Classical Mechanics: From Newtonian to Lagrangian Formulation
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Lecture – 37
Rigid body dynamics - 11

Torque free motion of symmetric top is what we were discussing in the last lecture. So, we continue on the same line.

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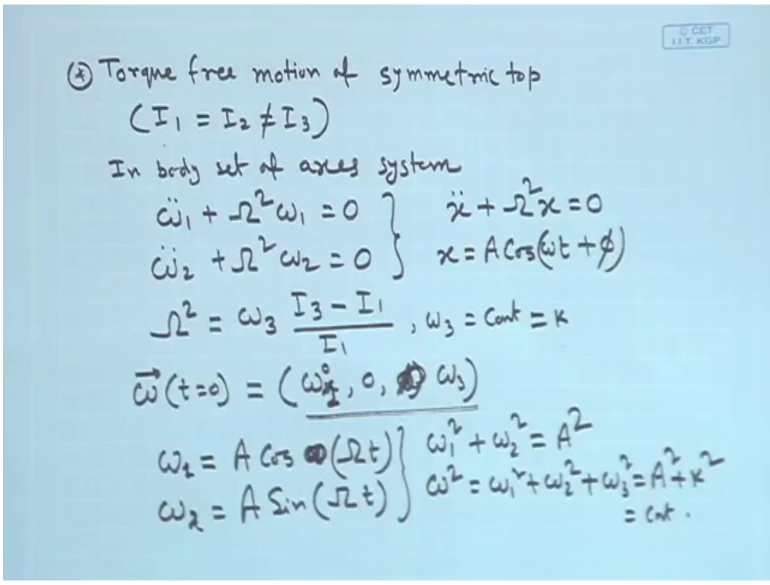

Now, if we remember torque free motion of symmetric top, where symmetric top essentially means I 1 is equal to I, I 2 which is not equal to I 3. And when we write the sets of Euler equations in body set of axis, we get omega 1 double dot is equal to capital omega square omega 1 equal to 0, and a very similar equation for omega 2. Where this capital omega is omega 3 times, I 3 minus I 1 divided by I 1, and also omega 3 equal to constant. That is what we got by setting this in the this condition in the Euler's equation for torque free motion.

So that means, in the Euler's equation we have to set the right-hand side uniformly equal to 0. Now in order to solve this 2 equations. So, one is already known, one component is already known. So, the other 2 components you have to solve for, and this equation is extremely familiar if we compare it with the equation of simple harmonic oscillator. We get which was of the form x double dot plus omega square x equal to 0, which will give you a solution of x equal to a cos omega t plus phi, something like this.

You can also write a plus a sin omega t plus phi, phi is the phase term and a is the amplitude. Now in order to solve this set of equation, what we can do is we can set that omega at time t equal to 0 was at omega x 0 0 and 0. So that means, at time t equal to 0 we can set the time t equal to 0 such that the only the x component of omega existed, and of course, this is a constant. So, you have to put sorry omega 3 here, right.

So, you can say that between omega 2 and omega omega 1 and omega 2, at x equal to 0 omega 2 was 0, sorry omega 1 should not omega x it is the same thing actually anyway. So, if you set this condition this initial condition; that means, between your first and second component of angular velocity at time t equal to 0, only the first component takes is a second component is equal to 0 you can set that. This is I will give you a physical interpretation of this very soon.

So, if you set that you get by solving this 2 sets of equation omega 1 is equal to a cos omega sorry, capital omega t and omega 2 is equal to a sin capital omega t right. So, there are essentially very I mean rotating with omega 1 and omega 2, they are executing oscillatory motion with same angular frequency omega and around the axis. So now, if we look at the physical picture this means if you take this form of omega 1 and omega 2, we see that omega 1 square plus omega 2 square, will always give you a square. And of course, omega 3 is a constant.

So, omega square which is omega 1 square plus omega 2 square plus omega 3 square will be a square plus let us call this constant k it will be a square plus k square, which is also a constant. So, what it means actually it is beautifully summarized in the book of Professor A. K. Roy Chaudhary this book, I hope you already you have copies of this book with you.

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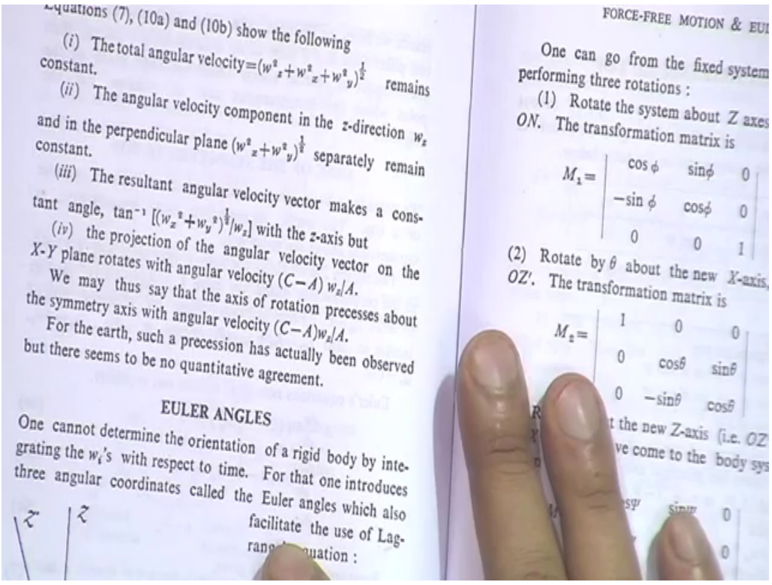
Equations (7), (10a) and (10b) show the following

(i) The total angular velocity $=(w^2_x+w^2_z+w^2_y)^{\frac{1}{2}}$ remains constant.

(ii) The angular velocity component in the z-direction w_z and in the perpendicular plane $(w^2_x+w^2_y)^{\frac{1}{2}}$ separately remain constant.

(iii) The resultant angular velocity vector makes a constant angle, $\tan^{-1}[(w_x^2+w_y^2)^{\frac{1}{2}}/w_z]$ with the z-axis but

(iv) the projection of the angular velocity vector on the X-Y plane rotates with angular velocity $(C-A)\,w_z/A$.

We may thus say that the axis of rotation precesses about the symmetry axis with angular velocity $(C-A)w_z/A$.

For the earth, such a precession has actually been observed but there seems to be no quantitative agreement.

EULER ANGLES

One cannot determine the orientation of a rigid body by integrating the w_i's with respect to time. For that one introduces three angular coordinates called the Euler angles which also facilitate the use of Lagrange's equation :

FORCE-FREE MOTION & EU

One can go from the fixed system performing three rotations :

(1) Rotate the system about Z axes ON. The transformation matrix is

$$M_1=\begin{vmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{vmatrix}$$

(2) Rotate by θ about the new X-axis, OZ'. The transformation matrix is

$$M_2=\begin{vmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{vmatrix}$$

So, I will just I mean without inter trying to interpret it in my own language, I will just refer to this few things a few lines. I do not know if you can read this, but let me tell you. So, the first line is the total angular velocity remains constant which we just proved here which we have just proved here.

So, this is one the second one is let us stick to the book for some time. The angular velocity component of the z direction point number 2, the angular velocity component of the z direction omega z and the perpendicular plane omega x square plus omega y square whole root, separately remains constant. This is also something we have seen. The resultant angular velocity makes a constant angle tan inverse omega square omega x square plus omega y square to the power half divided by omega z with the z axis. So, instead of omega x omega y and omega z, we have simply written a omega 1 omega 2 and omega 3, but it remains the same thing, right.

The projection of angular velocity vector on the x y plane rotates with an angular velocity which is c minus a omega z by a, what in in our notation this is I 3 minus I 1 omega 3 divided by I 1. C is for the I z as or I 3 and a is for I 1 and I 2. That is that is the notation this is used in this book. So, this is what it summarizes. We also got very exactly the same thing. Now what does it physically mean? It physically means that now if we try to draw these components separately one after I mean in in a single plane.

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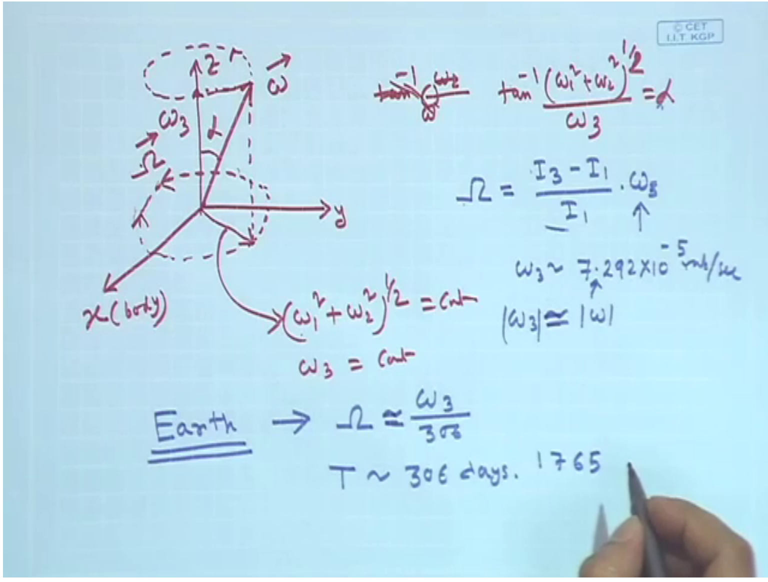

So, we have this is my first of all we have to show you that this is my body set of axis. Please understand that all these components are taken I am pressing it once again, that all these components is taken in body set of axis system. Now if this is my axis system x y and z which are body frame let us assume that we have our rotational velocity vector omega. So, our results which first result we got was what omega 3 is equal to constant, now that this result means the z projection of this angular velocity is always a constant.

So, this is your omega 3. And this result means the total I mean projected vector in the x y plane, the magnitude of the projection of your omega in the x y plane remains constant, but the projection precesses around the you know body z axis with a velocity angular velocity omega. So, if I take the projection of this angular velocity in my x y plane. This the magnitude of this will be omega 1 square plus omega 2 square whole to the power half which is also a constant, and this means this whole thing is rotating around the body z axis with an angular velocity of capital omega.

So, this is rotating keeping the same magnitude with the angular velocity of capital omega. Now if this one is rotating, and this one is fixed; that means, what is summarized in professor a k Roy chaudarys book; that means, this angle is constant some alpha which is given by tan inverse omega z divided by omega sorry, sorry this is not I just writing it wrong. It will be omega 1 square plus omega 2 square to the power half. So, half divided by omega 3, right.

So, this angle is constant. This is equal to alpha right. And keeping this same angle constant if this x y component x y projection of angular velocity is rotating in a and in some direction, could be clockwise could be anti-clockwise we are not commenting on this. Because you know direction of omega is arbitrary, and we need to have some proper measurement data in order to comment whether it is going in this direction or in this direction. We do not know that, but we know that there is an uniform velocity either angular velocity, whether is it is in this direction or that direction does not matter.

Now if this happens. So, z component is fixed, and x y component is rotating; that means, the entire omega vector itself is rotating around the body z axis. It has to be otherwise this 2 separately cannot be 2 true. So, we have this one constant, and omega 3 is equal to constant. This 2 and this whole thing is we have seen that omega 1 and omega 2 both are rotating with the same angular velocity. All this leads you to the picture that the actual angular velocity vector is rotating around the body z axis with an angular velocity of capital omega, right.

Now, look at the expression of capital omega, capital omega is given by I 3 minus I 1 divided by I 1 times omega 3, right. So, we can put the numbers in here. And omega 3 is the angular velocity. So, this is now let us take a system which describes the situation. Now it so happens that our earth our planet is an example of a symmetric top which is moving under torque free motion.

Now, why? Because we all know that earth is moving under the influence of sun, it is gravitational attraction around it, but there is no net force which is acting on earth. Because it is in equilibrium in the orbit. And it is spinning at the same time. Now this spin is a rotation I mean it is a rotational motion of a symmetric top. Earth, we can consider to be a symmetric top, it has a it you know it is something like an oblate shape. And so, z axis is the axis of rotation of the earth, which is which has a slightly different you know moment of inertia compared to x and y axis which due to it is cylindrical symmetry of the x and y direction we can choose it arbitrarily.

So, it is an example of a symmetric top. Now it so happens that this type of rotation now if we go back to this expression here, omega 3 is the rotation of you know the standard rotational velocity of earth, and this is given to be precise as 7.292 into 10 to the power minus 5 rads per second right. That we have seen. So, you see that omega 1 and omega 2 these components are very minimal. I mean we will see because this alpha is actually very small angle for earth, because alpha depends I mean. So, it all depends on the symmetry of the system, and earth is pretty much you know the angular momentum component in the z direction is pretty much dominant over omega 1 and omega 2. So, alpha is a very small number.

Now So, omega 3 is approximately equal to in magnitude. So, omega 3 we can say that the magnitude of omega 3 is approximately equal to the magnitude of omega, which is this right. And I 3 and I 1 we can I mean this is this has been calculated for earth shape, and we know the numbers. When we put this in we get for earth we get an omega which is approximately omega 3. So, if we put this I 3 minus I 1 by I 1, you will get 306 approximately equal to. That means, the time period of such time period of such rotation is approximately 306 days.

And this has been predicted by Euler by solving the Euler's equation and putting the values of I 3 and I 1, and I 3 and I 1 back in 1765 I guess. So, let me just quickly verify the numbers yeah; so, 1765.

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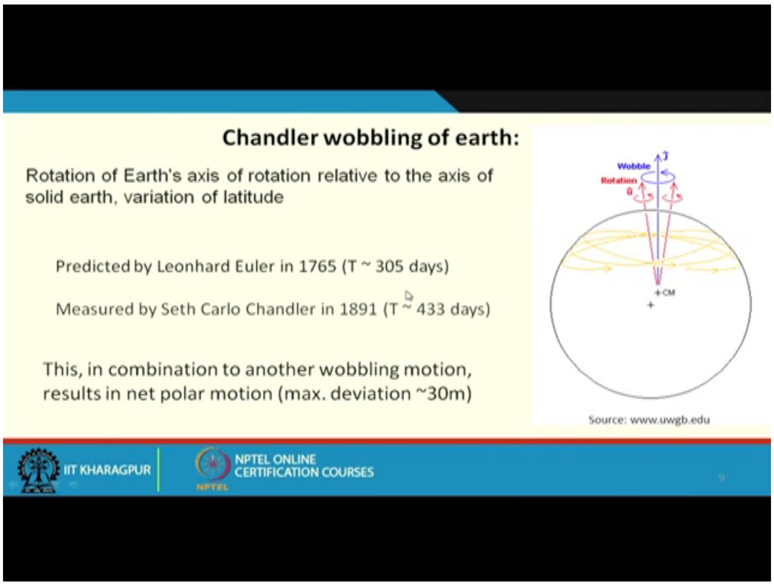

Now, there is a I mean it turned out that this rotation is called a chandler wobbling of earth which is a measurable quantity, it was predicted by leonhard Euler to be in 1765 to be around 305 days, but it was measured in 1891 by carlo chandler, and the actual measured value was found to be close to 433 days.

So now this in combination with another wobbling motion result in a net polar motion, which results in a maximum deviation of the of the north pole of approximately 30 meters. Now it needs a little more explanation I guess. So, let us come back to this.

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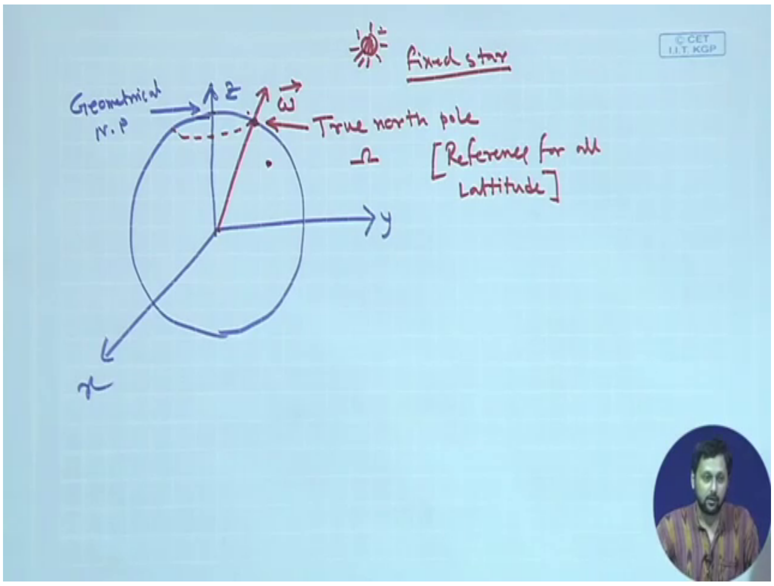

So, let us say this is your earth surface. Now this rotation what does it mean? That means, body z represents the geometrical north pole of the earth. And this one what does it represent? I will tell you in a moment. So, let us say; this is the geometrical north pole of the earth. So, it is slightly oblate in shape.

So, you have z, x and y, and please remember that x and y is arbitrary. We can choose it in any 2 perpendicular directions in this plane right. Now this is the geometrical north pole. That is coming from the shape of the earth. And this one where the angular momentum vector or sorry not angular momentum, the angular velocity vector omega crosses, I mean intersects with the earth surface it is the so called true north pole, right. So, according to the results we have just obtained, if capital omega the angular velocity is recessing is making a precision motion around the you know geometrical z axis; that means, the true north pole is actually rotating around the geometrical north pole, with a time period of capital gamma a capital omega.

So, this is the meaning of it. Now that means, all the latitude true north pole is what? Which sets the reference point true north pole is where the angular velocity vector cuts intersects the earth surface. And this sets the reference point for all the latitude and longitude. So, true north pole is the reference for all latitude, sorry not longitude longitude has a different reference point. So, it is the reference point for all the latitude. So, if the true north pole is rotating around the geometrical north pole; that means, the latitude will also change.

So, in order to measure the measure this movement for chandler did he actually try to measure in any point of the earth, we do not have to go to the poles we can sit on any point on this earth and try to measure the latitude variation with reference to some fixed the star in the sky, it is not sun by the way. It is some fixed star, let us say the pole star could be some any other star. So, let us say this is my fixed star and if I try to measure the latitude there are ways of doing it. There are even ancient days they were they were a pretty accurate instrument in which by which you can measure the latitude.

So, if we measure this with respect to a fixed star that will give you a variation of north pole right. Now it turned out his measurements were pretty accurate, and he measured for a 400 period of 433 days. Now in modern day this measurement. So, this is the situation described this red line the red axis is the rotation axis, which is the true north pole and blue line is the geometrical north pole. So, if the true north pole rotates around geometrical north pole a latitude will have this following change. You see this yellow line here on this. So, this is the change in latitude.

Now, in modern day using GPS technique, this situate this angular or this variation of north pole could be measured in a more accurate manner. Of course, there it is not as predicted by 305, as 305 days by Euler, and this is because earth is not a rigid body to begin with it has lots of fluid one third of the earth surface is fluid also inside earth. It is core is mostly in liquid state. So, all in all it is not a rigid body and this is where the deviation comes. Of course, it is not very easy to explain the deviation just by a simple model people are trying and there is also another type of wobble which is due to some other effect combination of that will give you a maximum deviation of true north pole which is measurable to be 30 meters.

So, this is what we have learned about chandler wobbling and the rotation of earth. Let us look at symmetric a asymmetric top. So, we have a system where which is given by which has an arbitrary shape; let us say.

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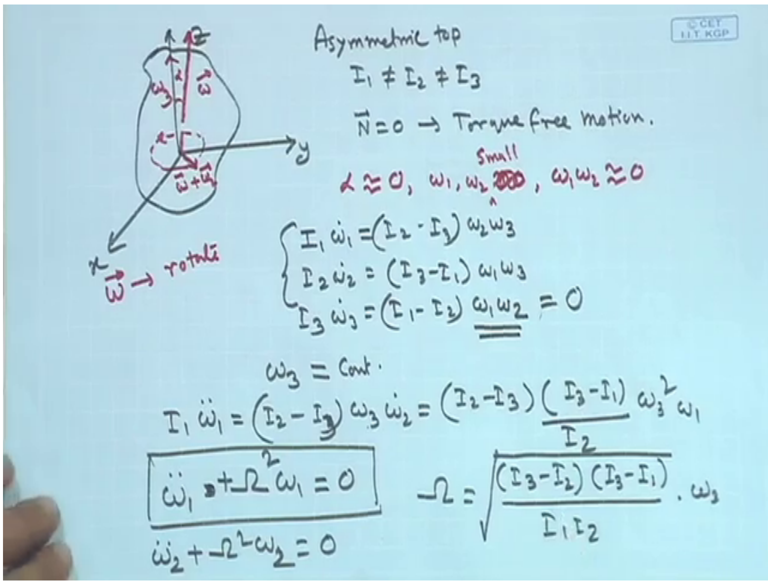

So, we have x, y and z. Now for this system we have I 1, not equal to I 2, not equal to I 3. And let us look at the torque free motion; that means, n equal to 0 for this system which is equivalent to equivalent of saying torque free motion. Now we in this in order to analyze the motion we will make an assumption, which is let us assume this is our. So, we have marked the body x, y and z axis.

Now let us assume this is our angular velocity vector angular velocity vector omega which is making an angle alpha with the body z axis. Now our assumption is alpha is very, very small. So, alpha almost equal to 0. So that means, the axis is. So, the axis of rotation is pretty much coinciding with the body z axis and it is rotating. Now if we start writing Euler's equation for this system, we get I 1 omega 1 dot is equal to I 2 minus I 3 omega 2 omega 3 I 2 omega 2 dot is equal to I 3 minus I 1 omega 1 omega 3. And I 3 omega 3 dot is equal to I 1 minus I 2 omega 1 omega 2.

So, these 3. Now as the angle is very small. Of course, if I can resolve the angular velocity into 3 components what we will get is the following. We will have a omega 3 which is in the z direction, and we will have a omega 1 plus omega 2 that will be somewhere in the x y plane. Now as the angle alpha is very small this if we are following this assumption, this essentially means omega 1 omega 2 both are also very small. Very small means we assume that they are like I mean. So, we are not assuming that they are 0, but what we are saying is they are very close to 0.

Now, if this is the case both of them are equally small. So, if we multiply 2 small numbers. So, the product omega 1 omega 2 will also be very, very small and we can or rather I would write omega 1 omega 2 small So that means, their product also goes vanishingly small. So, if this is the case then we see that the last term in the last equation this term we can we can put this equal to 0. So, the entire right-hand side goes to 0 which gives you immediately that omega 3 is a constant.

So, if the angle between rotation axis and body axis is small, then omega 3 is essentially constant. Now once again we can So, we are stuck to these 2 equations the third equation gave us omega 3 equal to 0. So, we can substitute this in we can substitute omega 3 equal to 0 in this equation, and then what we can do potentially is same construction like we did for symmetric top we can set omega 1 double dot is equal to I 2 minus I 3 omega 3 omega 2 dot. Now omega 2 dot will be I 2 minus I 3 multiplied by I 3 minus I 1 divided by I 2. That is the value of omega 2 from this equation. And also, omega 3 here omega 3 here will give you omega 3 square; right.

And of course, there is omega 1. So, this again we can reduce to the form omega 1 double dot is equal to capital omega, omega 1 or rather plus capital omega omega 1 is equal to sorry, capital omega square omega 1 equal to 0, where capital omega 1 is given by what will it that be? So, I 1 goes here. So, it will be see there is I 2 minus I 3 and I 3 minus I 1. Now what we can do is because we are changing sides we can write it as I 3 minus I 2 or multiply it by I 3 minus I 1 divided by I 1, I 2 whole root. And there is a omega square omega 3 square. So, omega 3 will come outside this root.

So, this is the expression of capital omega. We can might as well write I 2 minus I 3 I 1 minus I 3. It is similarly if we do the same procedure for the for omega 2, we can write another equation which is equivalent to this equation, which will be omega 2 double dot plus capital omega square omega 2 equal to 0, we can do that. Now this 2 equations from this equation like just like we did in case of symmetric top from here we wrote 2 oscillatory solution, solutions and then we said.

So, the omega 1 plus omega 2 component is moving in a oscillator, I mean there it is executing an oscillation around the body z axis. But unfortunately, here it is not that straightforward, because this will give you oscillatory solution if and only if omega is a real number. Now in case omega is not a real number this will give you. So, the what I mean is if this whatever under this root if this is positive definite, only then omega becomes a real number and this one gives you an oscillatory solution.

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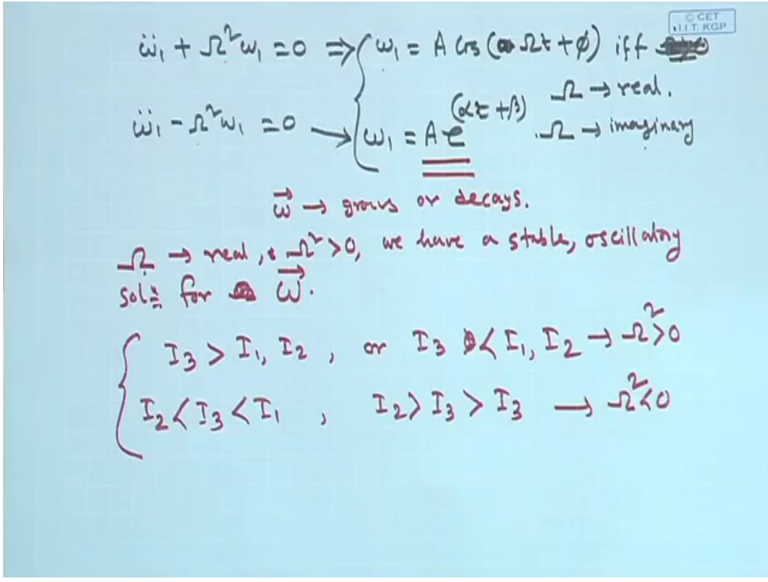

So, omega 1 double dot plus capital omega square omega 1 equal to 0 gives a solution of the form omega 1 equal to a cos sorry, capital omega t plus phi if and only if capital omega is greater than 0 right. Or rather capital omega square, what I would say is if and only if omega is a real number right. And it will give you a decaying exponentially decaying solution of the form e to the power alpha alpha t plus beta if omega is imaginary right. So, including these 2 what we can say I mean keeping these 2 things in mind. So, what I mean is if we have a solution of the form omega 1 double dot minus omega square omega 1 equal to 0. So, that will give you a imaginary solution, I mean this will give you an exponential solution could it be exponential growth could it be exponential decay depends on the sign initial condition in sign of alpha and beta.

So that means, this corresponds to imaginary omega, this corresponds to real omega. But it could be exponential I mean from physical principles we know that it could be mostly exponential decaying solution does not matter really. What is important is if it is an exponential solution, so that means, either omega 1 and omega 2 will yeah. So, if it is an if we get a solution of this particular form; that means, omega 1 and omega. So, similarly the same thing happened with omega 2 also that means, omega 1 and omega 2 will not have oscillatory nature. That essentially means omega 1 and omega 2 will not oscillate around or in a in a circular motion or execute a circular motion around the body z axis.

That means the entire omega vector, if it only rotates around this if omega 1 and omega 2 rotates then omega will also rotate right. But if omega 1 and omega 2 either decays or growth exponentially, then the total magnitude of omega also grows or decays in which case the motion will not be stable. So, if and only if capital omega is real; that means, capital omega square is greater than 0, we have a stable oscillatory solution for sorry for omega. In other case we will get a decaying solution. That essentially means now what conditions drive this weather omega will be real or imaginary it is given it is hidden in this particular term.

So, in this particular term if we have I 3 greater than both I 1 and I 2, or I 3 greater or less than both I 1 and I 2, then this term remains a positive definite. So, if I 1 is greater than both I 1 and I 2 then we have this term positive this term positive multiplication gives positive. If I 3 is less than both I 1 and I 2 then this term is negative this term is negative, both combination of both gives you positive, and then we have an oscillatory solution.

But if I 3 lies between I 1 and I 2 in this fashion or in this fashion, Then So this one gives omega greater than 0 and this one gives omega less than 0 or rather omega square greater than 0 and omega square less than 0 right. So, this will tell you by this simple consideration we can all immediately comment that the motion of an asymmetric top, please remember we are discussing asymmetric top here. This is this is stable if and only if we have a motion which is around one of the axis of highest or lowest moment of inertia. If it is not around a axis of highest or moment a lower in moment of inertia, then the motion is not stable.

So, with this we close our todays discussion on symmetric tops and motion of torque free motion of symmetric and asymmetric top. We will continue with the problems in the next lecture onwards.

Thank you.