Classical Mechanics: From Newtonian to Lagrangian Formulation

Prof. Debmalya Banerjee
Department of Physics
Indian Institute of Technology, Kharagpur

Lecture – 43
Lagrangian Formulation – 1

In the first discussion on rigid body, I promise that we will come back and discuss constraints and the nature in a lot more details, now the time has come. So, now, we are moving slowly into the domain of Lagrangian mechanics we have done our bit of Newtonian mechanics for now, definitely we will come back to that as and when required. The concepts are very important of Newtonian mechanics concepts are extremely important we cannot forget it ever, but now it is time that we go little more into the details of Lagrangian dynamics and to begin with let us start with the forces of constrain.

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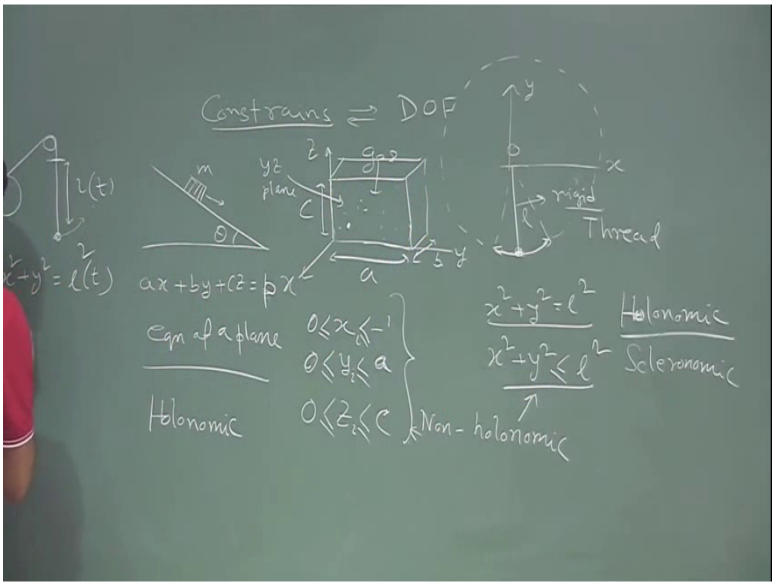

Now, constrain if you recall is anything that restricts the motion of an object could be a rigid body could be point particle anything which is limited by certain restriction is called a constrain. So, for example, if mass is a very familiar problem right the mass of m is sliding on an inclined plane theta. Now, it is because it is moving on this particular inclined plane, it is always there on that. So, the is a constrain of motion that this particular mass has to stay on this plane that is the constrain. I am standing on the ground that is the constrain I have I have to follow I mean if of course, parts can fly. So, they have additional degrees of freedom that is I think that is how you can understand this two.

The term constraint and degrees of freedom, they are in terms acronymous to each other. So, if you have more constrain then you have less degree of freedom if you have more degree of freedom you have less constrain. So, in this case, it is a very constrained system it is it is bound to move on this particular line. Also similarly, if we have a closed box and we have many gas particles in it. So, this is gas inside and we know this model this is the particle in a box model. We have solved this I mean we have learned this model during our discussion on thermal I mean in the thermodynamic courses. And these are also constrained in a sense they are allowed to move only within certain lengths let say this box is a 3 D box, so it has dimensions a, b and c. So, this gas particles are constrained to move inside these limits.

Another example, which is also very common is a simple pendulum, which is swinging. So, a simple pendulum can go here, go there, anywhere on this circle of radius l; l being the length. So, this is also a system of constrain, because it has to move, it has to stay at a length which is within this length l. Now, if we have a proper simple pendulum by definition a proper simple pendulum is not only that the length I mean it has to move I mean what I mean is the for a proper simple pendulum, this length or this connecting length l is a rigid rod. What I mean by this is the length is fixed. So, not only that it has to stay within this length, but it has to stay exactly at the length l from this origin o.

So, instead of rigid if we have a thread, for example, then in principle it can go anywhere in this circle, you know. What I mean is if I have a instead of a rigid let us say we do not have rigid, but we have a thread. Now, you swing it, you swing it such that it goes up here and then (Refer Time: 04:59) with energy all the energy and it falls. So, it can stay anywhere. So, the bob in principle can stay anywhere in this circle if it is a thread, but if it is a rigid rod it has to move exactly on this circle. So, all these are examples of constraint systems compared to a system of free particle where we have one particle which is free to move anywhere in the 3 D space, so that is a free particle and these are all systems of constrain.

Now, coming back to this examples are this can you classify this constrain also. We can have a situation where we have a pendulum for example, where there is a pulley and there is some kind of arrangement in by which the length of this pendulum L can be made a function of time. What I am trying to tell you that we can have I mean, so the pendulum can rotate you know it can execute oscillation, but at the same time the length can be changed there, if there could be some kind of an arrangement, we can do that. It is pretty easy we just have folding one side of the thread, for example, as the pendulum moves, so I am just shortening or I mean giving it more length we can shorten the length we can do that. All these are examples of constraint system compared to a system of free particle.

Now, there could be simple relation in case of certain constraint. But before going into that let us assume, let us talk about the classification of constraint. We have discussed four cases, where first case is a simple pendulum with variable length, second is an inclined plane, third is the particle or gas particle in box, and the fourth one is a simple pendulum which let us assume that it has rigid rod connected. So, these let us say let us assume that these four cases we are discussing. These are all constraints right. And due to this due to the nature of the different forms of constraint we can classify this system, depending on the nature of different, different types of constrain we can classify this system into different categories.

For example, this one, the constrain condition can be given in terms of an equation. What I mean by this is we can write the see any plane in a any arbitrary xyz, I mean any Cartesian xyz coordinate any arbitrary plane can be given by the equation ax plus by plus c z equal to 0 or d equal to no. It has to be equal to 0 right ax plus by plus c z it will be equal to 0, that is the equation of a plane. So, this will be the equation which can represent this constraint condition. I am sorry it has to be actually some constant p, it not equal to 0, not equal to 0 is a special case of a plane; in general, it is ax plus by plus c z equal to p right. So, this is the equation of plane. So, in this case, the object is moving under certain constraint condition which can be put in form of an equation.

Similarly, here also when we have a rigid rod, we can write this equation for any let us say we set a Cartesian coordinate x and y here at the same plane in this 2D plane where the pendulum is executing oscillation. And for any arbitrary orientation of pendulum, we can write x square plus y square is equal to l square that represents the constraint condition. But this case can we actually write the constraint condition in form of an equation? The answer is no. What do we can say that for any particle any gas particle any arbitrary gas particulates let say this particle. So, if I set my origin let us say my origin is here all depend again all depends on how you set your origin. So, this is my x, this is my y, this is my z.

So, basically this surface the surface which of the box which is facing us is the surface, basically is the y z plane. If this is the case, so this is y z plane this surface. So, if this is the case then we can write the constraint condition for this particular particle to be x has to be limited between 0 and this is a right y or sorry y has to be limited between sorry this is x, so sorry my mistake. So, x has to be limited between. So, if this is b, so it will be 0 to minus b yeah again it all depends on where you choose your origin. If you shift the origin slightly instead of minus b you can have plus b also if you just shift it to the back end of the box. We are not doing that just to give you illustration of what we are doing, we will just stick to this.

So, y on the other hand will be limited to 0 and a sorry 0 and a; and z will be limited to 0 and c right. So, these three sets of equation and this is for any arbitrary particle ith particle with coordinate xi, yi, zi instantaneous coordinate and this ith particle could be anything. So, all the particles inside this box gas particle has to follow this three sets of relation I would say these are not equations, but these are inequalities, but this inequalities defines the constraint.

Now, here for this particular case, we can similarly write x square plus y square equal to l square plus please remember l is a function of time itself. So, in this last or in this case we can still write an equation, but which explicitly includes time because length of the pendulum itself is changing with time. So, we see that and also again if I replace this rigid rod with a rigid string or sorry with the flexible string what I mean is that was the not string I would like thread just to avoid any confusion. Now, if we do that, once again we will get back an inequality right, oh sorry here should be equal sign on as well right. So, if we just replace this rigid string, a rigid rod by a thread we get back this inequality. So, it could be anywhere your length total length could be anywhere less than equal to l.

So, now let us assume I mean let us try to classify the constraints depending on this properties. This one and this one, these are equations; and whichever constraint condition can be expressed in terms of equations, we will call them the holonomic constrained I should write it somewhere I do not know, but let us write it here holonomic and this one is also holonomic. h o l o holonomic n o m i c holonomic right that is the spelling. So, these are called holonomic constraints which can be expressed in terms of equation. This and this for example, these are not equations these are inequalities and the constraint condition, which are not equations in this particular I mean in this form or we can so we will give more examples of that, but let say this is one case which are non-holonomic; this and this.

Now inside, so we have two basic broad classification of constraint; one is holonomic which can be put in terms of equations; another is non-holonomic which can cannot be put in terms of equation in this particular integral form or integrated form I would say. I will explain what is that in a moment. And inside this holonomic constraints also there could be two types of cases; one where the equation does not in I mean the constraint condition does not change with time; and the one where the constraint condition can change with time.

So, inside this holonomic case also we have two branches one is where constraint condition is time independent, these are called scleranomic. So, let us write it here, sorry I will just check the spelling scleronomic constraint, I am not sure about the spelling I mean I hope I will rectify myself in the next class. So, this one on the other hand is changing with time and it is called rheonomic. So, we have four types of or rather two broad types of constraint one is holonomic, another is non-holonomic. Inside holonomic we have scleronomic and rheonomic. So, all in all there are so this is we have a scleronomic case, here we have rheonomic case here.

So, there are four broader classes of constraint. Now, for most of our discussion will be sticking to holonomic constraints; we will take off maybe one or two examples where we will be dealing about non-holonomic constraint. Sclerohmic constraint is something they were we will focus on; rheonomic constraints we will not touch in this discussion. So, we will just leave it like this. Of course, we will do a general formalism in the next few lectures we are going to do a very generalized formalism, which will also include some cases of you know rheonomic cases, but we will not you know in general we will not bother about much about this.

So, this is where we stop in terms of constraint, of course, we will be you know visiting I mean we will be repeating these terms holonomic, non-holonomic, rheonomic, scleronomic a number of times. Now, we stop here and next let us move back to Newtonian mechanics once again. Now, just (Refer Time: 17:00).

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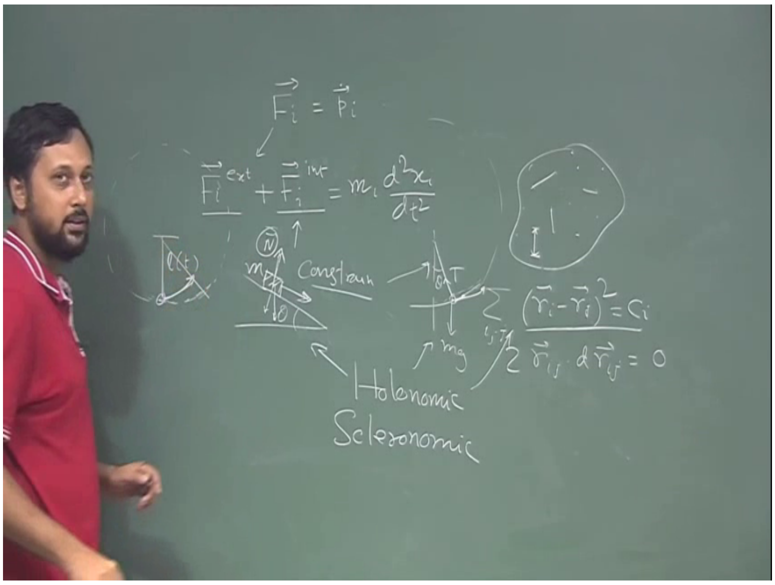

So, in Newtonian mechanics, we have an equation which is of the form F i is equal to p i F i is the sorry pi dot; F i is their total force on ith particle and p i is the momentum of the ith particles we are all familiar with this. Now, this F i has two parts to it. One, I think we have already discussed this particular form that we can write F i as F i external plus F i internal and this is equal to p i. So, this is actually a vector equation. So, this is equal to p i dot. I think we have discussed this when we were discussing systems of particles. We broke this total force on ith particle into two parts; one is due to the external force one or a applied force, another one was due to the internal forces.

Now, if we rewrite this equation F i minus p i dot is equal to 0, we can do that I mean it is just taking it into one side. Now, if we consider this the both sides as a dimension of force. So, one is the applied force; one is the so one is actually the acceleration due to change of momentum due to force which also has a dimension of force. Now, for ith particle if let us say this total displacement of ith particle it is an infinitesimally small displacement which is given by delta or rather d ri, if I multiply that with this one we can easily write dot dri which will be equal to 0. And there is no harm if I explicitly put the sum over all the particles, I mean here what we are doing is a pure mathematical manipulation, I am not saying there is no new physics involved in here not right now we will see that there are things which will come very handy, we will start from here.

So, essentially what we did is we took the Newtonian equation for ith particle or a system of particle, actually this is also valid for a system of particle. So, we took thus this for a system of particle just wrote it into this particular form. Keeping in mind that f has two components; one is the external force and one is the internal force. Now, we have very interesting property of these internal forces, which will explore. This internal forces most of the time these are the forces or yeah, so there could be two types of internal forces. One is the dissipative forces which are called so called frictional forces or viscous forces, viscous force is also kind of frictional force which could cause power dissipation. And the other type of force which can act internally is the forces of constraint.

What I mean is so for example, again let us go back to this case of an inclined plane and we all know how to write this force equation we have a mass m here. So, there is a mg sin theta component, there is an mg cos theta component which has to be equated with the normal reaction n from there we can get an angle of angle the value I mean an estimate of the normal reaction. And of course, we can calculate the acceleration into in this particular direction. This n is purely coming from this internal effect. And if there is any friction present which will act opposite to the direction of motion that is also an internal force. Let us assume that there is no friction at present. Let us assume that there is no dissipate dissipative forces at present, we will add them I mean once we you know go ahead with our formalism and then we get to the final form, we can add them later on we will see that.

Right now, let us assume that there is no such frictional force. So, the only force which is internally acting on this system is this normal reaction m and this is purely due to the force of constraint. Now, question is how do we integrate these equations. Finally these are the starting points in order to gain the insight we have to integrate these differential equations. And finally, we are willing to get x i as a function of time this is our final scale. Once we know the position of each and every particle of the system as a function of time, then we can say that we have solved the problem completely and we can get gain this by integrating this equation which is the right hand side is nothing but the second derivative of position vectors we all know that.

Now, in order to integrate this equation in Newtonian mechanics we have seen that. So, many times though to you have to somehow you know get the details of these forces of constraint in this case n has to be equated with one component of weight which is in this particular direction, which is mg cos theta in this case. So, from there we know that amount of normal reaction is created and then we can you know essentially get rid of this force and then we can integrate.

Similarly, if for a simple pendulum, if this is my mg and this is my theta, once again there is a tension t like this normal reaction n, this tension t is also a part I mean also generated by the forces of constraint. So, once again this is the constrain and this t is generated due to that. We have to get rid of this constraints explicitly in order to solve the Newtonian equation. Question is can we go into a set of we can go into a formalism where there will be no forces of constraint to begin with, is it possible? Answer is yes, we can get rid of these forces of constraints, and we can write equations, which will explicitly include only the applied forces. I hope you understand the problem.

The problem is, forget about this for a moment this is not something we are discussing right now we are discussing this. And right hand side is mi d 2 xi dt square, we have to integrate this equation for let us assume for each individual particle. We know all the details we know the forces of con constraint or let us say there is only one particle in the system even then we have to we need to know the details about this internal forces. Without knowing this internal forces we cannot explicitly integrate this equation and of course, we have taken two examples which are very simple and we know exactly how to get rid of this internal forces. So, we are good, but not always we are so lucky.

Sometimes for example, we do not know the internal forces explicitly, but we know their effects. Take an example of a rigid body. What is a rigid body? Rigid body is a system of particles we have many particles in accumulated in one object, and the property that makes it a rigid body is the mutual distance between any two particles in a rigid body is fixed. So, in this case, we do not know the constraint, but we know that effect. We cannot say what is the magnitude of or probably we do not bother about the magnitude of force which is acting between these two particles or I mean in principle we can know if we know how to define this two point masses. If we first of all sometimes rigid body is a continuous system we cannot we can assume that it is a in a you know collection of point masses, but we cannot give you a proper dimension of this point masses it is not possible because it is a continuous system most of the time.

So, it is very difficult to measure the actual force between two actual particles, but all we know that rigid body is rigid because all these forces of constraint will keep this I mean keep this distances mutual distances between any two particles constant. So, we can write the constraint condition for example, as r i minus r j square vector of course, yeah r i minus r j square is equal to some c i. So, this can be written. And although this is something we can write we can never measure the force exactly. Most of the time we cannot and most importantly we do not have to. What is important is a property I mean once we write it in this particular form then we can you know manipulate it slightly and I can show you we have done it already that the work done by all these forces of constraint for a small displacement is 0.

How, just take this, just take this, take a derivative of this then it will be two times if I write this as r ij it will be r ij dot d r ij equal to 0, take a derivative. So, if d r ij is the small displacement between these two particles and r ij is a vector which is pointing we know that the force nature of force is central. So, r ij is along the direction along whose force will act this is equal to 0 so; that means, they are always and of course, what we know for sure that f ij will always be in the direction of r ij. So, if r ij dot d r i equal to 0, f ij dot d rij will also be equal to 0 we have proved that in a more formal way for system of particles we have proof that without going into this without writing this equation. We have shown you that system of particles the internal work done due to or that combined work done due to all the internal forces is 0. Of course, there is a please remember there is a sum over i and j. So, you have to include that into this calculation and then it will you can immediately see that this is 0.

Now, similarly in here also we can write we can see immediately that the only possible motion is along this particular direction, and the forces of constraint is perpendicular to the direction of the motion. What I mean by direction of motion is the allowed direction under this particular constraint. If I lift it up from here then this constraint has no meaning; if I am forcing this object to stay on this particular inclined plane then the forces of constraint is normal to the direction of motion. Similarly for a pendulum,

pendulum works I mean the pendulum works stays on this circle yeah this big circle and this circle at any point the direction of motion is tangential to this circle, and the forces of constraint is along the radial direction. So, these are perpendicular to each other.

So, what we see that in this three cases all these three are holonomic cases. Please remember that all these three examples are examples of holonomic constraints. These two we have shown you how to write the equation previously this one is the equation of constraint for this system rigid body and these three all three are holonomic systems. And in we have seen by taking three examples of course, there cannot be a general proof of it or even if there is I am not familiar with it that. But what we see here is the forces of constraint is always perpendicular to the direction of motion or rather I would say the statement is the forces of constraint in all these three systems does not do any work. I think I can I can prove it to yeah I have I have proven it to you.

If you are not clear about it then you can always write back to me, I can give you more examples and most importantly please read any of the standard text books you will get a better understanding on this. This is very important to understand that for holonomic systems, we have shown you explicitly that forces of constraint do not perform any work. Now, what happens in case of non-holonomic system there is a different issue all together. Non-holonomic systems we are not dealing, but also there is catch here, this is not the systems are holonomic, but it is holonomic plus scleronomic. So; that means, none of this cases in that the constraint condition does not change with time. So, these are fixed constraint the slope if the angle of the slope is changing with time, I mean this is something which is because if the object stays on the plane any time if the angle is changing the per normal will be perpendicular to the direction.

But take the example of this pendulum again. Let us say the length l is a function of time when it moves let us say it for this particular instant of time, it should move on this particular direction or this particular circle. Now, you are shortening the length as we go, so it will move in a path, which is not following this circle. So, any point on this path the displacement will not be perpendicular to the force of constraint which is the tension t right, tension is along this direction along this direction. So, t and displacement, the displacement and t is not perpendicular at as the case of a static pendulum I mean static means I mean the where the length is not a function of time.

So, what we have shown you is the case of holonomic plus scleronomic constrain. We have not used, we have not discussed the case of rheonomic constraints, but it turns out that rheonomic constraints can also be taken into account. So, we will stop here today in this lecture. We will continue with our discussion on rheonomic constraints and we will discuss something called the virtual displacement that is in the next lecture

Thank you.