Classical Mechanics: From Newtonian to Lagrangian Formulation

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So, we wrote an equation which is f u equal to some u is u is equal to cos theta and fu equal to u double dot. Now we examined the rule, we started examining the case where fu u dot equal to 0.

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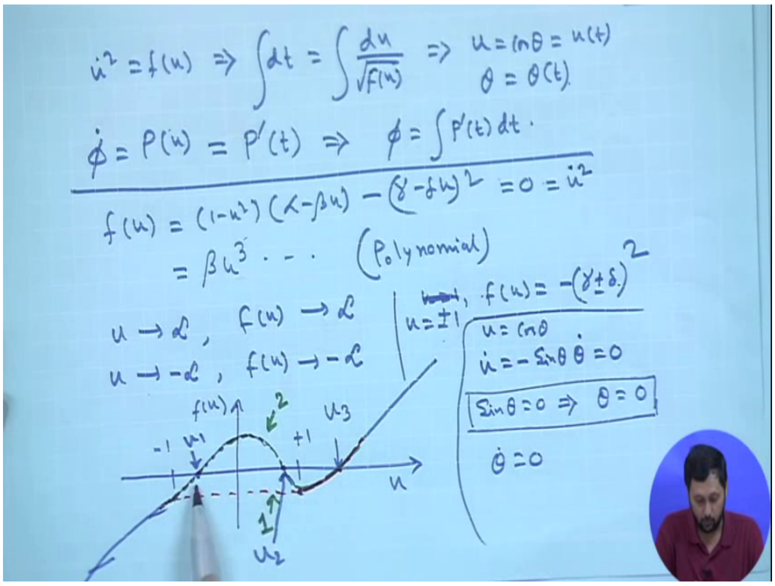

So, essentially we were looking at the roots of this polynomial which is fu and we essentially what we did was we set fu equal to 0 and try it; we argued that the roots or the we argued that at very high u or very low u fu will be going to positive infinity and negative infinity and we have also seen that u equal to plus minus 1, this term uniformly vanishes for u; u equal to plus or minus 1 leaving behind this term which is a negative definite term.

Now, for this negative definite term, it will definitely have negative value at u equal to plus 1 and minus 1. So, what would be the possible shape of u in the f; f of u in this region whether it would be like following the green line or it will be following this red dot that we have to examine right so, but before that let us try to understand it more you know in a physical consequence; what is this f u equal to 0 means. So, essentially what we did was we. So, if you recall the original equation was fu is equal to u sorry; u dot square. So, we essentially set when we put a fu equal to 0 that essentially means u dot equal to 0; what does it physically signify; and what are we looking for by examining the roots of fu equal to 0.

So, let us look at it in a more physical manner. So, there are 3 types of possible motion in a symmetric top the first motion is the spin motion.

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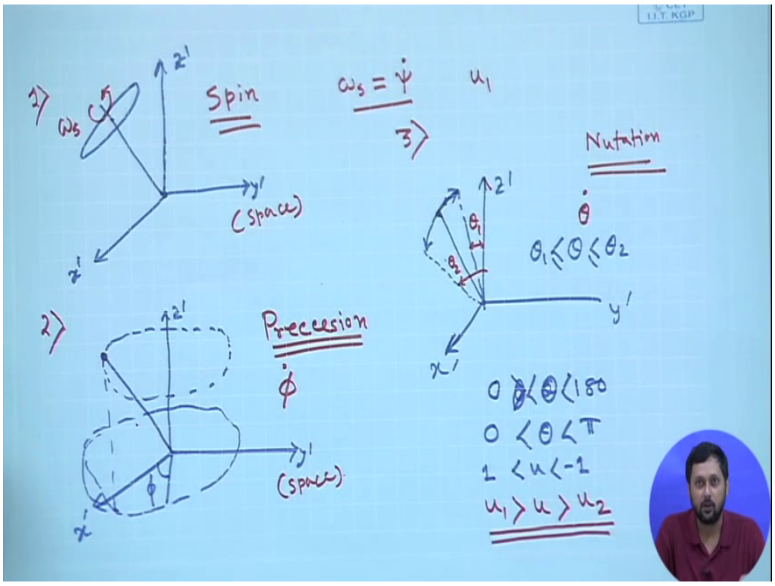

So, let us say this is my top; I am just us; I mean drawing this top like this we have a disk and we have forced you know mass less rod through the discs for example. So, we have this is my space. So, now, we are describing all the motion with respect to the space set of axis. So, this is my X prime this is my y prime; this is my z prime, right.

So, the first possible motion is the spin motion of the top spin at it is around its own axis which is which is given by let us say our spin velocity omega s right or we can also call it omega z that is up to you. Now in terms of Euler angles please realize that that omega s is very well represented by the last Euler angles time derivative. So, omega s is your psi dot, right. So, this is one motion of top which is the spin motion which is very nicely taken care of by the third Euler angle.

Now, there are 2 other motions possible if we trace the top axi, I mean the axis of this top and we see; how it behaves with time as the top spins if you recall a child stop; it also rotates around its own axis and this rotation is the. So, if I just draw it separately. So, this is my first case now for the second case; here I am only drawing the axis of the top; I am not even drawing this is our disk. So, if we follow the tip motion of this tip. So, it also rotates around the space z axis. So, this is my X prime this is my Y prime and this is my z prime.

So, the axis of the tip of the top also processes around the z axis now this precession if I draw it in a I mean if I project it in the xy scale; it will be kind of a closed motion I am not saying it is a circular motion I will just say it is say some kind of a closed motion in the xy plane which is very well described by phi is not it because we essentially we have a motion which is around the axis around the space z axis.

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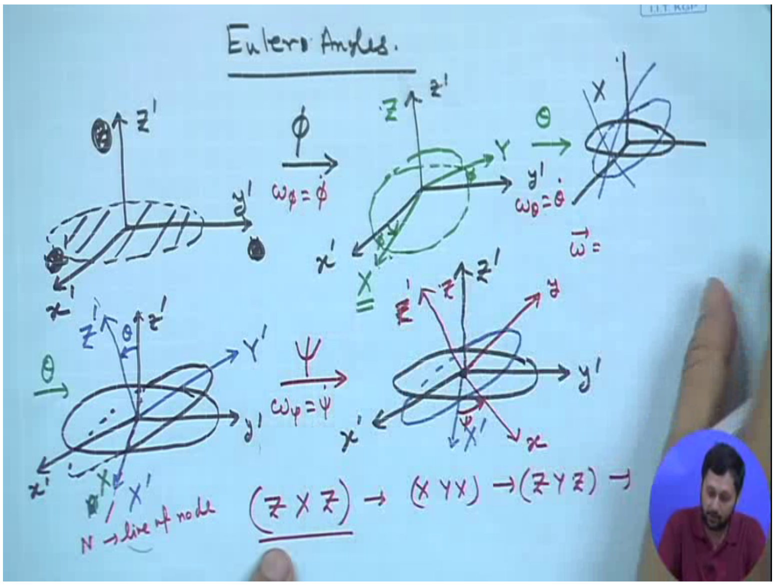

And if you recall our initial discussion on Euler angle; the first of Euler angle phi is a rotation around the space z axis.

So, this angle phi is very well describing this particular type of motion. So, we see that the second of the motion which is called. So, let us give the name. So, this is my spin motion this is this motion is called the precession; precession, it is called the precession motion precession means when the axis of the top rotates around the space z axis. So, these all these axis are space axis here and here and the third one the third type of motion is called a nutation; nutation is when you have let us say your axis system; once again X prime, Y prime, Z prime and your I am just again drawing only the axis of the top.

So, what happens is this top with time it falls or goes up. So, essentially it can let us say come down to this position. So, it can come down; I mean make an angle theta 2 or we it can go up to make an angle theta one. So, essentially this axis can draw can move up or down with respect to again the space z axis and this so; that means, it is a motion which is well represented by the motion of the second Euler angle which is theta. So, if you recall the second Euler angle was a rotation around the modified x axis which is once again can be represented looking at this looking from the space set of axis is. So, if some object I mean if some axis is moving up and down in this direction this can be very well be represented by this theta this particular Euler angle.

So, we see this nutation is represented by theta dot I mean the nutation velocity is represented by theta dot whereas, the precession velocity rep represented by phi dot whereas, spin we have already seen is represented by I dot. So, 3 Euler angles in case of symmetric top represents 3 types of motion possible in a top right and I think we probably nutation is obvious when you are looking at a child stop mutation is very obvious precession is also; I mean nutation is not very obvious, but if you look carefully you will see that the acts when the top is rotating it is not first of all its not exactly vertical its slightly tilted of course, you see this precession; precession, frequency I mean precession motion spin of course, it is obvious if it is not rotating; it is not working. And finally the nutation; when it goes ups and up and down; it is not very obvious, but sometime you might have seen it if you have played enough with a child stop well I did. So, I could easily identify all these 3 motions.

Now, let us try to understand this once again now when I say u dot is equal to 0 in this; in the in the expression of fu equal to u dot. So, if u dot is equal to 0 if you recall u dot is equal to minus sin theta; theta dot. So, if this is equal to 0 we can either have sin theta equal to 0 which implies theta equal to 0, because theta is what theta is this particular angle it can so that means, the axis of the top remains totally vertical along the z axis now that is a very special case. So, we will just keep this aside I mean we will discuss this case, but we will not. Now it is a special case we will call it the slipping top will come back to that or what we can have is theta dot equal to 0; that means there is no nutation motion. So, there is no essentially no nutation motion, right.

Now, if we said theta or rather it is not about nutation motion; I would say if during the motion if it encounters a turning point. So, what I mean is see if the axis of the top is moving between 2 limits theta 1 and theta 2. So, for any theta which is greater than theta 1 and less than theta 2 less than equal to theta 1 and theta 2 in this for this theta the top motion is possible. Let us assume for a moment that this is the range of nutation. Now if this is the case, then the axis of the top or when the axis of the top reaches theta 1 and theta 2; that means, it has to change the direction; it can go up to here and as it reaches theta 1; it has to start coming down again because this is see where to get this range; how to get to this range will; I mean that is that something will discuss later, but let us assume that this is the range which is defined physically.

Now, if this is the case; that means, it has to go; I mean it the top has to go up, then it comes down it then as it reaches theta 2, it once again starts coming up. So, these are the turning points. So, theta 1 theta dot equal to 0 means these are the turning points of the nutation motion for precession; it is not a bound motion; when we talk about the precession motion which is the rotation of the axis it is not a bound motion. So, this there is no upper or lower limit. So, theta dot equal to 0 means there is no precession motion.

Similarly, spin is also not a bound motion spin is about its own axis when we says high dot equal to 0; that means, there is no spin, but for nutation theta dot equal to 0 means; it is a turning point of the nutation motion ups when the axis goes to certain limit, it turns back and that is at that point theta dot equal to 0 of course, we can also have theta dot equal to 0 if there is no nutation motion that is also a possibility, but somewhat I am trying to tell you that theta dot equal to 0 need not necessarily mean that it is a; its not a nutation, I mean there is no nutation motion, it can also mean it is one of the turning points of the notation.

Now, with this knowledge let us come back to this one; now what is u essentially u is equal to cos theta and that is how. So, let us u is equal to cos theta. So, that is how u dot e is equal to minus sin theta d theta. Now we have treated u the when we were examining the roots of Fe equal to 0. So, we have treated u as a continuous function and that we had to do it because you see the leading term of this function this polynomial is u cubed. So, it has 3 roots right. So, we know that if it says if it is a polynomial of order 3, then it has 3 roots if it is a polynomial of order n, then it has n roots out of that n roots may be m number of roots are real and there could be n minus m number of imaginary roots.

So, this is all comes under the theory of polynomials and all and some of you might be familiar with it, but what we know immediately from this if the functional dependence of F u is that there has to be 3 roots and from the physical consideration and. So, looking at the looking at the functional form we see that the physical range of motion of u is between plus 1 and minus 1, right. So, if cause theta equal to 0 what is the physical range of cos theta.

So, now let us come back to this once again where is it. So, what is the physical range of cos theta or physical range of theta theta can vary between 0 and 180 degree, right? So, theta has to be greater sorry theta has to be within this range. So, that is to say 0 theta pi. Now in this range cos theta can have a value of 1; so, when we check the range of u; so, it has to be between 1 and minus 1. So, that is for any physical possible motion please remember u is equal to cos theta. So, u has to be within this range.

Now, we see that from the consideration we have from the functional form we have for f u there has to be 1 root which is above between u equal to plus 1 and positive infinity that we have seen in the last day in the last class itself because otherwise the this cannot be satisfied that when you goes to positive infinity this goes to infinity u goes to negative infinity; this has to go to negative infinity and when u equal to plus minus 1; this has to be negative definite these 3 conditions tells us that one of the roots of u has to be imaginary it cannot be you know visualized fear in a real world.

So, this is our; we call this root u 3 and the other 2 roots u 1 and u 2 for any physically possible motion it has to be between plus 1 plus one and minus 1 in this bound it has to be within this bound. So, the shape from this physical argument we see that the shape is actually this not this right. So, we have interception points with the u axis and these are called the roots these are the roots of the system the third one we already called it u 3. So, let us call this u 1 and u 2.

So, we have 3 roots of u out of which one is imaginary 2 are real. So, u 1 and u 2 that corresponds to this 2 theta angles theta 1 and theta 2; so, we from this; if we try to correlate this 2 discussion for the physical motion of the top it has to be between u 1 and u 2, but please remember cos theta and cos theta with increasing theta cos theta decreases. So, the limit of theta is between 0 and pi and limit of u has to be between u 2 and u 1. So, u 1 is greater u 2 is less? So, these are the physical range of u in which we can have a motion physically possible motion of the top clear, right.

So, now let us try to explore the now. So, so what we have seen. So, far is. So, so what we have done here is we have set u dot equal to 0, these are the turning points from there we just follow the physical physically possible I mean we followed an argument to find out 2 physical possible roots of the top motion and these are correlated with these 2 angles theta 1 and theta 2 in the nutation.

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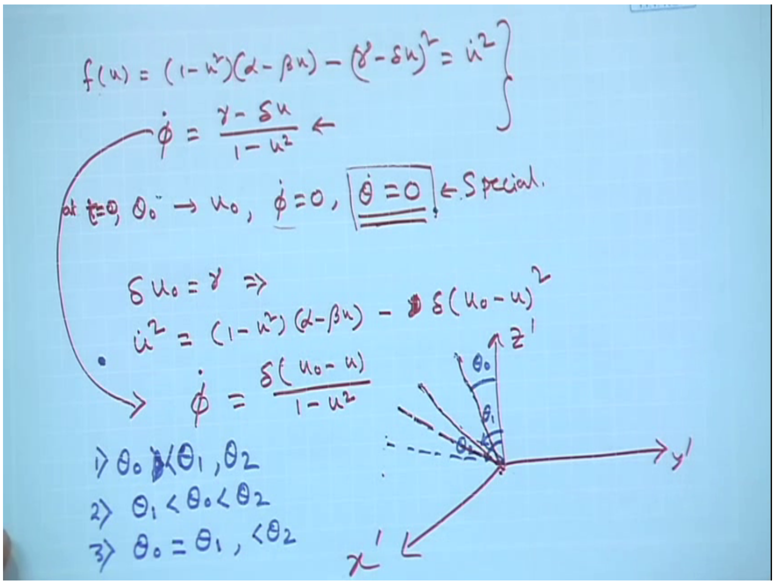

Now, also there is an additional condition if you recall this additional condition is if; so, one condition is f u which is equal to 1 minus u squared alpha minus beta u minus gamma minus delta u whole square equal to u dot square. So, this is one relation; we have the second relation is phi dot is equal to gamma minus; sorry, just give me a second what is it; gamma minus delta u n divided by one minus u square. So, this is the second relation.

So, let us focus on these 2 relations and see how does the top motion that axis of the top or tip of the axis of the top behaves in space when we set certain initial condition in terms of phi dot and theta dot or basically the initial conditions, we set certain sets of initial condition and see how this motion takes place, right. So, let us come back to this phi is the precision; we have seen that phi is this motion, right. So, let us assume that initially; we are releasing the top from a certain angle theta 0 which corresponds to a u equal to u 0 for which we have initial conditions. So, this is at t equal to 0 at t equal to 0 theta is theta equal to theta 0 which is corresponds to an u equal to u 0 and when we are releasing.

So, it is like this let us assumed that the axis of the top is rotating. So, so it is spinning of course, if there is no spin there is no motion. So, if it is spinning and we are just holding the tip; so, let us say this is the origin this is the point where it touches the origin. So, it spins it spins around its own axis and you are holding the top at this position that this angle which it makes with the vertical is your theta 0 angle and; that means, there is no precision, right.

So, you just release it from that point. So, what happens is it starts falling and then at set at the same time when it starts falling. So, if you write down the energy equation total energy is kinetic energy plus potential energy potential energy is what mgl cos theta. So, when it starts falling cos theta increases; that means, mgl cos theta this total potential energy contribution decreases as it decreases, it has to gain; I mean the total energy has to be constant.

So, if that has to be done. So, there has to be some additional kinetic energy to get gain that additional kinetic energy the precision motion is triggered. So, it starts moving sideways moment it touches the lower limit it goes up. So, the new I mean where as it starts falling the nutation is already there. So, then the precision motion also starts and it keeps moving like this follow executing all 3 motions of course, it is it has to do spin without spin nothing can work. So, it has to have spin additionally moment we start just from physical argument we can see that it starts precision and nutation together, right.

So, the initial condition is at t equal to 0 theta 0 equal to u and phi dot equal to 0 of course, we can have initial condition when theta dot is also equal to 0, but we are not taking this as a general case. So, I will show you; a short video later on of on gyroscope which is also an example of symmetric tough I will show you how we can have cases where phi dot equal to 0 and theta 0 is still not equal to 0 right anyway. So, right now this is a special case. So, in general we are just considering at theta t equal to 0 theta phi equal to theta 0 which is corresponds to u 0 and phi dot equal to 0 substituting in here we get u 0 is equal to or delta u 0 is equal to gamma, right.

Now, which means; so, if we substitute this here in this equation if we write it down u dot square will be equal to one minus u square alpha minus alpha minus beta u minus. So, if I do this substitution it will be gamma times sorry delta times u 0 minus u whole square which is which is the new form of Fe. Now anyway, this is not terribly important what is important is if I now. So, similarly we can write phi dot to be equal to gamma minus delta u which where you know delta is gamma; gamma u 0. So, we can write this as gamma u 0 minus u 1 minus u square, right.

Now, let us assume that during the motion of during the top motion, it is the axis is heating. So, we start at a position which is higher than the initial bound of theta I mean the smallest of the angle of the initial of the bound in theta what I mean is we can release the top. I will just draw it again maybe that will be easier for you. So, this is my axis X prime, Y prime, Z prime; all the discussion in terms of space frame I am saying it over and over again.

So, let us assume that these are the angles theta 1 and theta 2 which are 2 bounds of the nutation motion and your u 0 it could be u 0 or theta 0 it could be. So, we have cases where theta 0 could be greater than or rather less than both theta 1 and theta 2, we can have a case where theta 0 is in between theta 1 and theta 2 right or rather I will just use this. So, it can be between theta 1 and theta 2 it can be less than theta 1 and theta 2 it can be equal to the upper bound right; it can be equal to theta one, but if it is equal to theta 2 and start falling from here or if it is less or greater than theta 2 and starts falling from there then the top motion is not physically possible.

So, 3 physically possible regimes are theta 0 is equal to theta 1; theta 0 is less than theta 1 plus theta 1 and theta 2 theta 0 is in between theta 1 and theta 2 and third one is theta 0 is equal to theta 1 which is less than or less than theta 2. So, these are the 3 possible physically possible initial condition what happens once again what happens if we have theta 0 equal to theta 2 for example, or greater than theta 2.

So, let us consider this; we have let us say; this is my value of theta 2, right which is the lower limit. So, it can oscillate between the axis of the top can oscillate between these 2 limits you start initially at theta 2. Now moment you start its starts dropping. So, even if it drops if you start at theta 2 already; even if it drops slightly it is with I mean it is outside the physically possible motion range. So, the top falls similarly if you start below theta 2 it anywhere falls. So, the physical consideration tells you we can have only 3 types of motion in this 3 initial condition.

Now, let us carefully examine the science of phi dot at different limits for this 3 initial condition.

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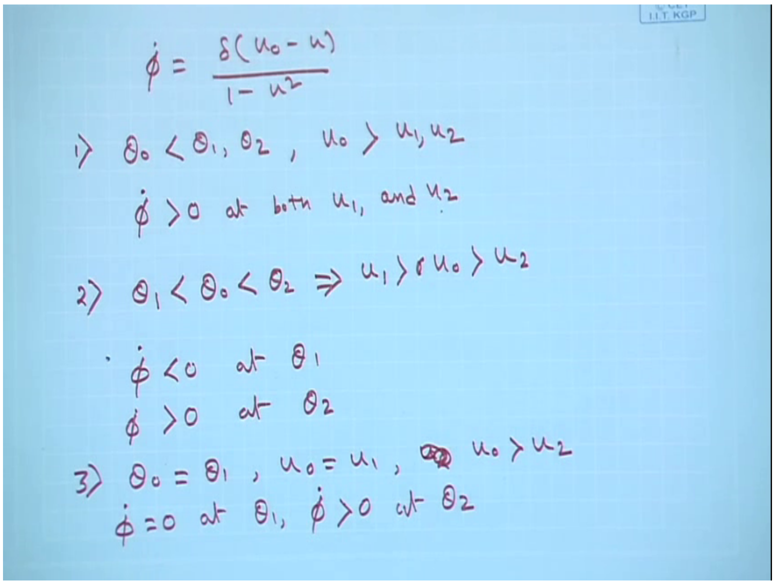

So, phi dot is equal to gamma u 0 minus u 1 minus u square, right. So, when case one when theta 0 is less than both theta 1 and theta 2 right which corresponds to u 0 greater than both u 1 and u 2 right because theta and u there kind of they are trained is kind of opposite to each other. So, if this is the case then at both the limits u 1 and u 2 phi dot becomes positive greater than 0 at both u 1 and u 2.

So, you have a precision speed at both the limits you of theta 1 and theta 2 to be positive right and what you get is the axis of the top.

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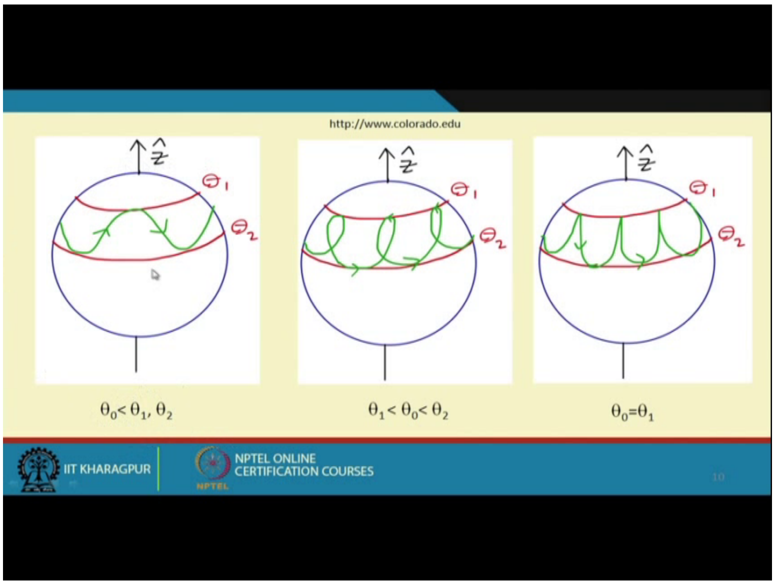

Now, let us switch to the screen here computer here. So, the axis c this describes the first case where theta 0 is less than both theta 1 and theta two. So, we have in the both limits precision speed greater than 0 so; that means, if I am like you know tracing the top of the tip as a function of time. So, the top because it is a fixed length it has to be on a sphere on the sphere surface if I mark theta 1 and theta 2 like this. So, my theta 0 is somewhere here. So, theta 0 which is unfortunately not marked here is up somewhere up here, right. So, theta 0 is less than both theta 1 and theta 2. So, start. So, the top starts falling initially from an angle which is less than theta 1 settles between these 2 limits and we have positive velocity or positive precision speed at both the limits and we have a tress is something like this. So, it keeps moving like this.

Now, if I take the second condition which is theta 1 less than theta 0 less than theta 2 which corresponds to u 1 greater than u 0 greater than u 2. So, phi dot; so, if you see u 0 is less than u 1, right so; that means, phi dot is less than 0 at theta 1 phi dot is greater than 0 at theta 2; That means, you have a precision which is positive at the lower bound, but negative at the upper bound which corresponds to the case second case.

So, you start from start from a position which is between theta 1 and theta 2; let us say some initial angle which is lying between here. So, your stop starts let us say from this point starts falling initially it has a positive velocity at the lower bound, but when it goes up and hits the upper bound it has to hit the upper bound, because these are the physical to physical possible roots physical possible bound the five precision speed is negative. So, it goes slightly in the opposite directions of sorry; sorry; starts falling again and then it keeps moving like this, right. So, we have a situation where; we have a tress which looks exactly like this.

Now, let us move to the let us go to the third case which is theta 0 equal to theta 1; that means, u 0 over 2 u 1 and of course, we have u 2 which is which is u 0 greater than u 2 right fine now in this case phi dot is equal to 0 at theta 1 phi dot is greater than 0 at theta two. So, we have a precision which is 0 at upper limit and greater than 0 at lower limit and we have a situation just like we have in the third case. So, it is s cusp; it is a; this particular shape is called a cusp, cusp, cusp and. So, this is the tress.

So, you might have family; you might have been already if you are looking at the textbooks you might be familiar with this shapes and many of you will find; it extremely difficult to understand what are these shapes physically mean. So, these are the physical interpretation for these shapes. So, all this can be taken out I mean all be can be understood in terms of this very, very, very simple mathematics where we only take we first find out the roots by setting fu u dot equal to 0 and then we slightly modify this equation for this particular initial condition to this and all these 3 conditions or all these 3 types of tresses are direct consequence of the simple mathematics, right.

So, this is more or less our discussion, I mean general discussion on symmetric top, we have 2 more cases of fast top and spinning top, sorry; fast top and slipping top and we will finish that net where and that will be the end of the rigid dynamics officially of course, we can you can always write me your questions if you have any I am obliged to answer the question.

So, after this, we have one more lecture on rigid dynamics, then we will continue with topic of Lagrangian dynamics.

Thank you.