Classical Mechanics: From Newtonian to Lagrangian Formulation
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Lecture – 40
Rigid body dynamics – 14

So, we have seen that the three components of omega can be expressed as; omega 1, omega 2, omega 3 in terms of the Euler angles, and we have seen that for heavy symmetric top. We are working in a frame which does not include the psi rotation, the last toiler rotation explicitly. So, what we are doing here is, we are setting psi equal to 0 to get the components of omega 1, omega 2, omega 3.

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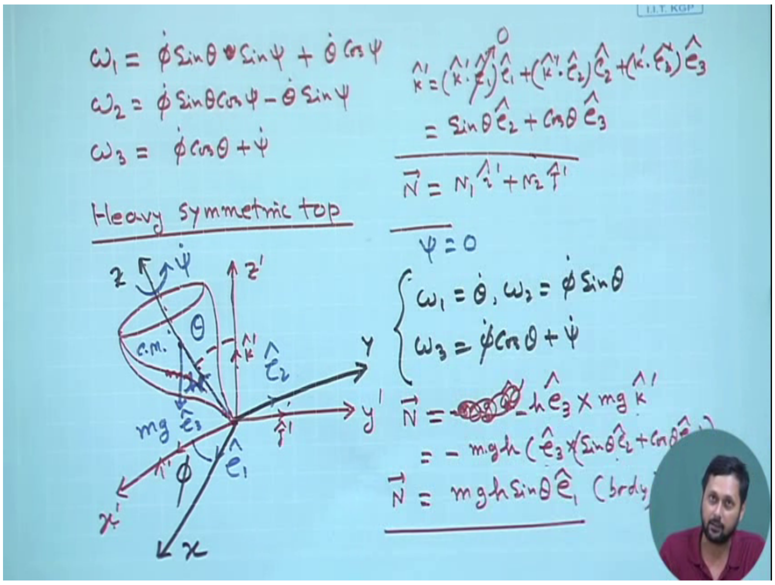

For this particular axis plus. Please understand that we are not putting psi dot equal to 0. So, we have these three components, and of course, we can write out the Euler's equations if we know the torque, right.

So, in order to get the torque, let us start by writing N which will be. So, mg is working through center of mass downwards. So, we can write mg to be equal to m g k capped minus; sorry, it will be r cross mg. So, r is along the body side, which we also have not marked. So, this will be our z axis. Let us call the;you know three unit vectors as e 1 capped e 2 capped and e 3 capped, right. So, your r cross N will be equal to r cross f f is mg minus k capped, which we just wrote and r is equal to h e 3 cap. So, we have h e 3 cap cross m g k capped k dashed cap with a minus sign. So, the minus will come here.

Now, as we have not taken the third Euler rotation into account, this has a, this has a significance, because it makes the mathematics. So, simple in this particular formalism. see one thing we can do is, we can also write, we can. I mean let us talk about it later, because right now no point. So, if we take psi is equal to 0, then my x axis the body x axis lies in the x x prime y prime plane; that means, it is in the plane of the space xy, right. In that case we can decompose this k 3 which is yeah. So, if I mark them as I or k prime. Sorry not k 3 k prime i prime j prime and k prime. So, in that case, we can decompose k prime in terms of e 3 and e 2 right. So, we can write a prime to be equal to k prime dot e 1, e 1 capped plus k prime dot e 2 times e 2 plus k prime dot e 3 times. Sorry, e 3 times e 3.

So, we can do that, and we see as the e 1 is always perpendicular to k, because it is in the xy plane. So, the first term is unanimously equal to 0. Whatever may be the value of phi, and if that is the case, then you see that this can be decomposed in this body y z plane, and we can write by considering this angle theta. I mean in terms of this angle theta, we can write this as. So, the first will be e 2. e 2 is the y y is sin theta right. So, we have sin theta e 2 capped plus cos theta e 3 capped right. So, when we put this back in the expression of n, what we N is minus what we get is this mg h . So, e 3 cap cross sin theta e 2 capped plus cos theta e 3 capped.

Now, once again the last term will vanish, because the last term is a cos theta e 3 cap term. I hope you can see this. So, last term is cos theta e 3 cap term, this will vanish, because it has e 3 in here, right. So, only the surviving term will be e 3 cross e 3 cross e 2, which is minus c 1. There is a minus sign here. So, this will give you m g h sin theta 1 capped. So, N is this, also not considering psi into account. I mean putting psi equal to 0 explicitly has another significance, when we look at it from the space set of axis. So, from the space set of axis, your torque will be. So, if we try to compute the torque, torque will be along. So, the expression of torque will be the same. So, it will be e 3 cross mg k capped, right.

Now this is important without decomposing it into this components of it. I mean without decomposing k prime into e 2 and a 3 we can see that the torque is a cross (Refer Time:

06:34). It has to be always perpendicular to k prime right. So, it could be anywhere in this xy plane, if we look at it from the space set of axis. Please try to understand x prime y prime z prime frame as N is equal to this, something cross k prime; that means, the torque wherever it is, cannot be along z prime axis. So, the torque, if I this is the expression of torque in the space set of axis. Now in the; sorry, body set of axis. Now in the space set of axis N can have a general form of N 1 I capped plus N 2 j capped, because the k capped comes in the expression of N itself. This can never have a component in the z direction. So, this has a significance we will see later.

Now we have this mathematics worked out. Next is writing out the Euler's equation. So, of course, we can write out 3 Euler's equations explicitly, but the first and second Euler's equation we will not do. I mean we will not write the first and second Euler equation.

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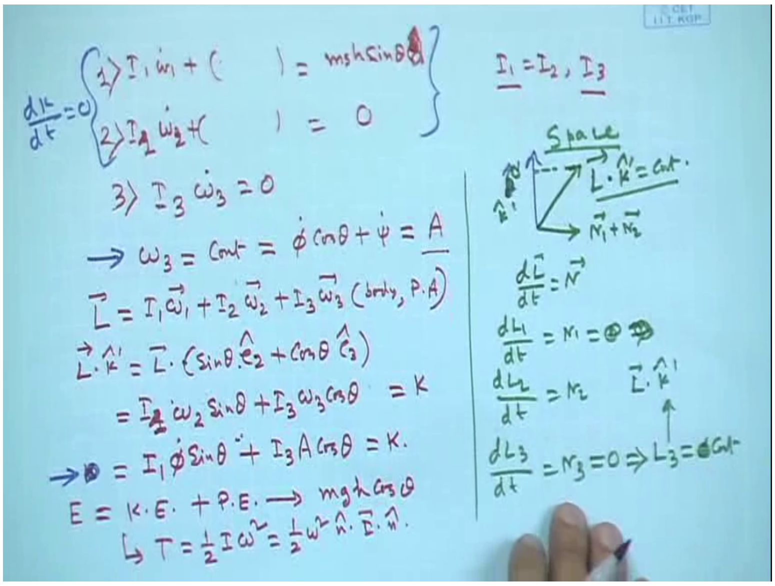

So, the first equation will be some I 1 omega 1 dot plus something equal to N 1 which will be mgh sin theta, you want. Sorry you want capped, we do not need. Similarly second Euler equation will be I 3 omega 3 omega 2 I 1 omega 2 dot, because I 1 is equal to I 2 and we have I 3. So, we will just write I 1 and I 3, we will not write I 2.

The second equation is also this plus something equal to 0, but the third equation is something that we will write out explicitly. It will be simply I 3 omega 3 dot is equal to 0. So, I am leaving it to you as an exercise. Why this will come 1. Once we put psi equal to 0. So, you will see that this exactly is going to happen. I mean of course, you do not even have to put psi equal to 0; the third equation will take this form, because the torque is 0 and I 1 is equal to I 2.

So, from here we can get omega 3 is a constant, and what was the expression of omega 3 we have taken. We have written omega 3 as phi dot cos theta plus psi dot. So, this is a constant, we see immediately from the third equation. This is the first, this is the second, this is a third. So, I am not writing the first 2 explicitly, because we are not it. So, the idea is the following, we will not try to integrate these equations. This would be integrable, definitely they are integrable.

We are not trying to do that. Although omega 1 and omega 2 they are written out explicitly in terms of 3 Euler angles phi. Sorry or 2 Euler angle phi, and theta and of course, we have psi dot, which is the spin into account. We are not trying to integrate it for heavy symmetric term instead. What we are going to do is, we will try to find out what are the conserved quantities we have, and we will write, we will try to give a quantitative description of the motion based on the functional form of certain parameter we will get. So, this is the strategy we are going to follow.

So, this is; essentially; we will see that these are, there are three conserved quantities we can write out; one is a, and again when we will be dealing with Lagrangian dynamics, we will revisit the problem of symmetric torque, and there we will see that these conserved quantities are nothing, but associated conjugate momentum to certain momentum, which is supposed to be conserved by which will be obvious from the Lagrangian of the system, but right now let us just look at the conserved quantities we are getting. So, 1 is this quantity which we call a.

Now, look at this once again, there is no component when we look from the space set of frame. There is no component of torque along z. So, if let us say, if this is your space z which is z prime, and let us say this is your angular momentum l; so, angular momentum, because there are torques in the xy plane. So, let us say this is the general direction of the torque, which is N 1 plus N 2 in the xy plane. So, angular momentum will change in the xy plane, but as there is no torque in the xy plane in the z axis that projection of angular momentum along z axis, which is l dot.

Sorry not z cap, this is k capped equal to a constant it has to be; that is from the physical point of view, because please remember angular momentum can change, yeah I can from space set of axis, yeah from space set of axis our equation is simply dl dt equal to N . So, I can decompose it into dl 1 dt equal to N 1 dl 2 dt equal to N 2 and d l 3 dt equal to N 3. So, if N 1 is equal to 0, we can immediately, say immediately tell you, or sorry if N 3 equal to 0 then we can immediately tell you that l 3 which is equal to, sorry which is equal to a constant and l 3 is nothing, but l dot a prime that is written out in the space set of axis, right.

So, this will be another conserved quantity k prime. Sorry right now one. So, l is I 1 omega 1 plus I 2 omega 2 plus I 3 omega 3; that is written out in body principle axis right that we have seen. So, we can take l. So, this is one expression which we can use, I mean does not mattered if we write the vector in this frame, on that frame this is something which we can use right. So, l dot k prime is l dot, and if you recall we have already decomposed this into e 1, e 2, e 3 the, you know k prime we can write in terms of e 2 and e 3. So, if we simply use this expression and write sin theta e 2 plus cos theta 3 right.

So, only these 2 components will survive, the first component will not contribute. So, it will be I 2 or I will write I 1, because we are writing I 1 and I 3 only. So, it will be I 1 omega 2 sin theta plus I 3 omega 3 cos theta, and you recall that omega 3 is a constant which is a. So, if we call this whole thing equal to some constant k, then we can say k is equal to I 1 and omega 2 is phi dot sin theta right plus I 3 a cos theta is equal to k.

So, we have our first conserved quantity here, which is a, which we have our second conserved quantity here which is k, k is the k to conserved quantities are already there. now they turned out that there is a turns out that for this particular system, the torque the force, applied force is not a function of time, because it is a mass; of course, it is changing with you know mg cos theta, but it there is no explicit time dependence into this force, and if there is no explicit time dependence, it can be proved that the energy of the system is conserved once again. There are proofs, there are classical proofs available from Newtonian mechanical side or also we can use Lagrangian dynamics to prove it, but it is not terribly important. I think we will all agree to the fact that the energy conservation is valid.

So, we can write e equal to which is a constant and e can have 2 parts; one have kinetic energy and one have potential energy. Potential energy part is simple, we can just write potential energy as mg h cos theta. Look at this picture, and you will agree to that. So, this is mgh. So, that the height is essentially l cos theta right. So, if sorry h cos theta. So, this vertical height of the center of mass; so, the potential energy is mg h cos theta.

Now, for the kinetic energy, please recall kinetic energy t is given by half I omega square, and if we or which is equal to half omega square N I N. Now if we dissolve it in the, it remember our discussion on ellipsoid of inertia. So, when we dissolve this N dot N dot I dot N in a principal component or sorry principle frame of reference, all the cross terms vanished.

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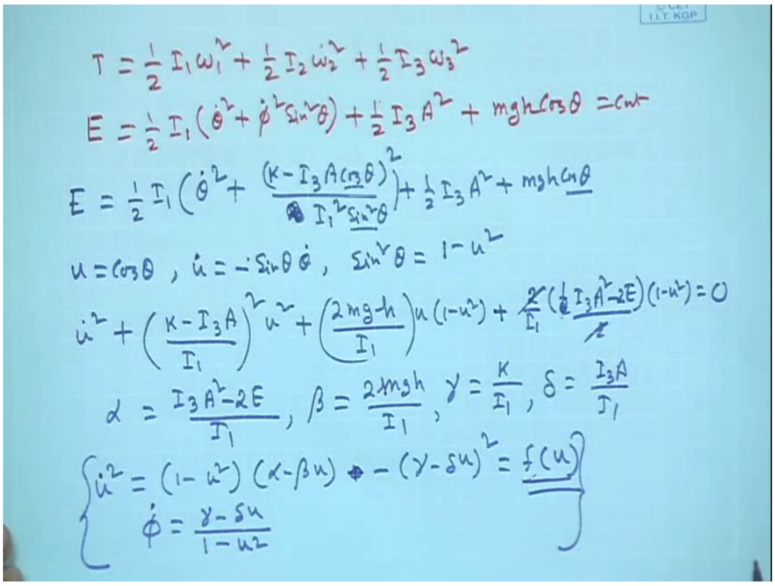

So, in that particular frame, the expression for t will be simply half I 1 omega 1 square plus half I 2 omega 2 square plus half I 3 omega 3 square right. Now omega I 1 and I 2 are equal. So, the first term will be simply half I 1. Now putting the expression for omega 1 and omega 2, it will be theta dot squared plus phi dot squared sin squared theta right. So, this is the expression for omega 1 and omega 2, and for omega 3 once again we can put a. So, the third you will be a I 3 I 3 A square.

Now this is the expression for kinetic energy. If we write it for total energy e. So, we just have to add m g h cos theta to it, and this term is also a constant that we all know, and total energy. So, we have three conserved quantity; one is this a, one is the k, and one is u. Now it turned out that we can also write this part in terms of k, please recall that k is given by this particular expression; of course, we have not derived, right.

So, in the second expression now we can start our manipulation what we are going to do is, we are writing e. So, I am using a different color of ink, because now what we are trying is right now what. So, far what we have done or simply, yeah we have simply written out the expressions, we have not done any mathematical manipulations. Now it is time that we start our mathematical manipulation.

The first one is by writing ie which is half I 1 c theta dot square stays theta dot square plus what we are going to do is, we are going to write phi dot phi dot square sin square theta in terms of k and a. So, if we do that, it will be k minus I 3 a cos theta by cos theta whole square by 2 sorry cos theta whole square by I 1 square sin square theta. Check the expression of k and e, this will be obvious to you and then this one stays as I 3 a square plus mgh cos theta. So, this bracket will close here right.

Now, let us put u equal to cos theta do not mix it up with u equal to 1 by r. We are not doing central orbit, its just some notation u equal to cos theta. So, u dot is equal to minus sin theta theta dot and sin square theta. We can write as 1 minus u square right. So, whenever we get cos theta we can call it u; for example, here whenever we get a sin square theta like here, we can call it 1 minus u square, and we have an u here as well right, and of course, you have u dot. So, once we do that see u dot has a theta dot term in it. So, theta dot what we have here, can be said to be u dot minus u dot by sin theta, and theta dot square will be u, u dot square sin squared theta sin square theta once again is 1 minus u dot.

So, doing this mathematical jargon; I mean it is just as some juggling algebraic juggling, its what we are doing here is pure algebra. We are not going to do differential calculus, although the equations we started with is Euler’s equation, which we did not use explicitly by the way this expression of k can also be derived by writing first. If I write to first to Euler’s equation explicitly, then from there we can, it can be shown that this 2 can be used to write dk dt equal to 0, which is worked out in classical mechanics by Spiegel you can have a look at it.

So, if I instead of doing this physical pictures thing, where we write it I mean we try to analyze l in terms of the space set of axis, we could have done this mathematics also, but I like the physical picture. So, that is why I did it this way. So, of course, we are not writing the Euler’s equation explicitly instead what we are doing is, mathematical manipulation, and it turned out to be pure algebra in nature right. Now we are not writing a single differential equation, we are just doing pure algebra of course, if you call that, you know there is a u dot here. I mean its if you call it its differential calculus, then we are doing differential calculus, but in my opinion its pure algebra.

Now keep doing it and essentially you can rearrange this entire equation. I am not doing it for you, you can do it yourself to 2 steps in mathematics as k minus I 3 a by I 1 whole square u square plus 2 mg h by I 1 u 1 minus u square plus. So, it will be 2 by I one half I 3 a square minus e whole times 1 minus x square equal to 0. So, this is what we can get from this. Now it is time to define some constants alpha which is equal to, let us say alpha will be what. Let us start from the last 1. So, if I take this 2 up here, this 2 will cancel out. So, let us call alpha to be equal to I 3 a square minus 2 I 3 a square minus 2 e by I 1; so, there is an I 1 here.

So, let us call this alpha let me see yeah. So, this will be your alpha, and there is an u 1 minus u square here right, beta will be equal to 2 m g h I 1 this part gamma will be equal to k by I 1, which will come here and delta will be equal to I 3 a by I 1 which will come from the second term. So, if we substitute this we can once again to slight methods. I mean little bit of mathematical manipulation and we can write this whole thing as 1 minus u squared alpha minus beta u. Sorry minus gamma minus delta u whole square and this is a function of u only. So, in this expression alpha beta gamma and delta all are constants. So, the only variable is u. So, we can write this f of u. Now it is u dot square equal to this. So, we can also try to integrate that.

Now also there is a second expression, if we start with the expression of k and sly I mean right try to write this expression of k by slightly mod I in a slightly modified form, in terms of all this newly introduced parameter alpha beta gamma, and delta, then we can figure out that phi dot is equal to gamma minus delta u times divided by 1 minus u square. So, these 2 are the expressions which we will be using generally give a physical description of tops motion. Now, if we try to integrate this equation, the first equation, second equation also we can integrate in principle.

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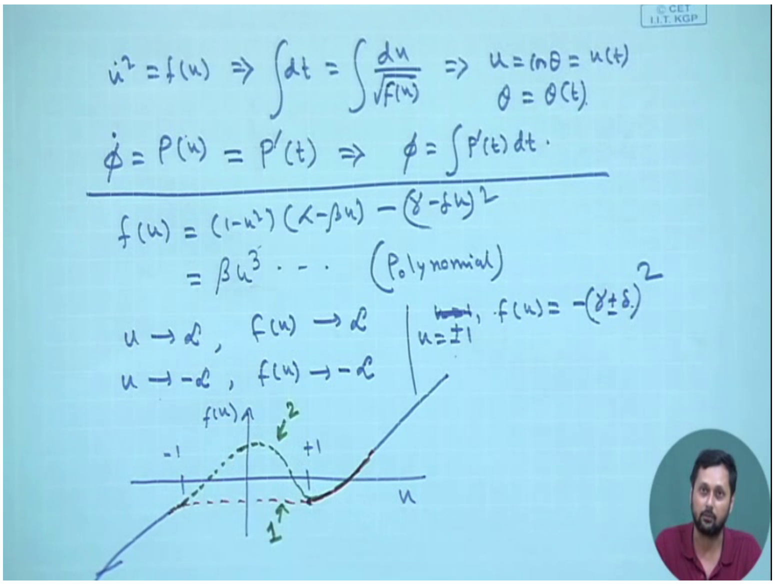

So, if we try to integrate these 2 equations what do we need to do. If take the first equation which is f equal to f u which will be integrated to give dt equal to integration, dt is equal to integration du by root fu right. So, in principle this will give you u, which is equal to cos theta as a function of time you as a u of t. So; that means, essentially by solving this you get theta as a function of time.

Now, plug it back into the second equation, which is phi dot is equal to some another function of u, which is actually this which is some function of u right. And now once you get u equal to u into theta in terms of t. So, you can substitute even you can define a new function which is p prime of t by substituting the u here right. So, essentially this can be integrated to give phi which will be integration of p star t, sound simple, but as I said during the discussion of central orbit we could ge, we can in principle also integrate in a similar manner, we can integrate r and theta to gain.

I mean to get the complete description of the orbit in terms of as a function of time, but we do not do that. We do not do that frankly, because it is very hard and this is also a very difficult integration. The first integration itself it is an elliptic integral, there is a class of integral which are called elliptic integral which are extremely difficult to execute.

Look at the functional form of u dot it is not. So, easy also, I mean it is might not be possible analytically, but numerically it might be possible of course, if you can do that, but also this is not strictly needed, what we are aiming at, is essentially we are trying to gained a physical insight of top motion by not solving any equation. We just want to see how it works. So, for that we need to, in strictly speaking we can just stick to these 2 equations. We try to examine the functional form of fu quantitatively and we will gain that insight.

So, let us start in that line f of u. See I am not writing the expression once more f of u is equal to 1 minus u square alpha minus beta u minus gamma minus delta u whole square. So, what is the leading term in this; the leading term is this, which is beta u cubed and then we have of course, there will be a gamma u squared term and all. So, it is a u cubed, I mean the leading term is alpha u cubed and it is a polynomial. So, now, if it is a polynomial in nature, which goes with a leading term of u cubed so; that means, as u goes to infinity f of u goes to infinity in the positive side as u goes to minus infinity f of u, because it is an odd function, u cubed is an odd function right. So, f of u also goes to minus infinity, right.

So, if we try to draw a quantitative description of it then and also you please understand; that if you put u equal to plus or minus 1 in this expression. So, both cases first term will definitely vanish. So, if the terms vanish at u equal to 0, if you put u equal to 1. So, f of u will be some number right. So, u equal to 1. So, f of u will be minus gamma minus delta u is equal to sorry, if you put u equal to plus minus 1 u equal to plus or minus 1 both cases the first term will vanish, because it is 1 minus u square and the second term will be minus of gamma plus or minus delta whole square. Now, does not matter, if this gamma plus delta or gamma minus delta is negative, the square will always be positive.

So, at u equal to plus and minus 1 f of u is negative definite. So, we have three condition on u, that at f equal to u equal to infinity, it goes to plus infinity u equal to minus infinity, it goes to minus infinity and at plus and minus 1, it is cause negative definite. Now, if I try to plot f of u. So, you have f of u, this is your u axis, this is your plus 1, this is your minus 1. So, we can say that it is going to minus infinity. It is going to plus infinity and in this case, these 2 cases, it is negative definite. So, some point, it has to cross 0.

So, we can have the following trends. We can either have it this or we can connect it this right. So, there are 2 ways either; this is my path 1 and this is my path 2, possible, 2 possibilities of u, I mean 2 possibilities in which f of u can behave between minus 1 and

1. We can stay in the negative direction or it can go to positive and come back to negative.

So, this is where we stop now and we will come back in the next lecture. We will start from this particular picture and see which one gives you a physically possible motion. From there we will go ahead and see what the possible physical picture of the top motion is, we can extract from this continuing on this particular line and then we will end this discussion of symmetric top or discussion of rigid dynamics, with 2 special cases of slipping top and heavy top.

Thank you.