Classical Mechanics: From Newtonian to Lagrangian Formulation

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Rigid body dynamics - 13

The frame we have attached with rigid body is called the body frame. Now this body frame is rotating along with rigid body and of course, there is a space frame space set of axis which is fixed in space. We have discussed it numerous times and whatever equations, we have written we have written it in respect to body frame, because typically the convention is we take the body set of axis or body frame to be to coincide with the principal axis system of that particular object particularly rigid body in question.

Now, if this happens then the equations greatly reduce it in complexity, all the cross terms, all the products of inertia term vanish and we have equation which is very nice looking we call it Euler's equations. But then there are problems for example, if we try to get gain the insight of motion with respect to the space set of axis which are the inertial frames then we have a serious problem because what we get in the Euler's equation were what we have is essentially 3 equations in terms of omega 1, omega 2 and omega 3, 3 differential equations.

Even, if we know the functional form of top or we know everything about the system integrating this omega 1, I mean integrating these equations will give us omega 1, omega 2 and omega 3 which once again I represented in body set of axis and if you understand the situation we have a rigid body let us say. So, this bottle is our rigid body this is I mean, this is executing some kind of a rotation about some arbitrary axis with one point fixed let us say the fixed point is at the center and the arbitrary axis of rotation is in this direction and so, it is moving like this and if I resolve this.

So, now if I write the equation in body set of axis, these equations will give us something which is the angular velocities, which are again expressed in body set of axis and it is rotating along with the rigid body. So, makes no sense. I mean it makes sense, but integrating those equations will not provide us what we are looking for we are looking for; for example, angular velocity. So, we need to have some ways of connecting this space set of axis with the body set of axis.

So, if we can and if you recall in the preliminary discussion of rigid dynamics we came out came up with the conclusion that the degree of freedom of a rigid body is 6, out of that moment we fix 1 point in the body we take out 3 degrees of freedom. So, we are left with 3 degrees of freedom of a system and these 3 degrees of freedom can be 3 coordinates. So far if you understand in this entire discussion we have written omega 1, omega 2, omega 3, but we never try to express this omegas in terms of change in some angle. Typically omega is some theta dot I mean theta being some arbitrary angle we never tried that. Now it is time we try that, so we have 2 things to do.

First of all we have to express omegas in terms of some angle number 1 and number 2 is we have to get a set of coordinates that connects the body set of axis along with the space set of axis and which are measurable I mean. So, essentially measuring the change in this angles we can talk we can describe the system in sitting at the space set of axis which is the inertial frame of reference.

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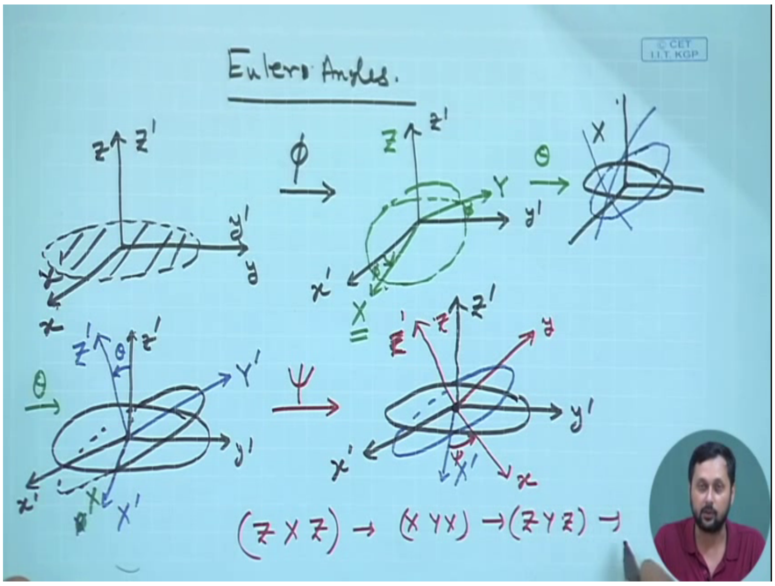

Now, these angles these are called the Euler angles or Euler’s, Euler angles. Now there are alternative definitions available many definitions available for Euler angles, we will stick to one particular definition. Let us describe; what are these Euler angles. Let us assume that we have at time t equal to 0 before we is have any kind of rotational motion we have 2 sets of axis coinciding with each other.

So, we have x, y, z which are the body set of axis attached to the body and we have x prime, y prime and z prime which are the space set of axis, right. Now we start rotating. Now we start the movement. Firstly, what do we do and let us assume that we have some let us just draw a laminar object just to give you a better understanding of the case.

So, let us say this is the object for which we have 2 sets of axis coinciding at time t equal to 0, right. Now as we move on the first rotation we get, we call it the phi rotation, what is this phi rotation? So, let us say the phi rotation is about and please understand this x prime, y prime and z prime these are fix primes. So, whatever we do these are not moving x, y, z when we are moving the object the moving with moving the object this xyz axis will be moving.

So, first is rotation in anti clockwise direction by an angle phi around this z axis. Let us, it does not matter, whether it is a body z or spaces z because at this point of time both are same. So, now, how does this new set of axis will look like? So, let us assume let us draw the first at first this is the x prime, y prime, z prime for new set of axis it is. So, this is phi, this is phi right.

Now, we will call this set of axis as, what should I do let us call it capital X, yeah just a nomenclature, capital Y and capital Z. And capital Z and small z prime they are coinciding at present. So, the object which we were discussing will also rotate slightly in the x y. So, it is a rotation of this object.

Next we rotate this object around this new X axis by an angle theta. So, if I try to draw this how what is the best way of drawing it let us try this out see. So, we have this object let us say. Now, we are moving it tilting it by angle theta. So, let us say this is the new set of. So, it was the initial position and once again we draw, so this is a bad drawing let us not use this there is a reason for it I will tell you in a moment. So, actually a good drawing will be this hopefully this is my object initially. So, this is my x right. So, I will explain a in a moment what it is just be patient right. So, once again we have z x y x prime y prime z prime. Now, let us use this is my y, this is my x and this is my z.

So, initially we rotated this axis system by in a anti, so all these rotations are anti clockwise direction. So, the rotation direction was this and next we rotated it by theta which the direction of rotation is these along this. So, forget about this along this new X axis. So, now, what happens in this new set which we call x prime capital X prime, capital Y prime, capital Z prime right and this capital X prime and capital X coincides with each other because capital X is after first phi rotation ok.

Now, because rotation is around this axis, this axis remains invariant, right. So, we have this (Refer Time: 11:04) and lastly. So, this is theta rotation and lastly we will have something called the psi rotation. What is the psi rotation? Psi rotation is about this new Z prime axis, so new z axis after the first and second rotation by an angle psi once again in the anti clockwise direction. So, how we will that be let us try to draw this first. So, this is my initial position of the object this is just for reference this was my second yeah. So, this was my position after 2 rotations capital X prime and now I rotate it by another psi around this z axis which is capital Z prime now it will go to. So, this is capital Z prime, it will go to small z and my new y will be small y and x will be small x right and original position x prime y prime z prime, right.

So, we have one theta phi rotation which is around the initial z axis where body and space I mean both coincides then the second rotation is around the new x axis, which is by an angle theta in the anti clockwise direction and the last psi rotation is around new y axis sorry new Z axis after this second rotation by an angle which is given here. So, this is the angle psi. So, phi theta and psi these 3 are my, so this is phi this is theta and this is psi.

So, this 3; please remember that this is only one definition of Euler angles. So, Euler angles can be expressed. So, this is a Z X Z definition it is called Z X Z because the first rotation is around Z, second is around X and third is once again around Z. So, we can likewise we can have X Y X, we can have Z Y sorry Z Y Z yeah. So, all these are possible combinations we can have. So, for our discussion will just. You can check in Wikipedia there is a very nice discussion like in many other articles in Wikipedia where the alternative definitions of Euler angles is quite.

Now, we have the angles what to do with it? We have defined 3 sets of angles and if you pay attention the first set of axis was x y z, I should not write this as my. So, first set of axis let us say was x prime y prime z prime let us not write this. So, and it was coinciding 2 sets of axis were coinciding. So, I will level them both as this.

Next set of axis is x capital X, capital Y, capital Z; third we have X prime, capital Y prime, capital Z prime and last we have small x small y small z. So, the flow of movement if we just go by the notation it is x prime y prime z prime, just to summarize we have phi then we go to capital X, capital Y, capital Z theta we have, capital X prime, capital Y prime, capital Z prime phi theta and then we have psi we go to x y and z. So, this is my first space set of axis and it remains unaltered and this is my final body set of axis which is attached to the rigid body.

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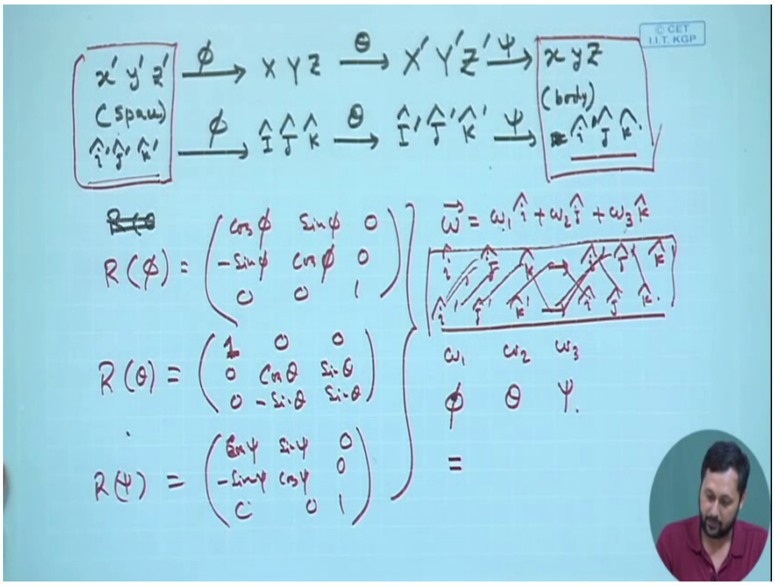

So, this 3 angles essentially the combination of these 3 rotations can describe any arbitrary rotation around of rigid body. So, these are the most general case we can have in terms of rigid body rotation. These are proofs, so there are proofs where which are more mathematical in nature and which we are avoiding all together, but we will do some mathematics of course, we have to do some mathematics. So, let us look at it in a slightly different way. Initially we have i prime j prime and k prime as our sets of coordinate then after the first phi rotations let us say, we have capital I, capital J and capital K as our 3 direction vectors. Then let us say, we have capital I prime, capital J prime and capital K prime for the next theta rotation and for the final psi rotation, we have small x or small i prime small or small I, small j, small k.

So, and as I said already each of this coordinate transformation can be expressed in terms of a rotation, I mean these are all rotations, so essentially we can link up these 2 sets of coordinate by some rotational transformation matrix. So, this rotational transformation matrix if I write it out, sorry. So, first we have to start with R phi which will be please remember that phi rotation is again against around z direction. So, we have cos phi sin phi 0, minus sin phi cos phi 0, 0 0 1.

So, similarly R theta which is again against an x axis about an x axis, so x axis remains invariant and we have cos theta sin theta minus sin theta sin theta. See you have 0 0 and 0 0 here and 1 in the diagonal term; that means, this direction is invariant. So, x for this second rotation x is invariant and for the third rotation R psi which is once again is around a z axis. So, will have this and cos psi sin psi minus sin psi cos psi. And everywhere it is a anti clockwise rotation anti clockwise, and please remember these are passive rotations because actually we are assuming that the can we call it passive yeah actually we can call it passive rotations right because the coordinate system itself is moving, right.

Anyway, now; what to do with these matrices? Thing is we can actually use this matrices to track down the change in this coordinates. So, essentially if we want to write a vector. So, let us say the our final target is we want to decompose omega into in terms of this final sets of system because we always try to end up by this set of rotation in an axis system which is not only the body frame, but also which is chosen to be the axis the what you call the inertial axis system. So, in this inertial axis system omega will be decomposed into omega 1 I cap plus omega 2 j cap plus omega 3 k cap.

That we know already and this is j and k they are continuously changing because this rotation see this values of phi theta and psi, if we take combinations of phi theta and psi we will get different values of i and j and k corresponding to the static frame this is not changing this is static this 2 things they are static. There is a space and these final things they are moving. Now using these rotation matrices systematically we can write out I, j and k this 3 unit vectors in terms of i prime, j prime and k prime; right. So, we can do that this is this is what well actually we do it in the other way around, but we can do that what we can do is we can actually start in a slightly different manner. We can start by writing you know we can start just give me a second I will write, right.

So, we can do that also what we can do is we can write i prime, j prime and k prime in terms of i, j and k. So, finally, what we are trying to do is we are we are trying to express omega 1 omega 2 and omega 3 in terms of phi theta and psi. Now if we can do that, so forget about this is strictly is not needed at this point to under probably it is what I said is not correct let us forget that. So, let us start once again from here we have omega 1 omega 2 and omega 3 where i, j and k is expressed in the body set of axis which are continuously rotating with the rigid body.

But if we can write omega 1 omega 2 and omega 3 in terms of phi theta and psi which is in some sense associated with the space set of axis by this sets of rotation, then in principle if we integrate this equations for omega 1 omega 2 and omega 3 and get the expressions of omega 1 omega 2 omega 3 terms of this 3 angles what we can essentially say is we can essentially comment on the nature of the motion because all these angles they are somehow connected with the space set of axis. Will take an example of symmetric torque under heavy symmetric torque and we will see how it happens.

Now, what can be done is what can be done is we can write this equation in an slightly alternative form see we can we are finally, aiming to write in terms of omega 1 i cap omega 2 j cap and omega 3 k cap, but what we can begin with we can start writing, the angular velocity corresponding to this rotation to be omega phi which is nothing but phi dot. Similarly we can write the angular velocity with respect to this as omega theta which is equal to theta dot and here we can write omega psi which is equal to psi dot.

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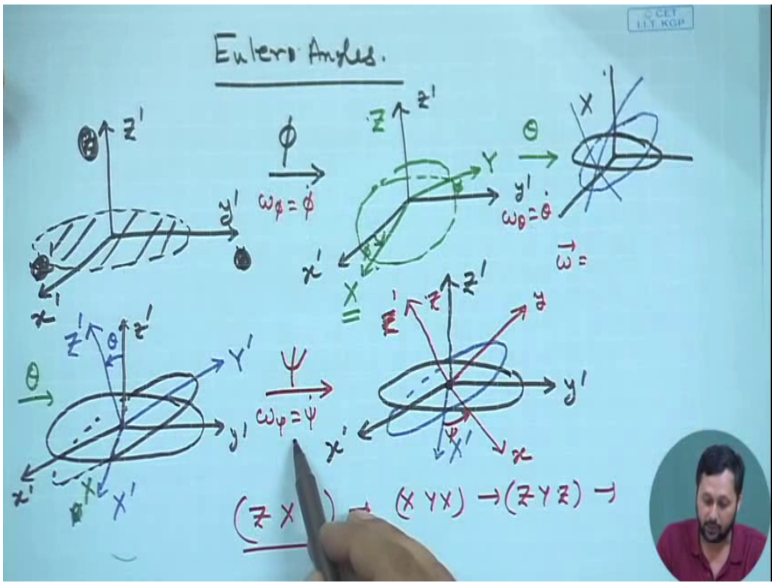

Now, if we write omega in this in terms of phi theta omega phi omega theta and omega psi we can write it as omega equal to omega phi capital sorry, omega phi small k cap plus omega phi theta write omega theta I cap plus omega psi K cap right which is nothing, but phi dot K cap or small k capped plus theta dot capital I cap plus psi dot capital K cap, right.

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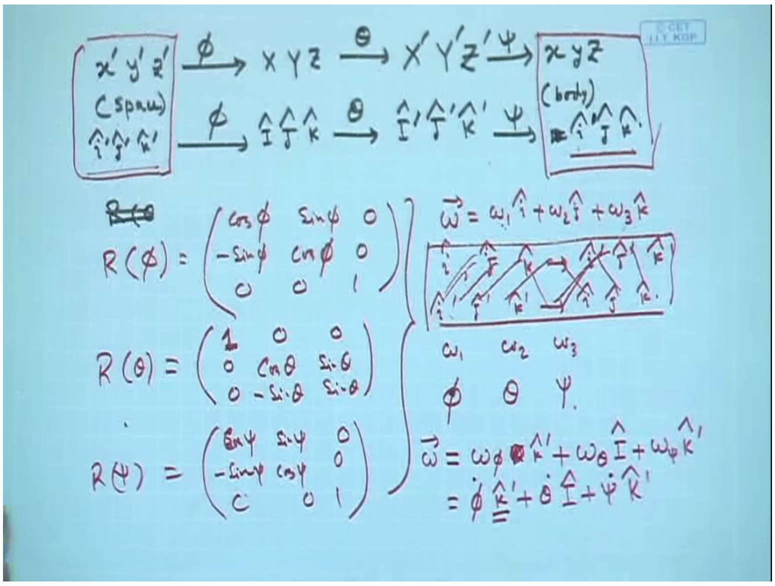

Now, this we have this transformation relations and using this transformation relations we can essentially express, this small k capped in terms of, so what is said here actually which is scratched what scratched out essentially this small i mean x prime y prime z prime or small i cap small j cap small k cap can also be expressed in terms of i j and k.

So, similarly this I can also be expressed in terms of i, j and k and K capped can also be expressed in terms of i, j and k which are the final body sets of coordinate. So, once we do that we can it is it can be shown please check theoretical mechanics by spy gel for a detailed you know detailed derivation we are not doing it, it is just you know it is the same thing we have to just do systematic matrix multiplications. And finally, we will end up in a relation which is omega 1 phi dot sin theta sin psi plus theta dot theta dot cos psi omega 2 is equal to phi dot sin theta cos psi minus theta dot sin psi and omega 3 is equal to phi dot cos theta plus psi.

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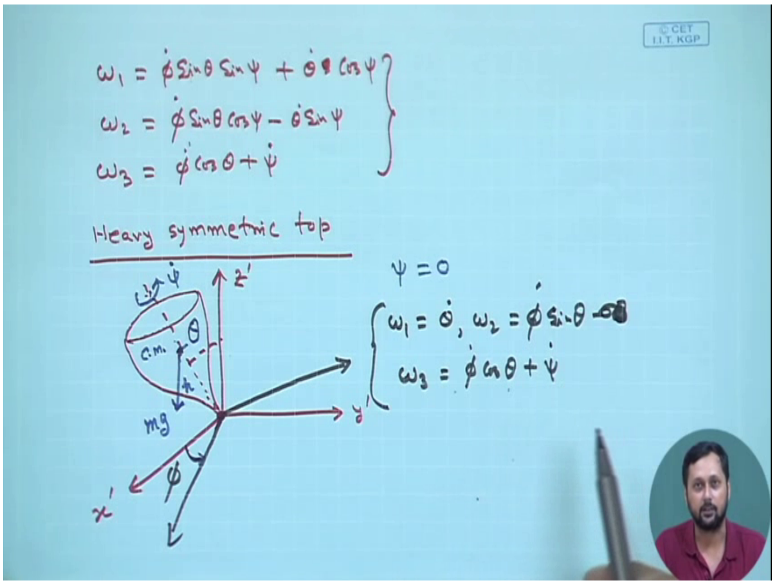

So, these 3 are the sets of Euler's, these 3 are the expression which correlates omega 1 omega 2 omega 3 expressed once again in body frame, but now we have 3 coordinates to track the changes in omega 1 omega 2 and omega 3. And these 3 changes are these 3 coordinates are somehow related associated with associated with a rotation around body set of axis I sorry space set of axis. So, this can be used in order to you know in order to determine the motion of certain object.

So, we will take the example of heavy symmetric top and figure out how to use these 3 sets of equation in order to express the motion of this. Now, what is heavy symmetric top? You must have seen you must have played with child top, right. So, heavy symmetric top one of the examples of heavy symmetric top could be a child top why it is called heavy because so far we have discussed symmetric top in top free condition now when a child top rotates on the table on a on a plane it also has it comes under the action of influence of gravity and that gravity is let us say if this is the axis of the top which is never I mean in very special case can stay just upright like this.

But even then there is a gravitational pull which is acting along the line of this top axis of this top, but moment I mean this is a very special case we will see that later, but in general the axis is slightly tilted and the torque I mean we know that the top is you know sometimes the axis is also moving like these and going up and down we have seen that if we have played with this child top. And moment it is slightly tilted from this vertical position there is a; I mean there is a torque which is acting due to its weight and we will see that component of this torque will be always along the body x axis we will see all this.

So, let us try to draw this situation, let us take child top as a case of heavy symmetric top and try to write the equations. So, we have this top which is moving let us take this is the axis, let us take the origin here so I set my z prime axis y prime axis and x prime axis like this. Now in order to write the equation we can also take this omega 1, omega 2 and omega 3 and try its start from here, but what is feasible because we are using we have to do a lots of mathematics in this, in the understanding of heavy symmetric top its better if we take a coordinate system which is of course, we have to take as coordinate system which is spinning along with the top.

So, let us say, this is my phi and this angle is this angle is theta see what happens if the top rotates around its direction I mean. So, this actually is the first Euler's angle this actually is the second Euler angle and about the third Euler angle we can just take the we can just assume that torque is I mean the top is spinning with an angular velocity of psi dot. So, what we are going to do is instead of using these equations directly we will be working in a frame in which psi equal to 0.

So, once we put psi equal to 0 and so that means, the code I mean we are not considering the third Euler angle directly into this system I mean why we do that we will see in a moment because this was greatly simplify our calculations. But please and please understand by doing this we are not I mean not losing any generality of the problem.

See what we need to understand that there are 3 types of motion possible in a heavy symmetric top. So, first of all why it is called heavy because let us assume that the center of mass is somewhere in this axis by the symmetry of the problem right now this center of mass has, so we have seen that for a system of particle we have we can assume that all the forces are acting directly at the center of mass. So, the moment are the, so the total moment of this force which is the torque this is acting which is acting in I mean the force is acting in this particular direction and if this height is h this length from here to here then we can calculate what is the exact value of the torque.

Now, there could be 3 different types of motion here - one is the spin motion of the top if it is spinning the second motion is the axis can actually go up and down with respect to the fixed z axis that is the space z axis. So, that is very nicely I mean nicely taken into account by the change in theta and it can also this axis can also rotate. So, let us say we have I have 2 pens in my head let us assume that this is this one is the vertical direction that is the fixed axis and this is the axis of the top.

So, the top axis can go something like up and down up and down which is taken by this particular angle theta also it can rotate I mean it makes a precision motion around this axis either like these or like this and this precision is taken care by this particular angle phi.

And there is a third motion which is the spin which is taken care by psi dot we can just put a psi dot for the spin part right. So, what we can do is we can take this equation we can put psi equal to 0 and then we write omega 1 equal to if we put psi equal to 0 this 1 equal to 1 this is become 0. So, with we have theta dot omega 2 is equal to minus theta dot, oh sorry, this will survive because we have phi dot sin theta minus, oh this will not survive and omega 3 will be phi dot cos theta plus psi dot. So, these will be my values of omega 1, omega 2 and omega 3.

So, we will start from here in the next lecture. So, will continue our discussion, right now we have to stop here because of time limitation, we will continue very soon with the next lecture.

Thank you.