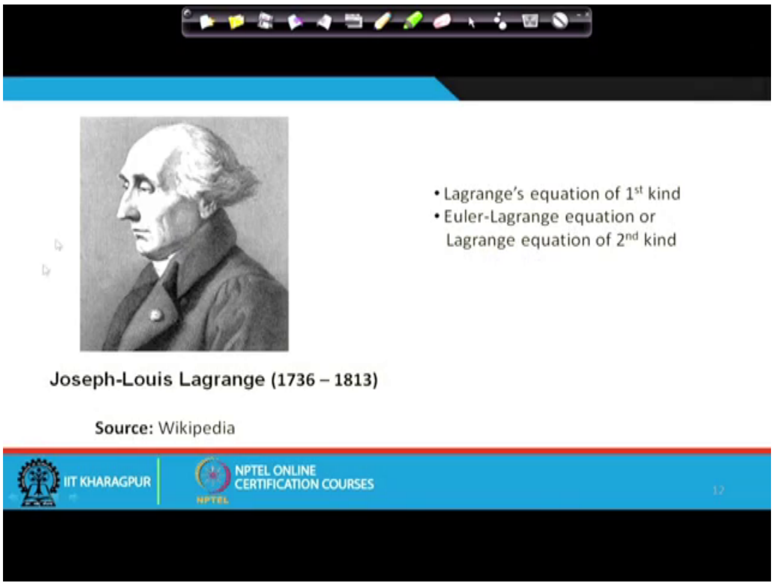
Now, the situation is following we have n number of equations. So, 3 n number of equations out of that only 3 N minus k numbers are independent, independent parameters. So, rest k number of equations what I can do is I can set, let us let us look at it this way let us say we have a series like this c one delta x1 plus C2 delta x2 up to Cn delta or C3 n delta x 3 n equal to 0 we have an equation of this form. Now because x1, x2's are not independent we cannot say that C1, C2, C3 up to C3 n vanishes independently, but if we can adjust our this alpha sorry, lambda such that out of this total number k number automatically vanishes. Now this, please remember this alpha is an arbitrary sorry lambda is an arbitrary constant. So, we can check the values of this 2 the actual real value of this quantity and we can adjust this quantity such that the total sum vanishes we can do that in principle.

So, if that we can we do to we do such that k number of coefficient automatically vanishes then we are left with 3 n minus k equal to n number of coefficients which are exactly same as the number of independent parameters. So, from this full set of 3 n coordinates we can, we can say easily say that out of this 3n coordinates 3 n minus k numbers are anyway independent because that is number of independent parameters. So, rest we can just manipulate this quantity and or this quantities rather and we can say that they are independent right.

So, this is the case if we can sorry we can said them equal to 0. So, if this is the case we can essentially say each of this parameters of C1, C2, C3 which has this particular form up to c3 n they are all equal to 0 individually. Out of them please remember out of that total number of parameter, total number of coefficients which has this particular form we are making k number of coefficients equal to 0 by adjusting this arbitrary constant and the rest are becoming independent as we are setting, that will match exactly sorry the rest are independent because they are matching exactly the number of independent parameters or independent coordinates and so, overall we what we have is totally 3N number of equations of this particular form equal to 0. So, this is the way of separating out coefficients from this particular equation. So, it is a modification of D Alembert’s principle, this particular the way of modification is we multiply this with a constant way as a certain constraint condition we multiply with a constant and add with the original equation.

This multiplier is called Lagrange’s undetermined multiplier, Lagrange’s undetermined multiplier and this equation the final equations we get they are called the Lagrange equation, I think I can write here right. So, they are called this family of equation is called Lagrange equation of first kind. So, this family is called Lagrange’s equation of first kind and these particular lambdas which are the undetermined constraint are called Lagrange’s undetermined multiplier. I am pretty sure you have used Lagrange’s multiplier in different for your math’s problems, sometimes you have a constraint equation you have to solve you know 2 equations in a particular I mean under certain constraint condition and this is where we use Lagrange's multiplier in general. That is why it was first introduced sometime in 1750's I do not remember the exact recall the exact date and time Lagrange.

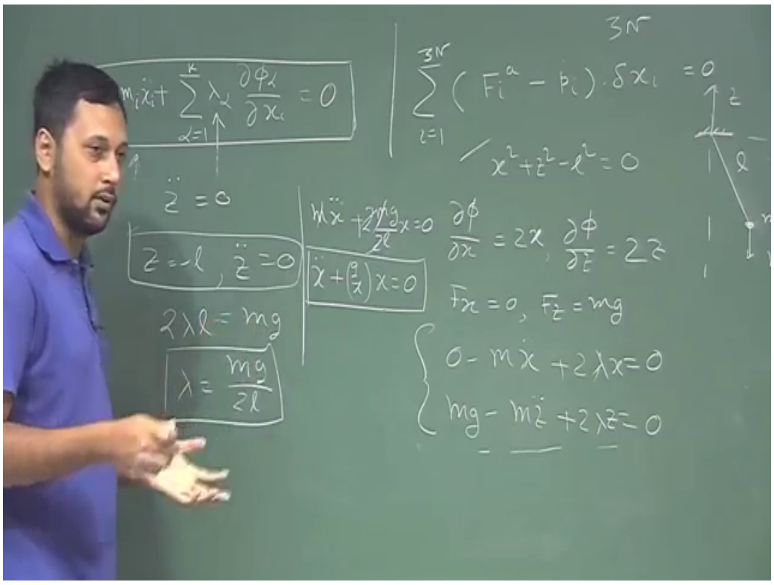
(Refer Slide Time: 18:06)




Joseph Louis Lagrange he was a famous he was a young mathematician at that time, he found this out this phenomena that actually D Alembert's principle can be used I mean directly from D Alembert's principle we can make you know all independent equations by applying an undetermined multiplier and I think somewhere around 1755 or something that time he was not even 20, he wrote a paper and send it to this famous mathematician, the famous math most famous mathematician of his time Euler's. Now Euler immediately saw the potential in this and he actually so he appreciated his work and not only that he started developing on this, he started building on this platform and essentially what we got is the Euler's equation of second kind or Lagrange equation of second kind or the Euler Lagrange equation.

That we will do in a moment, but before that will just take of this problem of simple pendulum once again, but this time will be using Lagrange's undetermined multiplier in the last class what we did we started with D Alembert's principle, we wrote the we wrote a wrote D Alembert's equation for simple pendulum and under certain assumption. So, once again in writing D Alembert's principle was never good enough, we had to use the constraint condition somehow in order to gain in order to separate in order to make this get an independent equation of motion from equation of this form. Now, what we are going to do is we will explicitly use this constraint condition to separate out 2 equations. So, once again simple pendulum has constraint condition of the form x square plus z square minus l square equal to 0, right, so there is only one constraint equation.

(Refer Slide Time: 20:05)




So, this is a good; that means, we have we do not have any sum, this sum actually goes from i equal to 1 to k, we do not have to apply this sum. So, there is there are 2 independent variables, rather 2 parameter x2 2 variables x and z and 1 constraint equation which will effectively reduce it to a single constraint equation, single parameter equation right. Now, if I apply this derivative del phi del x is equal to 2 right, del phi del z is equal to 2 and of course, there is no explicit time dependence we are not anyway bothered about this. So, and please recall now understand that this are independent standalone equations for each of this parameters. So, sorry this is a p i dot, this p i dot can actually we written as m i x i double dot, right.

So, these are equations of x i only, right. So, for simple pendulum we can write 2 equations of the form, for x there is no force so the force which is acting. So, this is my m and this is my l that is the length this is my support. So, whatever force is acting is acting on the z direction m g right. So, F x is equal to 0 and F z is equal to m g right; right. So, this is my x and this is my z and this is minus z right. So, if I write my first equation for x it will be 0 minus m x double dot plus lambda times del phi del x which is

2 lambda equal to 0, second equation will be m g minus m z double dot plus 2 lambda is equal to 0 right. So, we got 2 equations which we can now what to do. So, once again we have to make this assumption that z double dot is equal to, but 0, but that comes later first of all we.

So, we have 2 equations with whether the one way is we have to determine lambda from this to solving this I mean from some somewhere. So, we have one constraint equation here which is explicitly used or what we can do is we can just subtract one equation from the other and get rid of the lambda. So, either way this is the valid thing because this 2 equations are valid. So, whether we can get a value of lambda and substitute it or we can just get rid of the lambda from 2 equations that is up to us. So, let us do this in the other way. So, we have minus m x double dot plus 2 lambda, minus m g minus m z double dot minus 2 lambda is equal to 0.

Now, if we do that 2 lambda, 2 lambda goes and if we put z double dot is equal to 0 once again we have a sign issue how; sorry. So, this will be oh terrible mistake. So, it will be 2x and 2z that makes more sense. So, we have a 2x here lambda yeah so 2 lambda x and 2 lambda z yeah now this makes more sense right. So, in the first term we have to add del x del x I, del phi del x i right. So, first equation will be x 0 minus m x double dot plus 2 lambda x, right. So, now, this are 2 equations right. So, if these are my equations what I can do is I can from the second equation once again I can put z equal to minus l and z double dot is equal to 0, if I do that what I get is 2 lambda minus 2 lambda l or 2 lambda l is equal to m g. So, that will give you lambda equal to m g by 2 l. So, essentially we are not eliminating lambda, but we are getting a value for lambda.

Now, if I that is from the second equation now if I put it back in the first equation I get is. So, it will be lambda on the other side, right. So, it will be m x double dot plus 2 in to m g by 2 l x is equal to 0, 2 2 goes m m cancels which will be x double dot plus g by l x is equal to 0 right. So, we got back the standard equation for simple pendulum again under this set of approximation right, this approximation we have to do because other if you do not make this approximation our pendulum is not simple pendulum and this equation is not valid. So, I made some initial mistakes by just omitting these derivatives, but this is the general procedure; sorry.

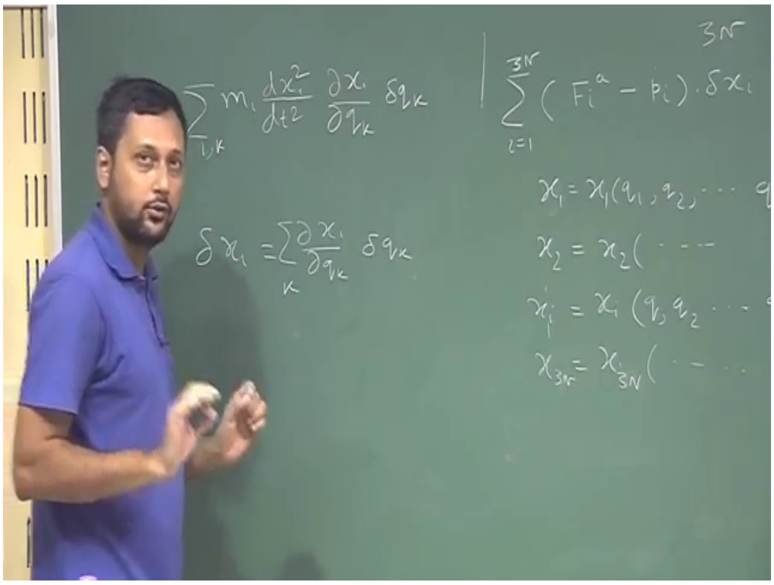
Now, although these are this is a very elegant method of solving or gain getting inside in to the system or essentially get getting back the equation of motion. So, what is the advantage first we have to discuss, the advantage is we are not worried about the constraint forces. Not worried about constraint forces means we are simply writing the force as it is like we are writing m g here instead of taking the component and on the other hand we have we have a disadvantage because this formulation whatever I mean it might be elegant compared to what we have in D Alembert's principle. It is very difficult for larger systems to every time if we have see in this case we have one constraint equation we can have systems which actually have 4, 5, 6 constraint equations. Now gaining this or so what we have to do is we need to you know put 5, 6 different values of undetermined multiplier lambda and in effect we have to solve for this multipliers this is a very tedious process and sometimes it is not very elegant.

So, that is why it will be even better and so, what we are doing essentially we are starting from the standard coordinates Cartesian coordinate system and we are ending up also equations writing equations in terms of Cartesian coordinates right. I mean not that the Cartesian I mean I should not say Cartesian coordinates or rather, I would say we are sticking to the old coordinate system in this method.

But we have already defined a set of generalized coordinates which are exactly the number of which are. So, many as many as many number as required by the system for a complete description. So, it is always better if we can write this in terms of independent parameters right. Let us try that and these are as this one is called Lagrange equation of first kind the one we are going to derive now is called the Lagrange equation of second kind.

Now, Lagrange equation of second kind right so we start with the, we I think we have to remove this as we need the space.

(Refer Slide Time: 27:40)



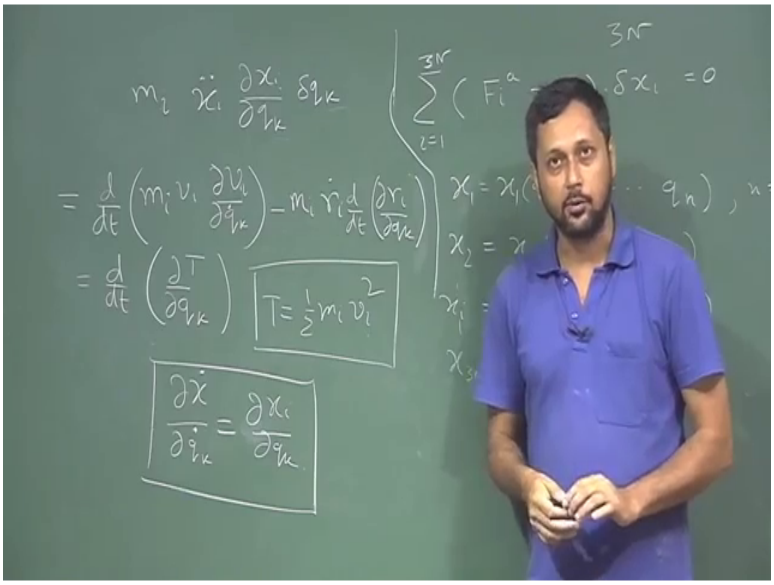

So, we start with this term here. So, we have forget about the minus sign we have p i or rather I would write m i del not del d d x d 2 x i d t 2 right times delta x i equal to 0. So, if i just or sorry i should not write equal to 0 so this is the second term right and there is a summation over i, right.

Now, if we start playing with this second term mathematically what do we get, it is just basically a mathematical series of mathematical manipulation let me just have a look at it right. So, what we can do is we can write this in terms of. So, this part is be tricky and I always need to refer to my notes because there are so many calculations instead of this curly x i what we can do is we can convert it in terms of the generalized coordinate, we have to add rather sum this operation I will justify now. Remember they were this transformation equations we wrote as r 1 equal to r 1, of q1, q2, qn n is equal to 3 N minus k.

So, first of all we will change r to x because we are not using r's anymore sorry similarly we have x2 is equal to x2 of what naught similarly we have x n is equal to x or x3 n is equal to x3 n of naught. So, if we look at the i th coordinate i th Cartesian coordinate which is again a function of q's right. Now if we take d x i like we just like we did for in case of constraint equations, we can show that delta x i, I am just not repeating this calculation delta x i is nothing, but del x i del q k delta q k where sum is over k.

So, we just did that additionally there is a sum over k, once again we do not have to write any of this explicitly why because both the indexes k and i; they are repeated index and a repeated index if in Einstein summation convention means that the summation is implemented. So, we will be following the simple simpler form we will not write the summation explicitly if these indexes repeated for example, here i is repeated right k is repeated. So, we assume; what do we have. So, I hope so you understand this why we changed delta x i to this.

(Refer Slide Time: 33:29)




Now, let us see if I have a quantity of the form d d t of m i r i double dot right, d d t of m i r i double dot right sorry these is just simplify it even more. So, I will just put double dot here. So, it is m i x i double dot del x i del q k delta q k right. So, now, this it can be shown with simple equation I mean simple mathematics that this is equal to d d t of m i x i dot delta x i delta q k minus delta x i delta q k minus m i r i dot and this third term will be delta yeah it will be d d t of del r i del q k this can be shown I mean again its looks very straight forward. There are some tricks in it because please remember that the derivative is also running over all the summations summation over k and i. So, it is bit tricky, but it is not nothing impossible to show it is not very complicated, but we have to be careful it is not very straight forward at as it might look like.

So, this is equal to this it can be shown there are 2 2 or 3 types in between I am just omitting that I am leaving it to you. Now, there is another thing I would like to point out d x i is equal to del x I, del q k, d q k plus del x i del t d t right. So, if I from here if I try to write the total time derivative which is d x i d t which is x i dot this will be this and this will come down right. So, x i dot is equal to del x i del q k d q k dot del x del x i del t from here if I write take another partial derivative of this velocity with respect to a generalized velocity.

So, if q k is a generalized coordinates. So, the velocity q k dot the total time derivative of a generalized coordinate is a generalized velocity and there is a reason why we can treat generalized coordinates and generalized velocities as independent parameters. Why so? Because you just take an example lets velocity in a central orbit what happens a particle is moving in a central orbit. So, we need to specify its position as well as it is in a one of its velocity in order to get the full description. So, although velocities are I mean velocities are function of position the time derivative of position it is purely not the position alone, a time derivative essentially means it is a division by one variation in variation in the coordinate and one variation in the time.

So, it is not only I mean a velocity although it is a dependent I mean it is a function of a position, it is not only the position I mean what I mean to say is it is not sufficient if you only specify the position of a particle to get the full description. In order to gain full insight you need to have both information of a position and a velocity. So, that is why generalized velocities are generalized coordinates they can be treated independent of each other.

Now, if I try to take partial derivative with respect to a generalized velocity it I s this one what we gain is del x del q k dot is equal to sorry del x del i del x dot del q k dot is equal to del x i del q k. So, this is a interesting result keep this in mind, this is you can by in short you can call it the cancellation of dots it is as if we are dividing this 2 quantity and this dots are getting cancelled it is not that I mean strictly speaking it is not that, but we it is easy to remember that when we are dividing 2 quantities the dots are getting cancelled at any and it is true for any other parameter which in this case we are just taking the time derivative of cut Cartesian coordinate, but even for a constraint condition.

For example, if a constraint condition is also a function of velocity we can do the same treatment will use this relation later on. So, this is one relation that we need to keep in mind. So, now, using this relation if I change this if i put 2 dots here it will be a valid you know modification and readily I can show you that this quantity is nothing, but partial derivative of total kinetic energy which has the form t is equal to half m i v i square. Now x i dot is nothing, but v i.

So, I can write this as v i right and t is half m i v i square. Now a good thing about energy is this are frame invariant if we change from Cartesian coordinate to generalized coordinate also the velocity the total form of kinetic energy does not I mean the value of kinetic energy does not change. So, we can use a kinetic energy which is defined in terms of Cartesian coordinate in this equation and take a derivative with respect to you know generalized coordinate it will not altered the problems, it will physically it is a valid description.

So, will start from here so we see the first term reduces to time I mean partial derivative of time sorry kinetic energy and the second term is what we need to see and we will find out what is the final form of Lagrange equation of second kind in the next lecture.

Thank you.