Classical Mechanics: From Newtonian to Lagrangian Formulation
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Lecture – 42
Rigid body dynamics - 16

So, we have already discussed the general motion of symmetric top. We have checked how does the motion be; I mean, how does the tip of the axis of the top behaves traces out? What are the traces, when we have 3 different initial condition? So, what we did essentially let us say this is the bottom of the top this is the axis of the top, so, we traced this tip of this axis as a function of time and of course, we understand because this length is a fixed, it is a rigid body if the wherever it does it has to stay on a sphere of radius equal to the length of this axis.

So, whatever traces we have seen, it is on the, you know on the tip. So, in principle you can solve this problem in few other ways also.

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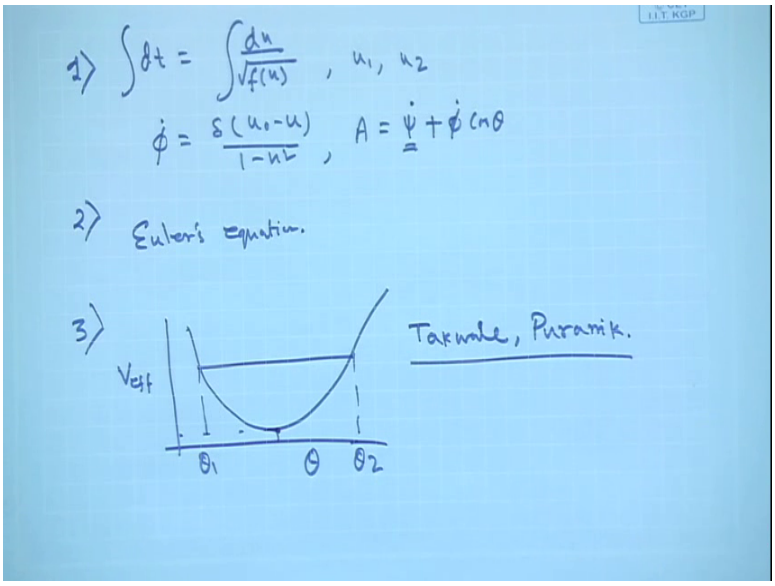

First of all numerically it can be solved by you know performing the elliptical integral; by elliptical integral like what we wrote in the last class like, something like this: Integral of dt is equal to integral of du root f of u. So, we can find out, we already know you know from the, just by plotting fu in a computer simulation we can find out 2 physical roots u of u 1 and u 2, we can do that and then numerically solving this integration is not so, difficult.

So, this is one condition what we can do and once we know u we can solve for in the second equation which is delta u 0 minus u into 1 minus u square, u 0 being the initial condition. Once we solve this, one can be solved and of course, we have A is equal to, what was it, phi dot sorry psi dot plus phi dot cos theta I think just let me check once, we always tend to forget this expressions. So, A will be, yeah phi dot cos theta phi dot cos theta plus psi dot. So, if we have theta and phi then from this expression we can always calculate phi so; that means, in principle, the equation can be solved. This is one way of doing it numerically.

Second way is you do not go into all this; you start from the original Euler's equations. You have 3 sets of Euler's equations, you set some initial condition; 3 first order differential equation, which can be reduced to second order differential equation by help of Euler angles and these 3 equations can be simultaneously solved by Runge-Kutta second order algorithm. So, that is also a doable thing, actually, I would say that is numerically less challenging compared to you know solving this elliptical integral because elliptical integral has many typical properties which might give you a very problematic approach. So, these are numerical solution.

The third approach is, there is a another way of looking into the problem physically that we can essentially reduce this problem in an equivalent 1 dimensional problem in an effective potential, which is, effective potential with respect to theta, like we did for the discussion of central orbit we reduced the problem in an effective potential of r. So, we can do the same thing here. We can find some potential energy surface, where theta 1 and theta 2 are 2 turning points of this particular potential energy surface, for bound motion and we can always see that there is a physically possible root situation where theta 1 and theta 2 coincides anyway.

So, this is a treatment which you can find in the book of Takwale and Puranik that is a very nice Classical Mechanics book by Takwale and Puranik which is also I guess listed in I am not sure if it is listed in this course, but you can always look into it. These are very easily available books. So, this treatment is there if you are interested you can go through it.

Now, what we are going to do is, I will just show you a short movie which I have taken from a YouTube site, it is an educational site so, I hope they would not mind because if I show you some movie for the education.

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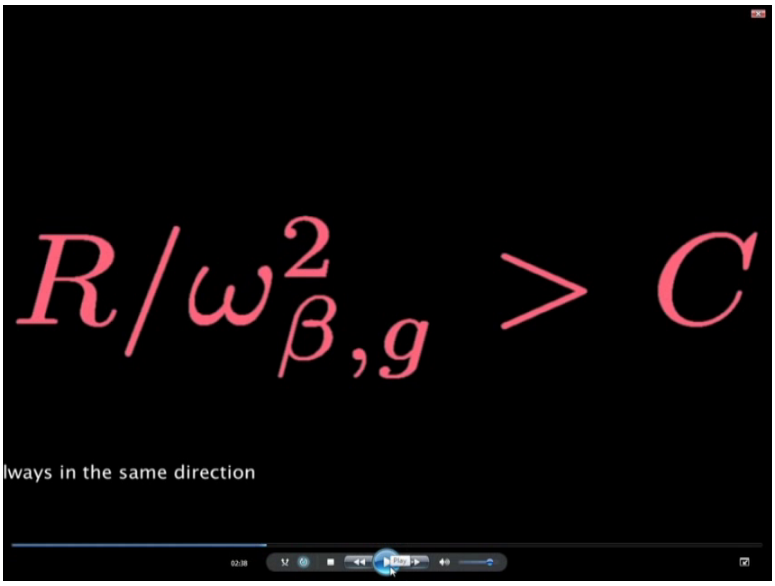

So, here what we are going to do is, I am going to show you the motion of a gyroscope play this.

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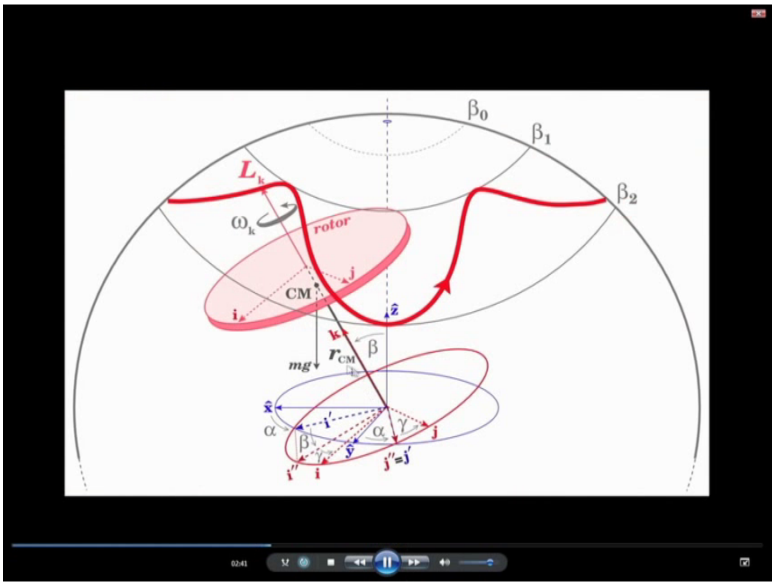

And their initial condition is slightly different; do not go by the equations. So, the first case is this.

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So, this is a gyroscope which is moving, I will just pause it for a second. So, what happens is, in a gyroscope there is this rotating bob. We can have a small motor driving this bob and this bob gives you the spin motion; please understand that, whatever may be the case we need to have a very good amount of spin to continue this motion. So, this gives you the spin motion, the rest gives you the. So, the gyroscope is actually you know it is an arrangement in which the spinning object can stay, I mean can move in a vertical axis by you know between angle 0 and 180 degree just like the nutation. So, it is a freely nutating system, if it want it can go anywhere and of course, it can spin around its own axis.

So, there is a very nice discussion on I mean very short and very nice discussion on gyroscope and how it works in the textbook of Spiegel - Theoretical Mechanics. So, you just I suggest you please read that. Now, here we have a taken an example of symmetric top; gyroscope is also an example of a symmetric top like child's top. Good thing about gyroscope is, in a child's top we can have range of theta between essentially between 0 and 90 in principal. Of course, 90 is not the recommended case, because at 90 the top falls on the ground because it is rotating on the ground. The good thing about gyroscope is spinning object you know, this is the axis and that the bob is spinning, it can go up and down in full 180 degree rotation and also it can move freely in the axis. So, that is the mechanical arrangement.

So, because it has this 180 degree rotational freedom we can actually have a physical range of theta anywhere between 0 and 180 not like in a child's top which is limited between 0 and 90 degree.

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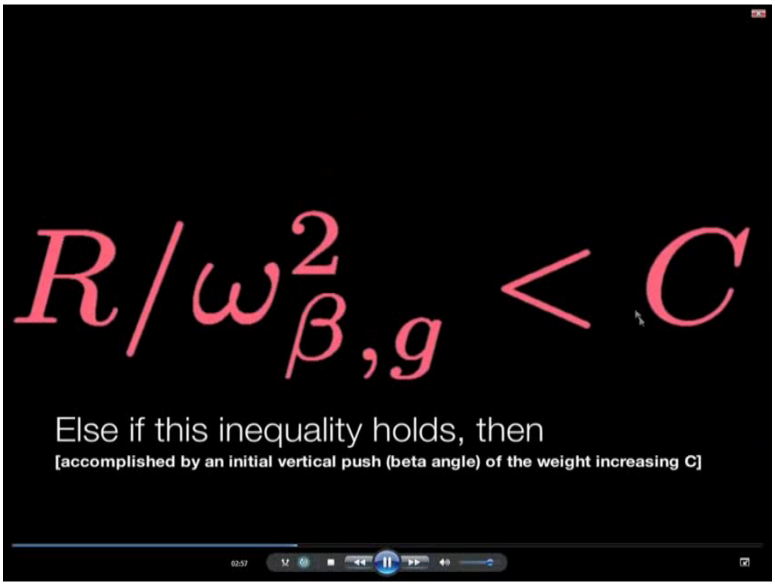

So, let us focus. I will slightly go back. So, what they have given here is the set of initial condition which will give you the more. Of course, their equations are slightly different do not go by the equation, but this initial condition means this type of motion where your theta 0 is higher sorry lower. You see we have positive velocity and both the cusps; of course, there is a damping in the gyroscope like any physical system.

Now, this is the situation where we are starting in between the 2 limits. Initially in the previous condition, we started above the 2 bounds now we are in the middle and it will have a positive velocity at one axis, negative at the other axis. This is how you initiate the motion once again, this is how you initiate the motion you have a positive velocity at bottom and negative velocity at top for precession and now when you start from one of the limits.

So, you have a cusp motion we see in the very next moment that there is a cusp motion where it is 0 at this point, positive at this point and this can be implemented like this. So, this video is found in YouTube, I will also share the link, I hope they will not mind if I use it in my lecture.

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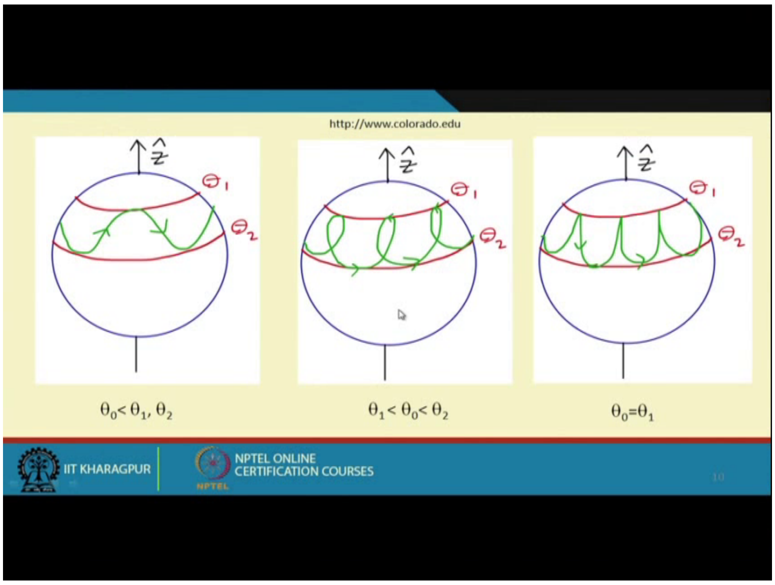

So, now let us move into the. So, these are the 3 motions we have demonstrated using gyroscope. Now, let us move into the more specialized cases of sleeping top and fast top.

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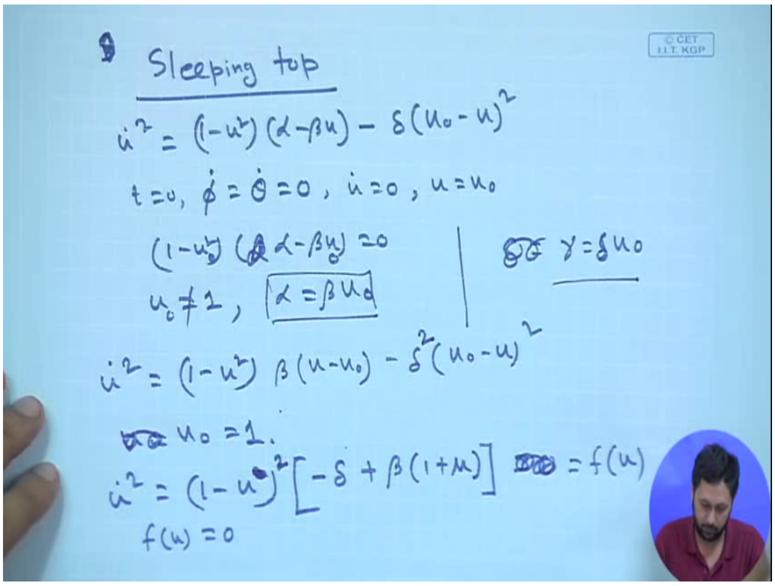

So, let us first go to the case of sleeping top. Let us start once again with this equation u dot square is equal to 1 minus u square, alpha minus beta u, minus gamma u 0 minus u whole square, u 0 being the initial condition.

Now, let us assume that we are starting with an initial condition that at t equal to 0, phi dot is equal to 0, that we have already seen and also we are taking theta dot equal to 0;

that means, at u equal to 0 u dot is equal to 0. So, if you put this into the equation, what we get? What we get is, the second term unanimously vanishes, so at and u equal to u 0. So, the second term immediately vanishes and the first term will give you 1 minus u square, alpha minus beta u equal to 0. If u 0 is not equal to 1, then we get alpha equal to beta u 0.

So, if you put, sorry it has to be u 0 alpha equal to beta u0. If you start with an initial condition when phi dot is equal to theta dot is equal to 0 and u equal to u 0, you end up. We have already seen that there is a condition of delta equal to, sorry, gamma equal to delta u 0, also we get alpha equal to and that is when only phi dot is equal to 0, but if we have additionally theta dot is also equal to 0, then we have alpha equal to beta u 0. So, employing this 2, the equation becomes 1 minus u square. So, alpha is equal to beta u 0. So, put beta outside u minus u 0 minus delta u 0 minus u or you can also write it is u minus u 0 does not matter. So, this is the condition we have.

Now let us consider the case of sleeping top; sleeping top is when u is equal to u 0 yeah sorry u 0 is equal to 1. Now, if you put u 0 equal to 1 in this particular equation what do we get. We can substitute u 0 equal to 1 and the equation becomes u 2 dot is equal to 1 minus u square see if you put u 0 equal to 1, so this will be u minus 1 and this will be u minus 1 whole square. So, it will be sorry; if you take 1 minus. So, u minus one whole square common what you are left with is minus delta plus beta.

So, this can be broken into 1 minus u, 1 plus u and if u 0 is equal to 1, then it will be 1 minus u whole square multiplied by this term will be 1 minus u whole square multiplied by 1 plus u. So, this is the equation. Now, if once again we try to look for the turning points of this particular equation, we have a new fu. Once again if we set fu equal to 0 and try to find out the turning points of this particular equation what we get?

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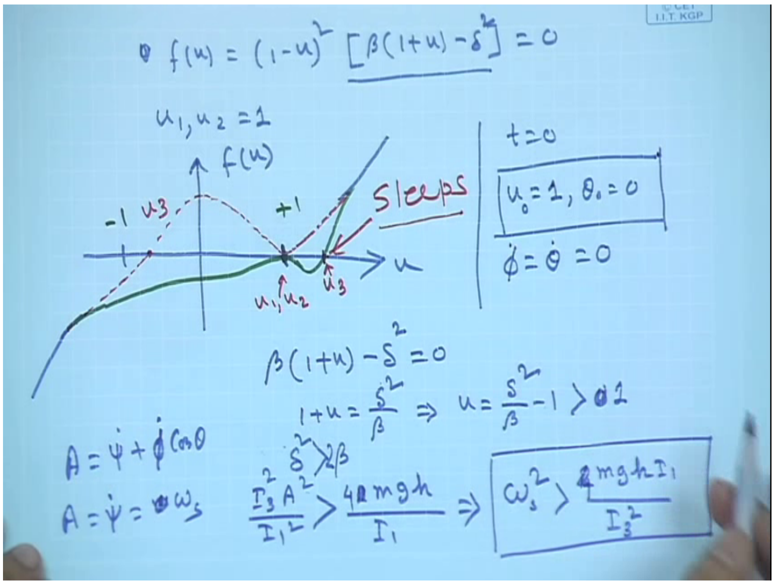

So, the functional form is f of u is 1 minus u whole square, beta 1 plus u minus delta. If I set this equal to 0 in order to find the roots, we have to see that it is a third order polynomial once again; because, we just manipulated with the parameters, we have not changed the order of the equation. So, it remains a third order equation. Now, for a third order equation we already have 1 minus u whole square outside; that means, 2 roots u 1 and u 2 is already given to be equal to 1.

For the third root we have to solve this bit of equation which can give you a third root. We can do that, if we try to do this. So, we have this situation once again. It is plus 1, this is minus 1 and the general trend continues I mean we again we just wrote this function f u in a slightly different manner under certain initial condition. The general trend will not change. It will once again go to positive infinity, so, the leading term remains u cubed. You have a u square from here and u from here, so, the leading term remains u cubed.

So, we have this trend that it goes to minus infinity for u equal to minus infinity positive infinity for u equal to positive infinity in between now we see that we have 2 roots u 1 and u 2 concentrated at 1 and what we are examining is the case of sleeping top. Now what is a sleeping top; by definition a sleeping top is a top in which there is no nutation. So, if we let us say if we start with this initial condition of the top you know the axis is vertical like this and if we just release it at time t equal to 0 from here it should stay like this. So, this is called a sleeping top so; that means, the sleeping means there is only 1 motion possible, that is the spin motion, that is psi dot.

So, now let us come back to this. We have 2 roots here; 2 roots concentrated at u equal to 1. So, this is possible. There are 2 types of extrapolation in between plus 1 to minus 1 possible. So, one is, that is a special case, because in general case we have negative values at both u equal to minus 1 and plus 1, but in this case it is a special case because we have taken u 0 equal to 1, that is why we have 2 roots coinciding at u equal to 1. So, we can either have a situation where the roots are like this. See, what happens is, it is touching the u equal to 0 axis, exactly at plus 1.

So, touching means and then it goes down and intercepts here once again at u greater than 1. This touches here, this touch here means it is this, there are 2 roots which are coinciding at that particular point. On the other hand, we can have a situation which is like this, so, it crosses 0 then it comes back touches here. In both cases we have u 1 u 2 here; in this case u 3 is here the third root and in this case u 3 is here.

Now, which one of these is the possible solution or possible I mean which one of this is what we are looking at looking for rather. We are looking for this solid line, not the dotted line, why because, if the dotted line is the possible shape of fu then we have a root once again which is between u equal to minus 1 to plus 1, so that means; cos theta between 180 degree and 0 degree and 180 degree. So, that is a physically possible I mean this is something that is not an imaginary root, it is a physical root.

Now, for a physical root to, I mean if we have another physical root in this physically possible range of motion, then the top cannot stay vertical. So, if we start the top at this particular position and we have a u 3 or theta 3 which is somewhere here down here, so, it has to come down, explore that root and go back. So, that way the top will not sleep anymore. In order to the top to sleep, we have to have the third root beyond u equal to plus 1, that is not a physically possible root.

So, that means; if we have this as the solution, the green curve as the solution to f of u, only then we can have possibility where the top actually sleeps. Because, this root is not physically accessible, this root is physically accessible. So, the sleeping top will sleep if and only if the third root is physically not accessible and if that is the case then we can write beta 1 plus u minus delta. So, if we try to get the root from here, this will be equal to 0, so, that means; 1 plus u will be delta by beta, which gives u equal to delta by beta minus 1, which is has to be greater than 0.

So, if this is greater than 0 or sorry this is greater than 1 then delta has to be greater than beta. So, this is the condition and if we now put the values of delta and beta, I hope I have it somewhere sorry I have so many papers around here, I will still try to find out, I have it. So, if I put all this, essentially we find out the value of delta to be I 3 A by I 1 greater than beta; beta is 2 m g h by I 1 which gives you a condition that it will be sorry this will be I 1 by I 3 or no I think, I am missing a square here somewhere, just a second just give me a second yeah. So, it has to be a delta square, I missed it.

So, in the first equation itself I missed it that is why yeah. So, it has to be a delta square. Delta minus beta u it has to be a delta square I guess yeah. So, this is delta square. Delta square greater than beta, so, that means; I 3 square A square I 1 square greater than beta. If this is the condition, then we have a situation.

Now, recall that A dot, A is what? A is equal to psi dot plus phi dot cos theta. Now, in this case phi dot equal to 0, cos theta equal to 1. So, in this case A is equal to psi dot only, so, that means; that is the; I would call it omega spin, I have introduced this before. So, this is the angular velocity of spin. A square is nothing, but the angular velocity of spin. We write omega s in case of A square and we get omega s square greater than 2 m g h I 1 by I 3 square. So, this is a condition, if I am not really wrong this is the condition we should get. So, again I am missing a factor of 4, of course, I am. Because, it means yeah, just give me a second yeah it means, it will be greater than 2 beta. So, there is a factor of 2, the 1 goes there, delta square; basically it will be delta square here, delta square here. So, delta square by gamma beta greater than 2, delta square greater than 2 beta, so, there is a additional 2 factor here. So, this is the final expression.

So, in order to have a sleeping top we should have the spinning speed; having u equal to 0 or u 0 equal to 1; that means, theta 0 equal to 0 is the initial condition. So, initial condition is theta 0 equal to 0. This is the must condition, also we have phi dot is equal to theta dot equal to 0 at initial phases, so, that means; it has to be released. So, let us say this is the top, this is the axis of the top; let us say this is vertical assume that, it has to be released exactly with a I mean very high precision, so, that means, there is no phi dot and theta dot triggered. Of course, it has to be exactly vertical, in additional to this to initial condition the spin speed has to be greater than 4 I mean spin also has to be a certain limit which where w s square has to be greater than 4 m g h I 1 by I 3 square.

So, this is the condition for a sleeping top in order to. So, if all these conditions are met, if you release your top exactly at the vertical position with a sufficient spinning speed, it will stay like this forever. But in reality what happens, see in this entire discussion we have always neglected the, what you call the friction. The point where the top meets the ground, of course, there is a finite amount of friction, now that friction and of course, there are air resistance and other effect that will slow down the top essentially it cannot run forever.

So, at some point the angular velocity even for a sleeping top if all the conditions are met for a sleeping top it will start losing rotational I mean it will start losing kinetic energy and the angular velocity will fall below a certain limit and the top will start moving from its sleeping position it will start moving out. So, this is one case.

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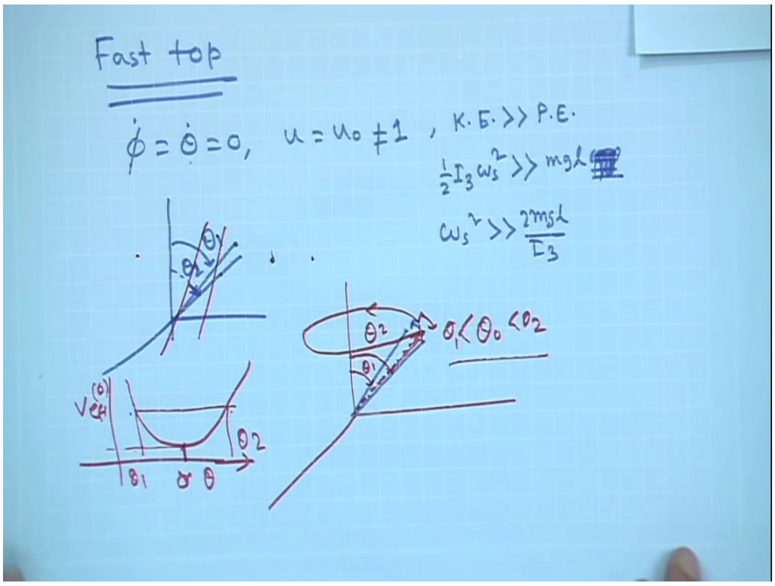

The second case is the fast top. So, this I am not discussing in details.

So, in fast top once again we have this initial condition let us say phi dot is equal to theta dot is equal to 0 and we start initially at an angle u 0 which is not equal to 1. So, if this is the case then it can be shown I am not doing it that your precision and of course. So, why it is called fast top? Fast is, if the kinetic energy is much greater than potential energy.

So, once again this maps to a condition where your, see what is the kinetic energy if the phi dot equal to theta dot equal to 0 ? The only possible motion is spin. So, your half I 3 omega s square has to be much greater than m g l cos theta and cos theta well we can just avoid the cos theta for now because you know it is something which is varying between 0 and 1 anywhere, it is not a large number.

So, this particular condition will give you omega s square much greater than 2 m g l by I 3. If you start from this condition; this condition is met. It can be shown easily with some simple calculations just take any standard textbook and follow the derivation. What happens is, the limits of precision or limits of nutation motion which is given by theta 1 and or theta 2 and theta 1 rather. I will just draw it again maybe sorry this is not a good drawing. So, this is the axis let us say it is executing some nutation in this 2 regime where this is your theta 2 and this is your theta 1.

So, it can be shown that if this particular condition is met, the oscillation between the nutation motions is a simple harmonic in nature and the simple harmonic motion is so fast sometime, that it seems like it is not executing any nutation at all. So, what I mean to say is the movement between this 2 limit is very fast, so fast, that if you look at with your naked eye it seems like that there is no movement in nutation, no nutation movement at all and you feel like that the top is doing only precisional motion with an average value theta 0, which is somewhere between theta 1 and theta 2.

So, this is why it is called a fast top. Do not confuse it with possibility I mean do not confuse it with a possibility, if you recall you just we just drawn the energy landscape. Let me find it out for you. Where is it? Sorry, I do not find it. If you recall that I just draw the energy I mean if the effective potential V effective versus theta, I said that it can be reduced to a equivalent 1 dimensional potential, V effective of theta and theta and. So, bound motion given is given by theta 1 and theta 2, but there is a possibility where these 2 limits exactly merging into 1 location. This is where we can have a true nutation free motion. it is something like similar to the condition of circular orbit. In a central orbit we have 2 bound motion between 2 limits r 1 and r 2 and we can have elliptical orbit.

But if the motion is confined to a single point, I mean if these 2 limits coincide with each other then we can have a circular orbit. So, something similar can happen here, but that is a different case. So, we are not doing any mathematics for this fast top case and now let us do let us follow the problem set, let us do some problem. So, we have 3 more problems in the set.

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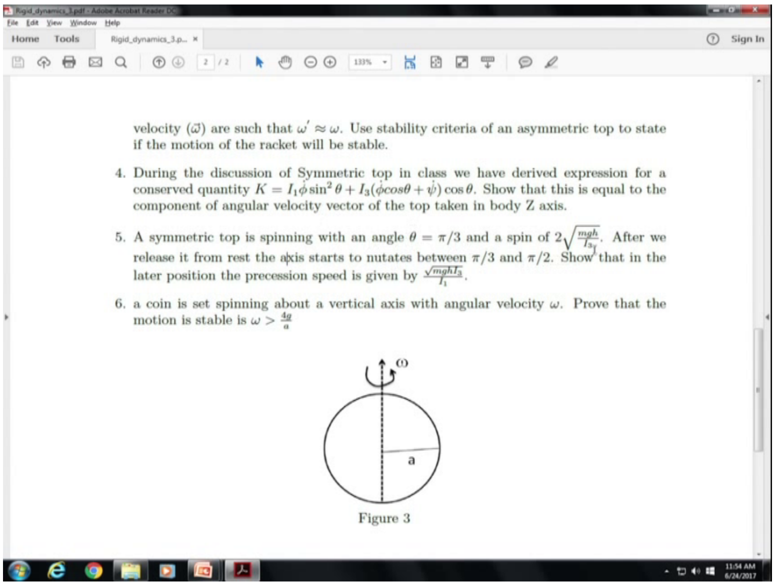

velocity ($\vec{\omega}$) are such that $\omega' \approx \omega$. Use stability criteria of an asymmetric top to state if the motion of the racket will be stable.

4. During the discussion of Symmetric top in class we have derived expression for a conserved quantity $K = I_1\dot{\phi}\sin^2\theta + I_3(\dot{\phi}cos\theta + \dot{\psi})\cos\theta$. Show that this is equal to the component of angular velocity vector of the top taken in body Z axis.

5. A symmetric top is spinning with an angle $\theta = \pi/3$ and a spin of $2\sqrt{\frac{mgh}{I_3}}$. After we release it from rest the axis starts to nutates between $\pi/3$ and $\pi/2$. Show that in the later position the precession speed is given by $\frac{\sqrt{mghI_3}}{I_1}$.

6. a coin is set spinning about a vertical axis with angular velocity ω. Prove that the motion is stable is $\omega > \frac{4g}{a}$

Figure 3

So, the first problem is what we have already solved. You have to show that the constant K is actually the z component. Show that this is equal to the component of angular velocity vector of the top taken in the sorry it should be space Z axis, I will correct it. So, it should be the angular momentum component in the space Z axis, which we have already done. So, that is no need for that.

Now for the second problem; it is a symmetric top spinning with an angle theta equal to with, it should be an initial angle sorry an initial angle theta equal to pi by 3 and spin of, spinning speed omega s is given. After we release it from rest, the axis starts to nutates between pi by 3 and pi by 2. Show that the later position the precession speed is given by this particular value.

Now, you have to be careful in order to solve this problem. It seems like initially that there is not sufficient information given, but let me tell you sufficient information is given here. Look at this; your initial release position is theta equal to pi by 3 and the top nutates between pi by 3 and pi by 2 so; that means, your initial release position is coinciding with your upper limit of the precession motion, so, that means; u 0 is equal to u 1.

Now, if u 0 is equal to u 1, let me see if I can find this just give me a second because that that way it will be very easy.

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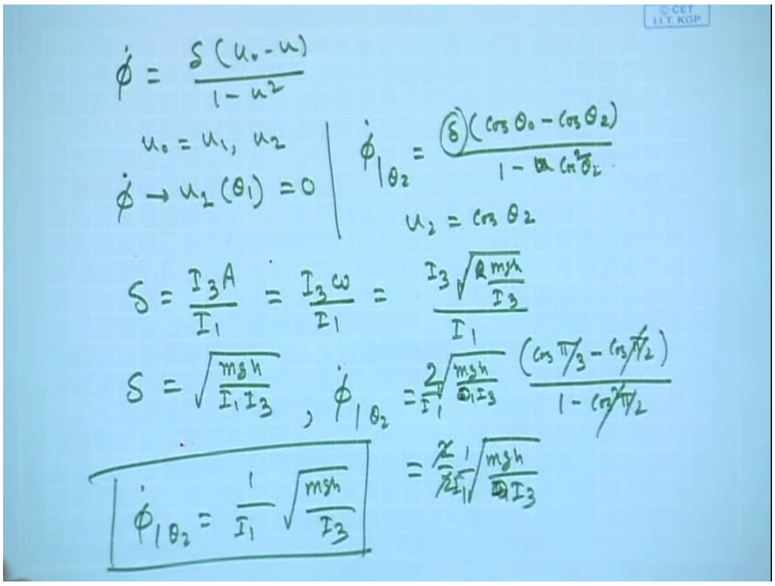

So, phi dot is equal to this. So, I will just write this expression once more, phi dot is equal to delta u 0 minus u, 1 minus u square. And we have a case where u 0 is equal to u 1 and we have to find out u 2. We immediately see that phi 0 or phi dot at u 1; that means, corresponding to upper limit theta 1 equal to 0 that we have already discussed. So, we have to calculate the speed at the lower limit. Phi dot at theta 2 will be delta u 0 is cos theta 0 minus cos theta 2 divided by 1 minus u, u square once again is cos square theta 2 because u 2 is equal to cos theta 2.

Now, we have to find out a value for delta, that is where the trick is. The spinning speed is given, now what is the expression for delta if you remember the expression for delta of course, I have to find it out for you yeah the expression for delta is I 3 A by I 1. Now, in case where there is no if it is released from rest we have just seen it will reduce to I 3 omega by I 1. So, delta is equal to. So, here the spinning speed is given which is I 3 root over m g h by I 3, it is inside root, divided by I 1. So, delta is root over m g h by I 1 I 3.

Now, all you need to do is, you need to put this into this equation which will give you phi dot at theta 2 is equal to root over m g h by I 1 I 3 and you kindly follow the values here. You know you have to put the values of theta 0 equal to cos pi by 3 minus, cos pi by 2; cos pi by 2 is equal to 0 and it is 1 minus cos square pi by 2. Now, cos pi by 2 is equal to

0. This is very easy, 0, 0. So, it is only cos pi by 3; pi by 3 is cos 60 degree, which is half. It is half root m g h by I 1 I 3. So, this is phi dot square, so, there was a 2 here, that 2 will get cancelled and we get this I think I have made a mistake somewhere, yeah of course, so, it will be I 3 here by I 1. I 1 will be outside. So, it will be 1 by I one, of course, phi dot theta 2 is equal to 1 by I 1 root over m g h by I 3 that is to be proved.

So, we have solved question number 5, last remaining problem of this set and the discussion of rigid body is problem number 6, which is a coin spinning with about a vertical axis with angular velocity omega. Prove that the motion is stable, the condition for motion is stable motion is omega greater than 4 g by a.

Now, for this what do we need to do I am not solving it for you I will just give you sufficient hint.

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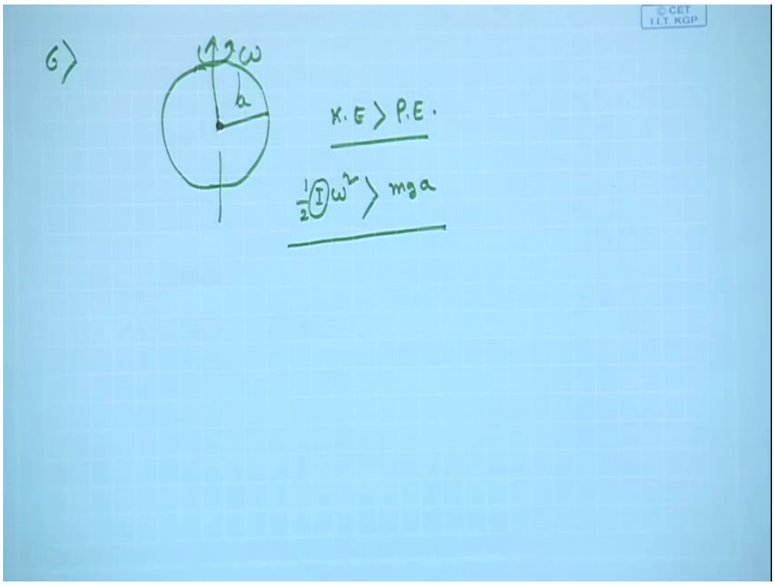

So, let us come back to this, we have problem number 6, where a coin is spinning with speed omega and it has a radius a. So, the axis, this is the axis of rotation. Now, this, please understand that, you can start by applying stability criteria of a symmetric top and etcetera because this coin is an asymmetric top. This system is an asymmetric top, but it is not needed. Please understand, any rotational motion about some axis is stable only if the kinetic energy of rotation is greater than the potential energy, especially for asymmetric tops; symmetric top, it could be a different story because it has certain symmetry associated with it.

So, specially, but again it can be stabilized for some time, but not for long like this. In order to have a stable rotation specially for an asymmetric top, it is mandatory to have kinetic energy greater than potential energy and kinetic energy is half I omega square potential energy is m g. So, where is the center of mass of this system? Center of mass is exactly at the center, so, it is simply m g a.

So, if I put this into perspective and then we get a condition omega is equal to this now you put the value for I and then you will get this condition. Maybe a root is missing m g a, could be that, I will check and correct in the final assignment if there is any mistake in the problem, but this is the only condition we need to apply in order to get the stability criteria of that coin for the last problem.

So, with this we end the discussion of rigid dynamics and it was I think you enjoyed the discussion. We learned it together. We started off from simple consideration of a systems of particle, essentially, went on defining moments of inertia, moments of inertia tensor, the different theorems for moment of inertia. Most importantly what we have learnt is how to project an inertial tensor in a certain direction to get the moment of inertia in that particular direction. At the end, we went into the specialized cases of symmetric top specially the torque free motion and with torque motion that is the heavy symmetric top and we looked at into certain examples of this whole system. So, please feel free to write me any queries you have regarding this system or the any of the classes essentially and we will start with Lagrangian dynamics very soon.

Thank you.