Classical Mechanics: From Newtonian to Lagrangian Formulation
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Rigid body dynamics – 12

So, let us continue with our discussion and let us solve some problems. To begin with we have the problem set for discussion on torque; I mean motion of symmetric torque or rather motion of systems.

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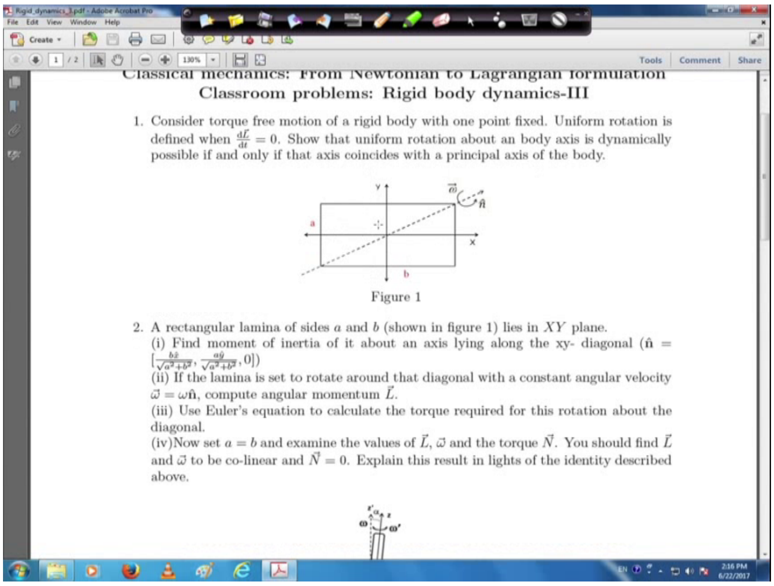

The first we start at first, we start with an identity. Consider torque free motion of a rigid body with one point fixed which we are doing in general, and torque free means N is equal to 0. Uniform rotation is defined when dl dt is equal to 0 So, that the uniform rotation about an about an body axis is dynamically possible if and only if that axis coincides with one of the principal axis of the body.

So, what is the statement of the problem? The statement of the problem is please first we recall that dl dt is different the expression for dl d ts depends on whether we measure it along the space set of axis or the body set of axis.

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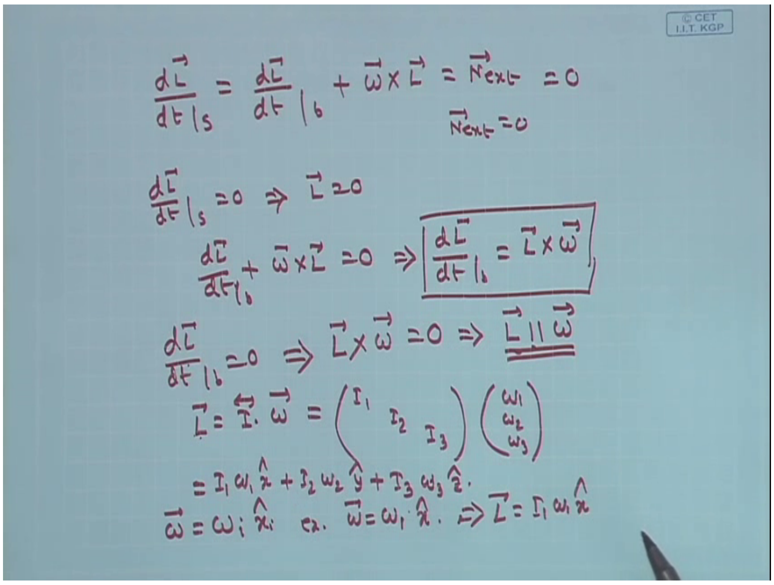

And we can write the equation dl dt space is equal to dl dt body plus omega cross L which is equal to N external. That is when one point of the rigid body is fixed, and it is executing a I mean rotational motion around that particular point.

Now, in our problem N external is equal to 0. So, if N external is equal to 0, then this whole thing is equal to 0 right. And we have already seen that from the space set of axis dl dt space is equal to 0, which implies L is equal to 0, which this we have already seen. Now what happens in body set of axis is dl dt plus omega cross L becomes 0. Which implies So, I am; so, we should write the L dt body. So, dl dt body is equal to L cross omega. Now when we are discussing uniform rotation; that means; so, uniform rotation around body axis will mean that dl dt body will also be equal to 0 right.

Because that is the definition of uniform rotation, when we are measuring from space set of axis then we are setting dl dt dl dt space equal to 0. When we are measuring from body set of axis with respect to body set of axis when we are discussing, the motion with respect to body set of axis we say dl dt body equal to 0.

Now, if we do that we see from this equation that dl dt equal to bodies in equal to 0, implies L cross omega is also equal to 0. And this implies L parallel to omega. So, we see that for torque free motion, uniform rotation is possible if and only if L and omega remains parallel, and from the relation we already know that L is equal to I tensor times omega, and only along a principal set of axis, if we if we resolve it around the principal set of axis what do we get? We get I 1, I 2, I 3 omega 1, omega 2, omega 3. So, this will give you around a principal set of axis L equal to I 1 omega 1 x cap plus I 2 omega 2 y cap plus I 3 omega 3 z cap, right.

Now, if L is parallel to omega, that is possible if and only if omega is omega I x I cap. So, omega has to be or the axis of rotation has to be one of this. If we have a general direction where all 3 components are present, then these 2 vectors omega and al and omega will never be collinear never, never be parallel to each other. Only if omega is given when omega is defined along one of the principal axis then the other 2 terms for example, if omega is equal to example omega is equal to omega 1 x cap, that is it. And then these 2 terms in the in the expression vanish is the second and third term, then L is simply I 1 omega 1 x cap. So that means, L and omega their code co linear to each other. And this condition is met and we have uniform rotation from this from as described in the problem.

So, I think this is clear from this set of calculation or set of it is not exactly calculation, but set of consideration, that only if we are describing rotation around a principal set of axis, then we can have uniform rotation for a measure please remember we are talking about body set of axis. So, uniform rotation around body set of axis is possible if and only if we are describing the rotation around one of the principal axis. So, this theorem is this identity is something which we are going to understand little bit more details in details when we are doing our next problem.

So, the next problem is a problem of rectangular lamina. Rectangular lamina which are sides a and b which is shown in the figure and it lies in the xy plane. Now in this plane the we define an axis of rotation which runs through the through the axis of the earth through the diagonal of this lamina. And so, first we have to find out and moment of inertia which along this particular axis of rotation. Secondly, we have to find out the angular velocity or angular momentum l, around this axis of rotation. And second we have and thirdly we have to calculate the torque which is required for this to maintain this uniform rotation around this axis. And finally, we have to take a equal to b and see what happens to all this quantity.

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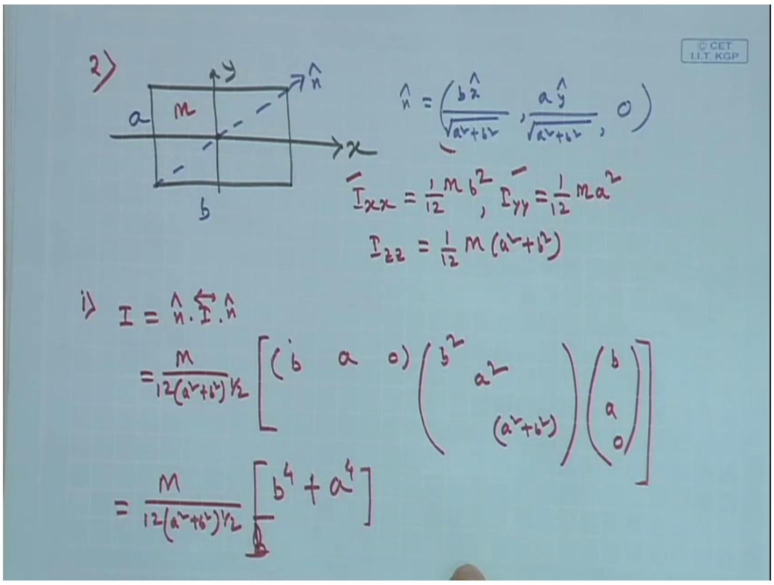

So, this was our problem one where we proved the where we got a condition for uniform rotation. Next, we have problem 2. Now for problem 2 we have our axis system, and these are this is my lamina. So, we have x we have y and the sides are given by a and b and this is my N capped right. So, N capped is given else; so, the diagonal vector when we have a and b is simply b x cap divided by root over a square plus b square a y capped divided by root over a square plus b square and 0.

So, if you take the magnitude of this you will see magnitude and direction if you consider it carefully you will see that it has a magnitude of one and the direction given in this axis. And then we have the and from the symmetry of the problem we know that we understand that x and y and the z axis which is perpendicular to both x and y. So, basically running out of they were running out of this pointing out of this plane of paper. These are the principal axis of the system. That is from the symmetry of the problem there is no mathematics needed. And I can tell you that I x x which will be one of the principal a moment of inertia is 1 by 12 m b square, which is the mass of m being the mass of this one yeah. So, this will be b square I yy will be 1 by 12 ma square and I zz will be 1 by 12, from the perpendicular axis theorem it will be m a square plus b square.

So, I am if I give you this one and this one the third one is obvious from the perpendicular axis theorem right. So, I am giving it to you to begin with you can always calculate this, it is very easy we can find how to calculate I xx and I yy in any of the textbooks. Now let us look into the first problem, first part of the problem. First, we have to calculate I which we know is a projection of the total moment of inertia tensor around this particular direction.

So, it turned out that this projection in order to calculate this projection, what we can do is we can take out a square by root over a square by b square m by 12 common. So, it will be m by 12 a square plus b square to the power half common, and inside we have b a 0 b square a square a square plus b square b a 0, right. So, if this is the case then 12 a square plus b square to the power half. Now you see this one it will give you b cubed a cubed 0. So, a to the it will finally, the result will be b to the power 4 plus a to the power 4, sorry, yeah that s it. So, this is the result.

So, there is answer to part one.

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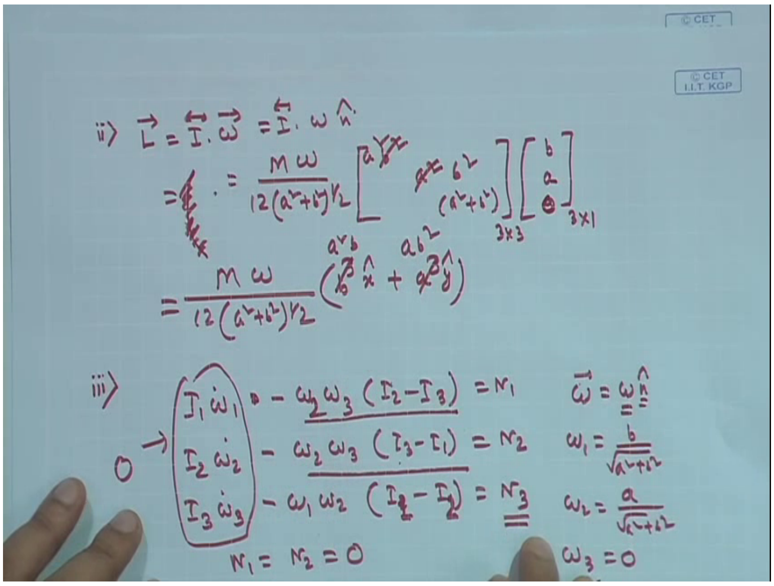

Now, answer to part 2 is where we have to find out L. And we know that L is equal to I tensor times omega once again. So, sorry once again I is given by this tensor multiplied by m by 12 pre-factor right. And omega is given by this vector multiplied by this pre factor. So, oh sorry, actually there is a mistake; so, because N is coming twice in this expression. So, this to the power half will vanish and we will have simply this, right.

Now, for this one it will be m by 12 a square plus b square. Now this half will survive, because this is nothing but I omega N kept. And there will be a omega coming up here this omega. And we have b square a square a square plus b square and for N we write simply b a 0 right. So, this will give you m omega by 12 a square plus b square to the power half b cubed x cap plus a cubed y cap. So, this is the direction of L. Of course, we see that L lie in the x y plane. There is no z component there is no z component at present that is. So, actually this will give you a vector I am just writing it as part the component, because it is 3 cross 3 matrix it is a 3 cross one matrix it will give you a 3 cross one matrix. I am just instead of writing it I am just putting it into vector notation. So, it is be cubic x cap plus a cubed y cap, right.

Now third part we have to write out the Euler's equations. And so, let us go back to the problem once. So, the third part is we have to use Euler's equation and find out the torque required for this rotation this particular rotation about the diagonal. Now we have already written in the previous lecture we already written the general form of Euler's equation I am not you know and I just do not want to write this. So, let us look at this for a while before writing anything. Of course, this equations are so, let us do it because this is for torque free.

So, it will be I 1 omega 1 dot is equal to I just give me a second do not recall this equations exactly. So, I just need to go to the relevant page of my notes right minus omega 1 or omega 2 omega 3 I 2 minus I 3 equal to N 1. Similarly, we will have I 2 omega 2 dot minus omega 2 omega 3 I 3 minus I 1 equal to N 2. Well, there could be some I no sign problem you can check that out omega 1 omega 2 I 2 minus I 1 equal to N 3 right, right sorry, it will be I 1 minus I 2, not I to minus I 1 I 3 I 2 3. So, it will be I 1 minus I 2 right.

Now, it is an uniform rotation. So, the rotation we have is omega equal to omega N capped and omega is a constant. So, if it is a uniform rotation these 3 equation, this all these 3 terms immediately vanish, all these 3 are uniformly equal to 0. Because omega is an uniform rotation. And we see N only has a and b components sorry x and y components. So, there is no z component of n. So, omega 1 is equal to b by what was it b by what was it b by a square root over a square plus v square. Omega 2 is equal to a by root over a square plus b square and omega 3 is equal to 0.

So, if omega 3 is equal to 0, then if we put it here in this equation. So, this term also vanish this term also vanish. So, without solving anything just by looking at it carefully we see this the 3 term vanish this to vanish. So, we get N 1 equal to N 2 equal to 0. So, only remaining equation is N 3, right. So, for N 3 we just have to write out the expression for N 3.

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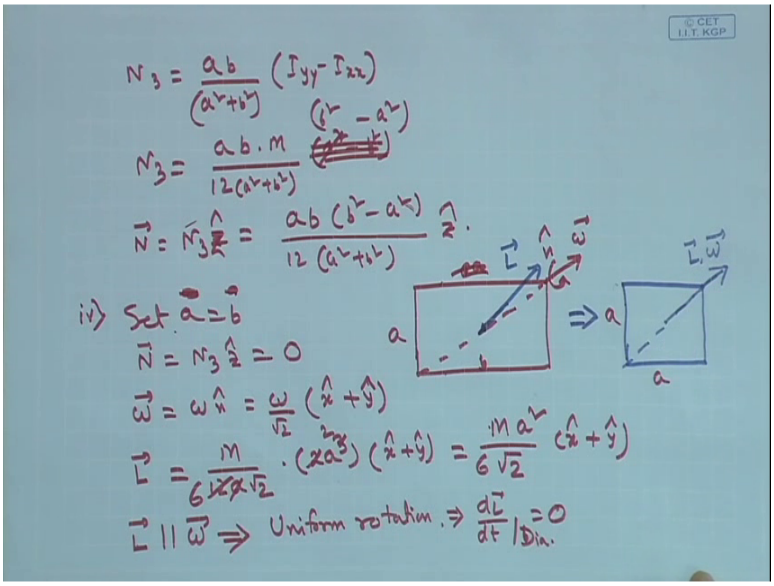

Which is by putting the values of omega 1 and omega 2 which is a b divided by a square plus b square. And so, there is a minus sign here it will be I 2 minus I 1, and if you. So, in this case I 2 minus I 1 is I y y minus I x x. So, if you put the values ab m by 12 a square plus b square it will be a square minus b square, I know. So, I made a mistake initially sorry where is it right I made a mistake. So, I xx will be ha I xx will be a cubed actually a sorry m a square, yeah this is this makes sense because it will depend on the mass distribution around x axis not along x axis. So, it will be a square and this will be b square. So, all this it will be a square and b square.

So, instead of these it will be a square b plus a b square yeah. Dimension remains same only some change. So, also for the expression of omega it will be our expression of L it will be a square here and b square here. So, it will be a square b right. So, it will be a square b and a b square right. So, this is the expression for L, omega and for the last part this will be b square minus a square because otherwise the torque becomes negative right; so, which is not possible. So, it will be b square minus a square here, right.

So, initially I wrote instead of we I by mistake I mixed up a and b. So, our final expression for N 3 is this a times b m by this term b square minus a square. So, we solved all of it.

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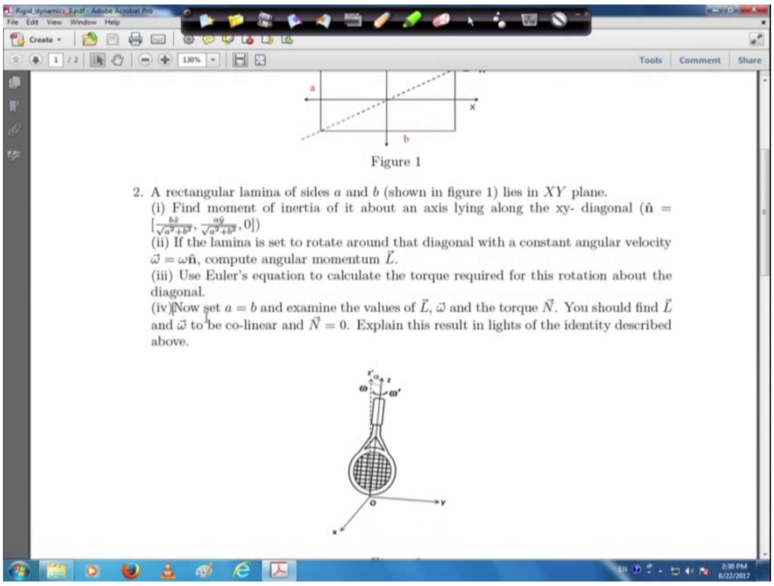

Now, let us go to the 4th part 4th part is really interesting, because in 4th part it says now set a equal to b and examine the value of L omega and N and you should find L and omega to be collinear and N equal to 0 right. So, this is very obvious actually, when we put when we look at the look back to the expression, say N is nothing but N 3 z cap. So, the torque is not entirely along the z direction. Please remember this z is the body z axis. So, if for example, if this is our lamina.

So, let us say if this is the lamina in question this book. So, if this is the lamina in question, it is moving along the diagonal like this. So, if the z z axis which is pointing upward through the center of the book. So, that axis itself is rotating with the z axis. So, do not do not think that the z, which we are talking about the torque which we are talking about is fixed in space the torque is also varying changing continuously the direction of it. Because this z is the body z axis, but whatever we can get an expression for the torque a square and this is z cap.

Now, 4th part we set a equal to b. Sorry not vectors a equal to b. What we see put a equal to b here? You get N equal to N 3 z capped which will be equal to 0. Now what happens to L and omega? So, let us examine omega, in the expression of omega which is given here. If you put a equal to b you see this first term will be a square plus a square second term is also you know and a is equal to b. So, your after cancellation your omega will be omega N capped which will be omega by root 2 x capped plus y capped right. And what happens to your L? Of course, we have an expression for L. So, this is well write out the final expression here.

So, you see set a equal to b this will be a cubed and a cubed. So, 2 a cubed this will be 2 a square to the power half. So, finally, we have m by 12 a root 2. And here you will have 2 a cubed, and the direction will be simply x cap plus y cap. Which will after cancellation So, it will be 6 m by 6 root 2 and this will be a square a square x capped plus y capped. So, we see L is parallel to omega, both are in the direction of x cap plus y cap.

So, initially when we had rectangular we started with a rectangular lamina. This is the axis of rotation and it is rotating with an angular velocity omega around this particular axis. So, the angular momentum at that time was not exactly coinciding with this is the origin by the way it is not exactly coinciding with omega staying close to omega, but actually omega was not I mean it is not an uniform rotation. So, omega will not be constant even from the space set of axis. So, it was somewhere around here which was again if I we have not calculated it from the space set of axis. So, I cannot comment how it was rotating at present, but it was not coinciding with omega, right.

Now, from here when we move to a case when we have a square plate then L and omega they follow the same direction and the torque required for this rotation vanishes. So, if we compare it with our; the result we got in our previous identity. This is the situation of uniform rotation of a rigid body. Now omega equal to l; that means, that essentially means dl dt around this particular set of body set of axis which is the diagonal, we will just write dl dt diagonal just to make it distinguished from any other dl dt body is equal to 0. So, this; what does it mean; what does it physically signify, it physically signifies a situation where the motion the axis of rotation is one of the principal axis, that is exactly what we have learned in the identity we started of the class with an identity in that identity it was said that uniform rotation about a body set of axis for the torque free case. Please understand that as we set a equal to b the torque vanishes.

So, the torque is also driven by the symmetry of the system. Uniform rotation to maintain the uniform rotation we needed torque only when the sides yeah sides a and verse rather sorry, this is b and this is a when a is not equal to b. At moment we have both the sides as a and a the torque requirement is also vanishing. And that is that all that indicates that the diagonal also becomes a principal axis of rotation. And it turned out it exactly it becomes a principal axis of rotation for a square plate.

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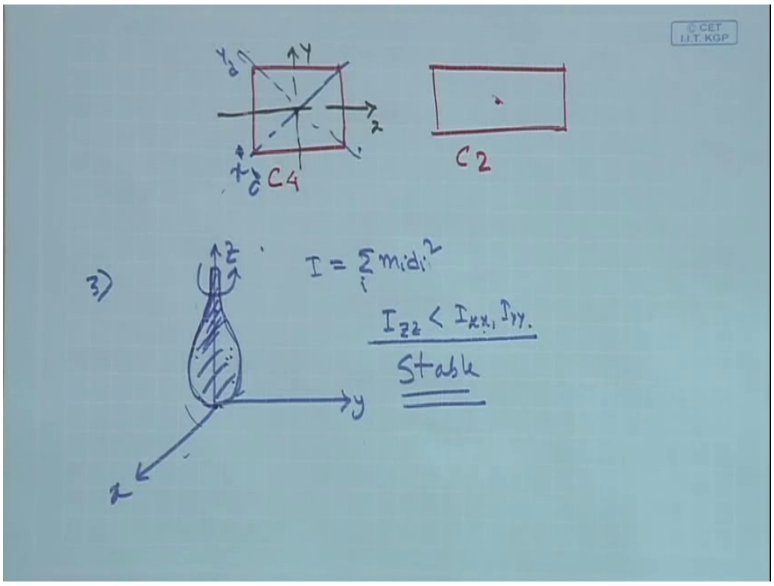

Now, what happens for a square plate. For a square plate the symmetry becomes fourfold when we have a rectangular plate of site a and v, we have a twofold symmetry; that means, if we start rotating it around this z axis which is pointing upward. So, at 180 degree it coincides with itself. So, in a in a or in other word in a total rotation of 360 degree it coincides with it is own image or own shape only twice, but for c for a square plate the symmetry becomes 4; that means, if you rotate it by 90 degree, 90-degree, 90 degree and 90 degree each 90-degree rotation it will coincide with itself.

So, in language of glue the group theory this is the c 4 symmetry point group, and this is c 2-point group symmetry. But that is not true terribly important to understand this. What is important is when you have a fourfold symmetry of the system in the xy plane there has to be 2 sets of principal axis present, 2 sets of symmetry axis present. Otherwise you cannot have fourfold symmetry. And this 2 sets of symmetry axis one is of course, this x and y, and the other one is just the axis running through the diagonal.

So, we can have either x and this rather, let us call it x d x diagonal and y diagonal. Either you can take this has your symmetry axis or you can take this as your symmetry axis, both will solve the purpose. And this is why because it the axis of rotation is a symmetry axis we have the torque vanishing or sorry not this, where is it? Yeah, yeah, we have the torque vanishing N becomes equal to 0 and omega is collinear to L fine. So, I think we get a better gain a better understanding of the situation by solve or rather gain a better understanding of the first theorem or first identity to be proved by solving the second problem. We quickly look at problem number 3, before we close this lecture. So, it is a tennis racket which is spinning.

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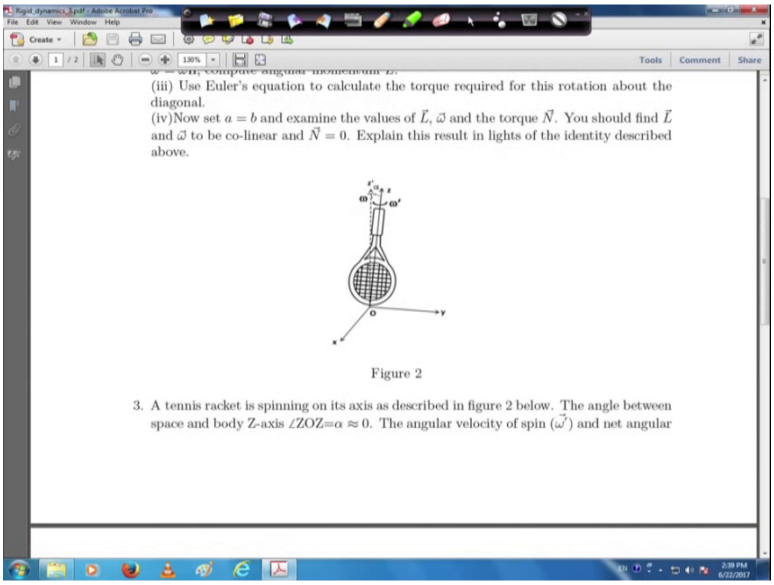

Now, with a; with an angle which is very close to 0, right. We have to comment whether this rotation will be a stable rotation or not, we have seen just now we have seen for the or in the previous lecture we have seen the condition for stability of an a symmetric top we can treat this as an a symmetric top. So, this is not a problem exactly in a sense you do not have to calculate anything, just by physical argument, you have to comment whether this axis of rotation which is along the length of the tennis racket. Whether it is a axis with you know highest or lowest angular moment or moment of inertia.

So, let us quickly look into it problem number 3. Let us assume for a second that the tennis racket, which my drawing of tennis racket is not very inspiring I know. Let us assume that it is a solid object. And you said the axis, this is your z, this is your x and this is your y. Now let us for a second under assume that this is my object, it is not exactly a tennis racket, but if I make a cut like this on this solid book, it will be a tennis racket for z. The rotation about z is like this, rotation about x which is pointing towards you let us say is like this. And the rotation about y which is pointing towards the line here it is like this.

Now, looking at the mass distribution around this 3 axis see, z axis around z axis see moment of inertia I is a function of mass distribution. If for a given axis is a function of mass distribution. From the symmetry itself, I can see that around z the mass distribution is minimal around x and y we have substantial, I mean the distance average distance from a mass point what is I? I is some of our i m i d i square. So, if mass is located far away from a particular axis for that axis, I can gain more moment of inertia. So, for z axis the average distance of mass points on this system is less compared to any other 2 axis. So, we can just by symmetry of the problem, I can say that I z set is less than both I xx and I yy. So, this is the axis of lowest moment of inertia. And we have seen from our discussion of symmetric a symmetric top, that if such a rotation takes place when a spin or spin I mean rotation takes place in an axis which is an axis of lowest moment of inertia.

Now in this case it is not exactly coinciding with z, but the angle is vanishingly small. So, we can very well assume that yeah, the point of inertia is around that particular axis which is described here, yeah it is pretty much the I zz and we can say from the symmetric criteria that this is the lowest axis. So, the x So, the oscillation will be or the moon on the rotation will be stable. So, we close this lecture here, and next lecture we will continue with the discussion of rigid body dynamics will be describing something called the Euler’s angle.

Thank you.