Classical Mechanics: From Newtonian to Lagrangian Formulation
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Lecture - 44
Lagrangian Formulation – 2

So the primary concern was whether the forces of constrain are always perpendicular to the displacement direction. It turns out that for olonomic plus scleronomic constrain this is generally it; this is in general true.

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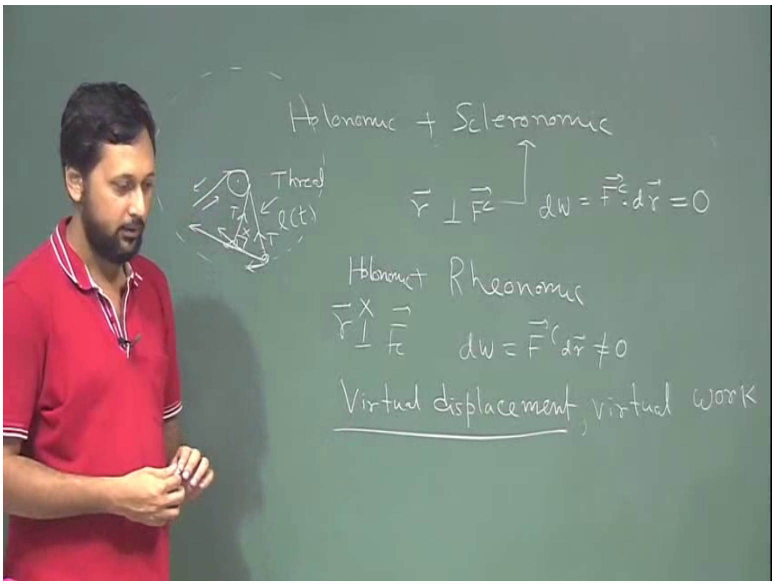

So, whenever we have a displacement of the system, let us say this is our system could be anything. Whenever it displace; the displacement is always perpendicular to the forces of constrain; it could be a bob of a pendulum, it could be a ball you know an object which is sliding it could be what other examples to give it could be. So, other systems we will see we will take up some examples and we will see.

So, wherever it is; if this is the direction of displacement r and this is the forces of constrain F c say. So, it is always seen that r perpendicular to F c for this case. For a holonomic and scleronomic constrain; by the way this spelling is correct. So, is it is s c le r o n o m i c. Now, if we have holonomic plus rheonomic; that means, when the constrain is changing; the condition of constrain is changing as a function of temperature. So, the example we gave was the following. So, we have a let us say there is a pulley here and has a pendulum other end of the thread. So, this is the thread of the pendulum which has a length l.

So, the pendulum at this point, let us say it is it should move on the circle of radius l right, but now this side we have some extra thread and we can either release it or we can pull it little bit more, because there is a it is a pulley let us say it is a frictionless pulley. So, it will not offer any friction to the system, but it will change the length, as we pull on this direction or release in that direction it will change the length. So, l is a function of t. So, if let us say the length is shortening and the pendulum is moving in this direction. So, as the length shortens on it is passage; during this passage it will end up here not on this, but on a shorter path.

Now, this; so the path is actually something like this. So, the direction of velocity at any instantaneous position is a tangent to the path, which will not be perpendicular to this direction. So, the tension T is always towards the center or the point of suspension. Let us say which will also slightly change, but if we I mean as it moves along the pulley, but let us assume that it is not changing, but this point is fixed, but what the main point is the velocity which is the direction instant direction of instantaneous velocity which is tangent to the path is not perpendicular. So, this angle is not 90 degree, but something else.

So, in that case we cannot say this and the work done due to the forces of constrain which is given by d w equal to F; we will write it as F c yeah F c dot d r yeah. So, this is not is in this case this is equal to 0, but if we calculate the same quantity for a holonomic plus rheonomic constrain, then del d w which is F C dot d r is not equal to 0, because r is not perpendicular to F c or I should not say r actually; I should say the velocity instantaneous velocity v. In this case it so happens that the velocity and no I should not write r because r has a different meaning, it is a displacement I do not know how should I write it? For a pendulum the displacement should be given by let us write it; r only just to be convenient.

Just keep in mind that this r is not the length of the radial vector, but it is the displacement from the equilibrium position. So, we can use r. So, this is not generally true that it is not perpendicular for holonomic and rheonomic constrain, when the length for example, the length of the pendulum changing, but also we have discussed a case where let us say; this is a plane which is inclined and this object is bound to move on this plane right. So, here also we see that this object is always the tangent at the normal force is always perpendicular to the direction of deviation.

Now, as we change the angle; let us say the constrain condition is the plane equation. Now if the plane is rotating; even if the plane rotates the instantaneous displacement direction stays perpendicular. So, this is an example it is a different type of system, but which is not true for this one, now; so we can have. So, what I am trying to tell you that we can have systems which are holonomic and rheonomic in nature the constrains, where this is true; that means, the force of constrain and displacement they are not mutually perpendicular. So, we can have nonzero work by the force of constrain, but we want to make the work done due to force of constrain; uniformly 0. across the all holonomic systems, that is our goal why and why will it necessary and how will it help us we will see very soon.

Now, in order to do that we employ something the concept of virtual displacement and see now I mean associated with this concept of virtual displacement is virtual work. First of all what is virtual displacement? Virtual displacement is a concept which is very much used in the Lagrangian formulation of mechanics and it is sometimes not so easy to understand. So, that is why I would like you to I mean not necessarily I will always know that you are paying attention. So, that is not the problem, but the problem is if you do not understand what I am trying to say right now you should you should not give up, but you should consult other books you should consult other materials you can always write to me, because this is a concept which is literally difficult to digest.

So, the situation is following; virtual displacement is a first of all why it is virtual; because virtual displacement there is no passage of time during a virtual displacement.

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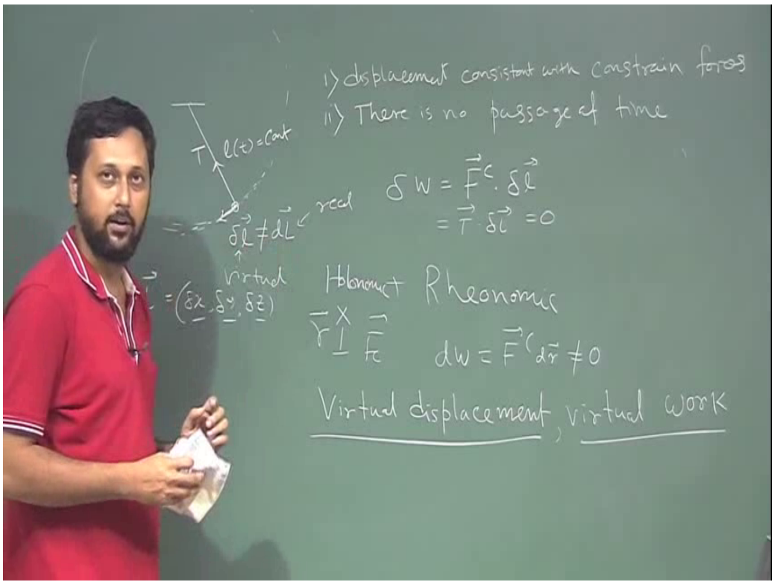

So, if I have to define virtual displacement I would say it is the displacement consistent with constrained forces that is one part of it and the second part will be. So, this part one and there is a second part and part two is there is no passage of time.

So, what I mean by these two statements? The first one is pretty much consistent I guess we will all understand this which displacement which is consistent with the constrain forces; that means, the displacements which are allowed by the constrain forces anyway that is the case I mean even if even if the pendulum is moving in this particular path; that means, this is; because the length of the a length is changing length of the thread is changing and length of the thread is changing length of the thread is changing means the constrain conditions are changing. So, depending on the constrained condition it will choose it is path; there is no problem with it. The second part is there is no passage of time and that is why it is called virtual.

So, it is not a real displacement to begin with what I mean is let us assume that again let us go back to the pendulum. Let us assume this is the pendulum with length; which is a function of time, at any instantaneous time point t right. So, at; at this time if we assume a situation where time stops here. So, so it is bit difficult to understand. So, we are assuming that actually, there is no time passage of time no time elapsed, but we are assuming there is a small displacement, which has to be keeping; this equal to constant which has to take place keeping this equal to constant, because l changes as a function of time and because there is no passage of time during that displacement l remains virtually constant.

What I mean is? If I draw a circle of radius l, in actual situation, because l is a function of time I have shown you already that the bob will not traverse along this path; along this circular path, but it will deviate from it, but if I am assuming that time is not elapsed during a displacement; that means, length which is a function of time remains unchanged. So, it has to follow this path. So, if I have a displacement delta l delta; please remember that we use this curly symbol to make it separate from the real displacement d L. So, d L and delta l they are not the same for a holonomic and rheonomic system for systems where there is no the force of constrain is not dependent on time delta l is equal to d L.

But in this case delta l is not equal to d L, delta l is virtual whereas, d L is real. So, this is real and this is virtual. So, this d L or rather delta l is always perpendicular to the force of constrain, because T is this and because l is not changing l of t is constant, because we are; whatever we are considering there is no time passage. So, that is why this delta l and t which is along l is always perpendicular. So, that is the beauty of virtual displacement. We introducing by introducing virtual displacement, we can always make even for a rheonomic system; we can always make the work done due to force of constrain equal to 0. So, if we try; now try to calculate the work done for since this system which is called the virtual work; which is given by once again delta omega, which is F c dot delta l, what is the direction of; by the way these are vector quantities, delta l and d L these are all vector quantities; could be vector could be scalar, because we can also break it into components every vector has in the 3d world if it has 3 components.

So, if curly delta; curly delta l is a vector, then it will have three components x; I mean three x, y and z component; each will we can I mean write with a curly symbol. So, what I mean is? Delta l vector is essentially delta x, delta y, delta z; where these three are the virtual displacements in three different direction, but anyway it is not very important to break it break the equation here, but what we understand directly that F of c is nothing, but your tension T and T is along l and l and d L they are of course, perpendicular to each other.

So, this is equal to 0. So, now, what we did here please understand carefully? What we have done here; we have constructed a system where or we have constructed a virtual displacement which is at part to the forces of constrain, and because there is no passage of time at that instantaneous time I mean at the at any instantaneous time, that particular moment forces of constrain has to be perpendicular to the direction of displacement. So, these two properties combined a virtual displacement always produce a virtual work which is equal to 0 right.

Now, with this construction and please remember for a scleronomic systems, so at the constrain condition is not changing we do not need to worry about virtual displacements anymore; because in that case delta l is equal to d L, but using this delta l formalism, what we have done is? We have included all holonomic systems, when we are writing virtual displacement and virtual work for all each and every holonomic system we are getting delta omega is equal to 0. So, this is the beauty of using virtual displacement we do not have to worry about; whether the system is scleronomic or rheonomic; if it is holonomic, we can always write delta w equal to 0 great.

So, now what do we do. So, we have understood; I hope you have understood this two terms virtual displacement and virtual work. Now let us move on from here; and try to see; where will this take us. So, you see we first started this lecture the previous lecture by writing this equation.

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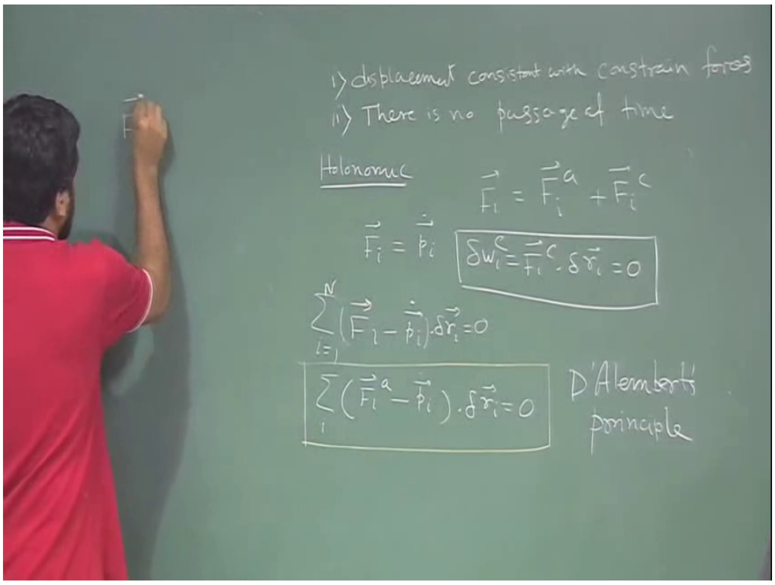

F is equal to m r double dot; or rather you should not write m r double dot; in general we should write p dot. Now, in order to and this is for a whole system, I can break it down into the system of particle formalism for each particle we can write p i dot this is also valid. We can also put sum on both sides which is essentially just another way of writing this equation this is also valid.

What we can also do is? We can F; we can do F i minus p i dot sum over I is equal to 0 and also, because this quantity; this left hand sides quantity will always give you will always be equal to 0 given that the system is in equilibrium, I mean; I should not say system is in equilibrium what we are doing is we are just bringing everything into one side and writing this side equal to 0. So, this has to happen it has to; I mean this equation has to be satisfied; no question about it. So, as this is this has to be satisfied, what we can do is? We can multiply for I mean; this is valid in both forms. Please understand this. This is valid in a summation form, also it is valid in individual form. If for an individual particle I can write an equation which is F i forget about the combined equation now, I can just concentrate on the individual equation which is F i is equal to p i dot right.

So, this is valid for each individual particle in a system of particles; that we have seen when we discussed in the system of particles. So, we can manipulate it in to this form. Also, now what we can do is as this is equal to 0, for each of this particle individually, we can multiply this with a virtual displacement delta r i and, then we is still remains equal to 0 not only that we can sum this over all the system of particles starting from I equal to 1 to N; whatever be the number of systems system of particles and we still get this equation which is equal to 0.

So, essentially what we did is? We started from one particle equation manipulated it brought it both into the one side set right hand side equal to 0. Assuming that there is no other forces accept; or rather all the forces are included in this term right. And then we just multiplied this with this one forget about this, multiplied this with this one without losing any generality and also summed over all. Now in each individual term in this summation is equal to 0, now and as the each individual term is equal to 0 of course, the summation will be equal to 0.

Now, please remember F i has two parts; one is F applied plus the other one is F constrain; and also we did that little bit; replace it here. So, replace it here and you can write F i applied minus p i dot delta r i sum over all I is equal to 0, because we have shown, already; that in case of virtual displacement F i C dot delta r i will be equal to 0 for each individual is; so for each individual particle, if we are considering virtual displacement; the virtual displacement has to be the work done due to virtual displacement has to be equal to 0 for each individual particle, that we have shown already right.

For any holonomic system; please remember all this formulation is for holonomic. Non-holonomic systems we will consider later. So, this is a relation which has which is a different form of Newtonian equation itself; please remember we always have to start from Newtonian equation not that we are developing a new physics altogether. It is that simply that we are starting from Newtonian mechanics, trying to manipulate mathematically the I mean; basically right now what we are doing is pure mathematical manipulation and we are trying to see if there is any advantage of writing Newtonian equation which started off in this particular form in this particular form is there any advantage; it turns out there is an advantage.

The advantage is here we have already; I mean found out the work done due to forces of constrain I should write a C here, and as this is always equal to 0; we have subtracted this from this equation. So, this equation is free from any force of constrain. F a is the applied force. So, whatever forces we have applied on the system we will we will be consider only those. And this principle has a name and this is called D Alembert's it is called D Alembert’s principle. Now this D Alembert’s principle gives you one advantage; that advantages it has removed all the forces of constrain from the system.

It also gives you one disadvantage. What is the disadvantage? Recall; that for Newtonian mechanics or Newtonian equations, let us say you have a particular form of equation which is F equal to for I mean; I am not writing it for individual particle, but writing the equation for a rigid body. So, let us say F equal to m; rather m r double dot right. Now let us say you are working on a Cartesian coordinate system and you know each component of forces individually.

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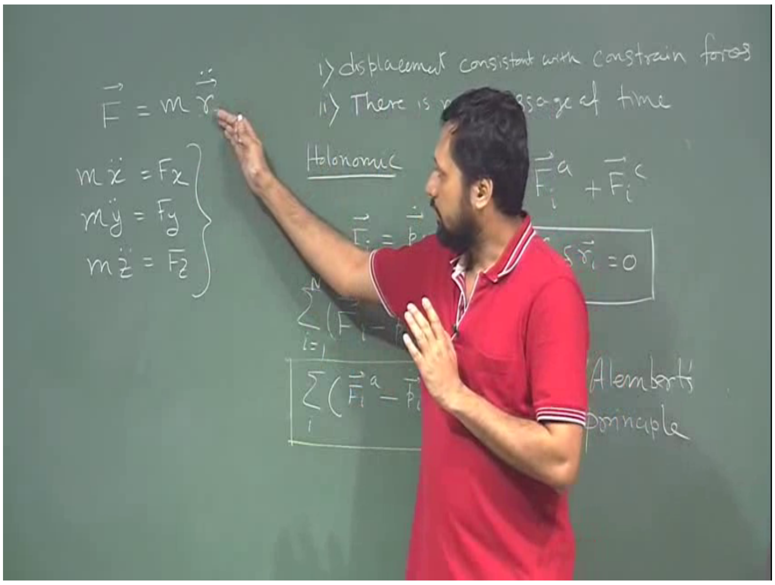

So, you may F x, F y, F z. So, what you can do is; you can break this equation into this simple. So, I mean not exactly simple, but these three equations and m z double dot is equal to F z; you can do that of course, whether it will be integrable easily or not that is a different question altogether; we are not going into that question at all, whether will it be it will be integrable, whether it will be difficult to solve, but at least you can decompose it into each individual part and assuming that these forces are independent of each other; that means, x force depends only on x component not y or z component. You can always influenceable you can always integrate this. And you can essentially get x and y and z as a function of time.

So, starting from Newtonian equation you can get individual equations for individual components, but starting from D Alembert’s principle, you have to understand that this is a single equation. There is no breaking into breaking it into component thing. Why not? Because, although we have removed the forces of constrains, we have not we are still working in a; with number of coordinates which are essentially; the number of coordinates required to describe the whole system. Let me elaborate; assuming that we have N particle system, we have N particle system; how many coordinates do we need? We need 3 N numbers right and this r is; please remember that, when we are writing it into component form I mean it is a vector notation.

So, each of this vector component essentially can be broken into three components right. So, we have if the summation runs from 1 to N we have altogether 3 N number of such components, because for each of this vector we have 3 scalar components. So, altogether we have 3 N numbers of r is or I mean.

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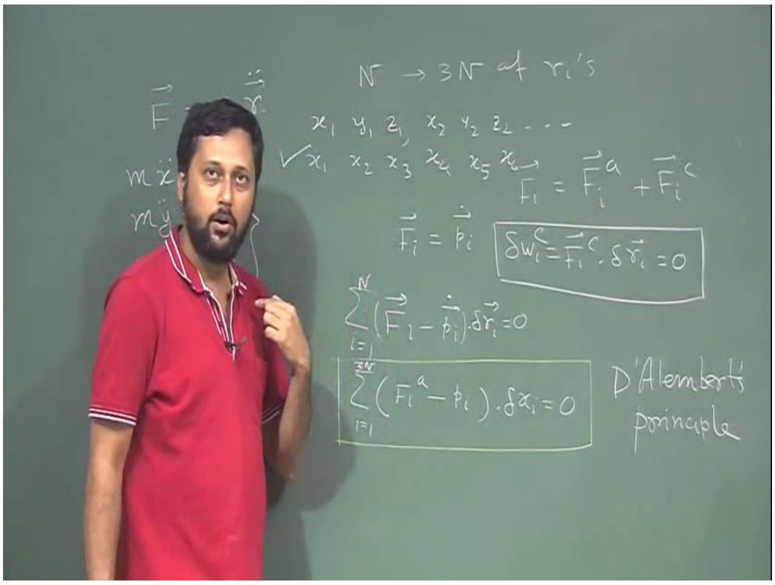

So, we what we are doing here slowly; we are adapting a notation where instead of writing x 1, y 1, z 1 yeah or rather instead of writing x, y and z components of each individually; what we can do is?

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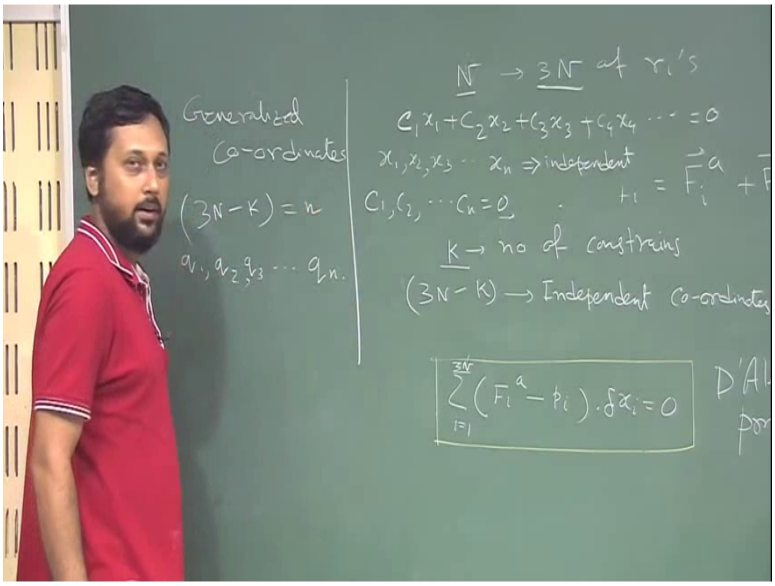

So, whether we can write x 1, y 1, z 1, x 2, y 2, z 2 things like that or what we can do is we can just write x 1, x 2, x 3, x 4, x 5, x 6.

So, we will be using this explicitly. So, if we have 3 N number of total coordinates needed, we if we have N particles we know we need 3 N number of coordinates, then we will just use x 1, x 2, x 3 up to x 3 N. So, what we are going to do is instead of writing vector equations; we will just reduce this equation into scalar and we will sum it from i equal to 1 to 3 N, this makes more sense and we will be using this notation uniformly from now onwards. Unless, otherwise specified; this index runs over all required coordinates right.

So, we have 3 N number of coordinates to begin with. Now out of this 3 N, how many are independent coordinates. Please recall that if you have an equation of like C 1, x 1, I mean polynomial or a rather linear equation of the form C 1, x 1, plus C 2, x 2 plus C 3, x 3 plus C 4, x 4 equal to 0. And if and only if your x 1, x 2, x 3 up to x n they are all independent; only if all these are independent coordinates, then and only then you can say this equation implies C 1, C 2, C n equal to 0.

What I am trying to tell you is? I mean a linear. So, this is a very basic condition for linear dependence and linear independence in your linear mathematical methods course. I am pretty sure you already know how to define a linearly independent system of coordinates, linearly independent system of coordinates, what we are trying to tell you that although we have 3 N number of coordinates, this 3 N number of coordinates are not independent right, because in it is not a I mean the system is not free system has N particles which needs three N number of coordinates.

But at the same time system has K number of constrains. So, this K number of constrains means, we need 3 N minus K number of independent coordinates. This we have discussed already, but this formulation breaks, when we are discussing a rigid body. Frankly; this is this I mean why it happens because N and K both approaches infinity there. So, this relation is not valid anymore, but in general for a system of particles where this number N and K they are both finite we can always write the total number of independent coordinate for a system with which has a N particle; that means, 3 N number of coordinates and we have K constrains.

So, the total number of independent coordinate will be 3 N minus K. So, now, now we have a situation, we have an equation which looks something like this, but we cannot say that all the coefficients are individually 0; that means, I cannot say that for each pair I mean; each particle F i a minus pi is equal to 0 not that; the whole thing is 0, but not the individual coefficients. Let us say these are your x 1, x 2, x 3, x n like this and these are your C 1, C 2, C 3. So, this is your C i term and this is your x i term. So, because x is are not independent, as if I mean as in this equation; only if they are independent we can claim that all c is are individually 0.

Similarly, if they are not independent, we cannot say that this term is individually 0. The whole sum might be 0. So, there is a problem, what do we need in order to circumvent this problem? We need something called generalized coordinate. How many generalized coordinate doing we need? We need 3 N minus K numbers of generalized coordinates. And these generalized coordinates; so these coordinates actually need not be coordinates as a whole what I am trying to tell you that it could be electric charge it could be mass it could be I mean some independent quantity which is not dependent on any other quantity which is used to describe the system.

So, in order to what I am trying to tell you is, we need for a system with N particles and K number of constrains we need 3 N minus K number of independent parameters that the just to say; and all these parameters they might not have the dimension of a length typically or angle for example, it could have it could be electric charge it could be electric current it could be mass it could be you know what else; it could be it could be a field it also. So, there are many examples we will deal with some of these examples.

Now, if I can somehow reduce this equation to in terms of generalized coordinates then I can individually set all the coefficients to equal to equal to 0, what I am trying to tell you that? Let us say there are 3 N minus K number of coordinates. Let us call this number or So, we call them q 1, q 2, q 3 up to q n let us call this number small N right. So, these are the independent coordinates. Now they are not I mean it not that there has to be some connection right, we cannot just define this q 1, q 2, q 3, q ns just like that I cannot just say this is one coordinate that is one coordinate there has to be some relation with the existing non independent coordinate system.

So, let us assume these are our orthogonal coordinate systems either spherical polar or a standard orthogonal coordinate any of this. And we want to move into an independent set of coordinates, where I mean a set of coordinates which is at par with the number of independent coordinate required and they are given by this there has to be some sort of transformation equation which says r 1 is a function of q 1, q 2, q n; r 2 is a function of q 1, q 2, q n and similarly r 3 N is also a function of q 1, q 2, q n right there has to be a relation.

Let us assume that take an example. We discuss central forces; central forces means we need two coordinates right we see that there are system has which is moving I can show you systematically; we will do that later which is moving in a circle I mean in a planar orbit basically has two degrees of freedom and that is pretty obvious from the way we look at a system. And essentially we can we have chosen r n theta. Now initially; if we will consider free particle in a 3d space we need x and y and z. So, there must be some connection between x, y, z and r and theta right and we know how this the how this equations are there are transformation equations that we can always write x equal to r cos theta y equal to r sin theta and of course, right now we are not considering z, but. So, our z is always equal to 0, but this is also an equation, right.

So, that way these are the transformation equations we are talking about here. Similarly we will we can have reverse transformation equations we can express this q 1s in q 1 in terms of r 1, r 2, r 3 up to r n; q 2 in terms of r all the r s like that. So, there is a set of transformation and reverse transformation exist which connects between the standard orthogonal coordinates with the generalized coordinate the that relation has to be there I am not saying that it is important to have a dimension of length or angle for this q 1s and q 2s, but there has to be some relation they cannot be just totally disconnected thus relation is very important.

Now, turns out the D Alembert's principle although they are not independent of each other. So, we cannot equally I mean we can equate this coefficient to 0, but what we can do is we can still use D Alembert's principle is in his. So, right now please remember we are working in the dependent coordinate system before going into generalized coordinate we are working in the dependent coordinate system.

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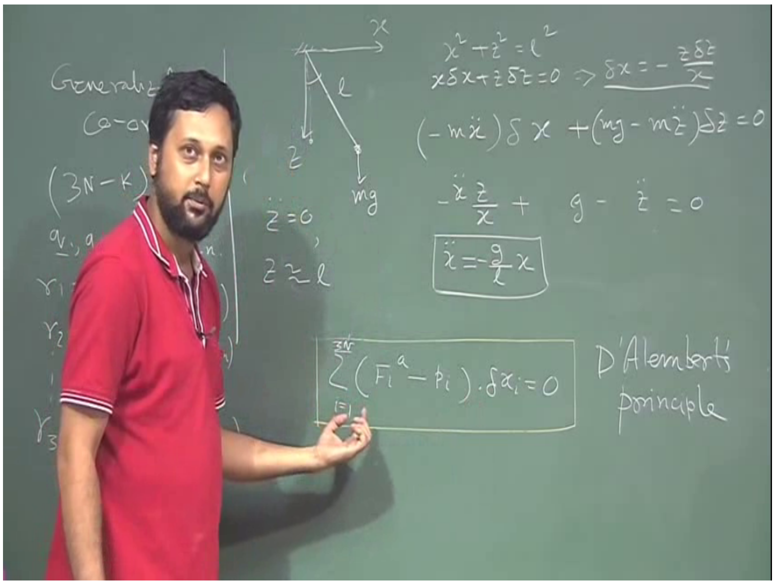

So, let us assume we have a simple pendulum right. Again, the simple example it is hanging there is a length l right. Now can I write the simple pendulum equation in D Alembert’s notation interns it turns out that we can of course, we have a constrain equation let us say this is my x axis and this is my z axis. So, x square plus z squared is equal to l square and if I now write the individual equations what are the applied forces.

So, applied force is m g and which is towards z direction is not it. So, what we can do is we can write the D Alembert’s principle like this. So, it will be see mg for the x component F f along x component is equal to 0. So, we can write minus m x double dot ha delta x plus for the z component it will be mg minus m z double dot delta z which is equal to 0 right; now this in this particular form this is what we get from D Alembert’s principle it is not extremely helpful.

But what we can do is we can use this equation and we can reduce this x; delta x plus z delta z is equal to 0 and this readily gives you delta x is equal to minus z delta z by x. So, what we are doing is as it is D Alembert’s principle is not very helpful frankly D Alembert’s principle just like this we can write a long equation which has all the terms in it if we have. So, if we consider a spherical pendulum we can do that in spherical pendulum we have to take another term which will have an y displacement as well.

Now, but it is not very helpful, but when we are using this constrain condition and please remember that delta and d has similar properties when we it comes for standard differential operation it is just a differential, but it is a virtual differential, because in a sense that if we sum anywhere if we find encountered delta t we uniformly put it equal to unanimously put it equal to 0. So, delta t is always equal to zero keep that in mind and just apply any standard normal derivative operation I mean; when we when we are using virtual displacement just use it as if it is a normal derivative plus please make sure that whenever you see an delta t put that equal to 0.

So, you do that and once we do that we substitute this for delta x we see minus mx double dot z by x plus mg minus m z double dot of course, m can also be taken out of th e system right delta z is equal to zero now if this is the case we can safely say delta z we can safely equate this whole thing is equal to 0, because now it is single parameter we do not have an equation where we have both delta x and delta z we have reduced it to a single equation with delta z and delta z and the coefficient of that can easily be equated to 0, if we do that we equate this equal to 0, right; so what do we get minus g z double dot is equal to 0.

Now, we have to assume we have to make this standard assumption that it is a I mean if we now want to see that the; if we want to reduce the it to our standard or equation for a a simple pendulum we have to assume that the displacement is small such that z double dot is equal to 0 and z is approximately equal to l the length; that means, the displacement is. So, small that the velocity in the z direction is almost negligible or acceleration in the z direction is almost negligible and the length z is almost equal to l. So, if we put that here what do we get we will get minus x double dot sorry x double dot is equal to g by l x and there should be a; that we can always check.

So, anyway; so we will get this equation or there should be a minus sign here you have to check the steps you can do that. So, finally, we will get back this equation if you make this assumption which is. So, you see D Alembert’s principle although as it is is not very useful we can always use the forces of constrain to construct and useful equation out of it also there is an way of there is a another way of doing it we can also use the force of constrains in a in a slightly different way we will do that in the next class. So, for now I will just summarize what we have learnt in this class in this lecture first is we have seen that virtual displacement can always be used in order to make the virtual work done equal to zero even for non holonomic sorry non even for rheonomics constrains.

Secondly we can write Newton's equation without involving the forces of constrain in this particular form then this can be used along with the constrain condition to get the standard differential equation in Newtonian form right the what I mean is the same equation we can get in the Newtonian mechanism also, but we have to consider this tension t right. So, we are not considered we are not bothered about this t anymore, but we can still get this familiar equation and also we defined some quantity which are called generalized coordinates which the number of generalized coordinates should be 3 N minus K which is the number of independent coordinates required and this generalized coordinates are related to the standard Cartesian coordinates by this type of transformation equations.

So, right now we have to stop here probably in the next lecture we will we will be able to derive the a Lagrangian; Lagrangian equation of first kind and second kind till then;

Thank you.