Classical Mechanics: From Newtonian to Lagrangian Formulation
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Lecture - 36
Rigid body dynamics - 10

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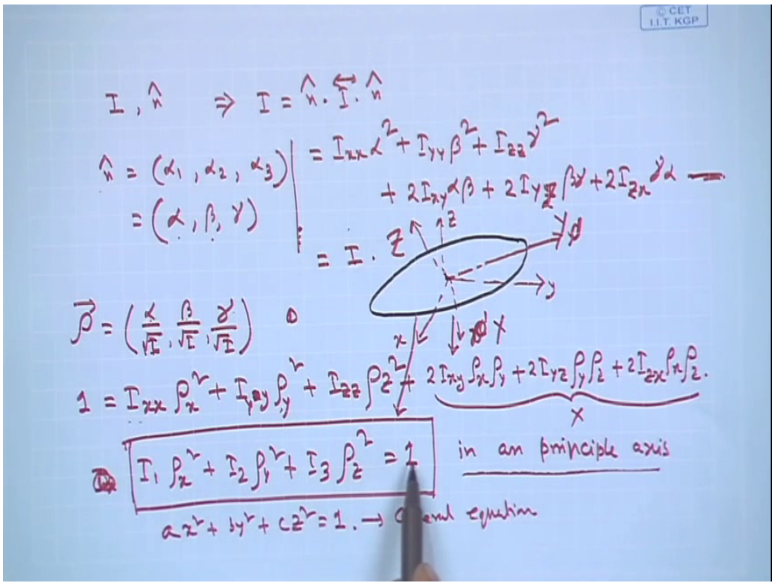

So, we continue our discussion on principle or ellipsoid of inertia. So, we got an equation which is this for any arbitrary direction. What happens is when we are when this alpha beta and gamma represents a direction of a principle axis where the cross the I xy, I yz all the non-diagonal terms vanishes that is why the equation reduces to this particular form. And in this particular form, this one is the equation of an ellipsoid, because the general equation of an ellipsoid in a 3D x, y, z plane is ax square plus by square plus cz square equal to 1. So, this is the general equation of an ellipsoid. So, we see we had an equation where all the cross terms were present now in a principle axis system this particular equation reduces to this particular form which represents the surface which represents the surface given by an ellipsoid.

Now, in this system also this is also an ellipsoid, but this ellipsoid where the axis system is not falling along the direction or different easy axis of the ellipsoid, what I mean by this is let us assume this surface, so this is an ellipsoid. So, we have both all the axis marked. Now, let us assume that this is the axis system we are working on it. So, this is my x, this is my y, this is my z. So, the surface which is tilted in this axis system is represented by this equation.

Now, when alpha, beta and gamma they are the principle set of axis, the axis system takes an orientation for this particular orientation alpha, beta, gamma, the axis system falls along the symmetric axis of the ellipsoid. So, let us call it the x prime, y prime or rather I would call it the capital X, capital Y, and capital Z axis system. For which the cross term because it the axis system is now following the symmetry of this ellipsoid surface, all these cross terms becomes 0 and the equation reduces to this. So, this is called the ellipsoid of inertia.

Now, if we have a system where we have certain symmetry present, for example, if we have spherical symmetry then I 1, I 2, and I 3 all this will be equal to some value I. And the equation will simply be rho x square plus rho y square plus rho z square equal to 1 and that gives you a surface of a sphere which is a special case of the ellipsoid. For symmetric systems we will have I 1 and I 2 will be equal and we will have more symmetry associated with this particular ellipsoidal shape and all.

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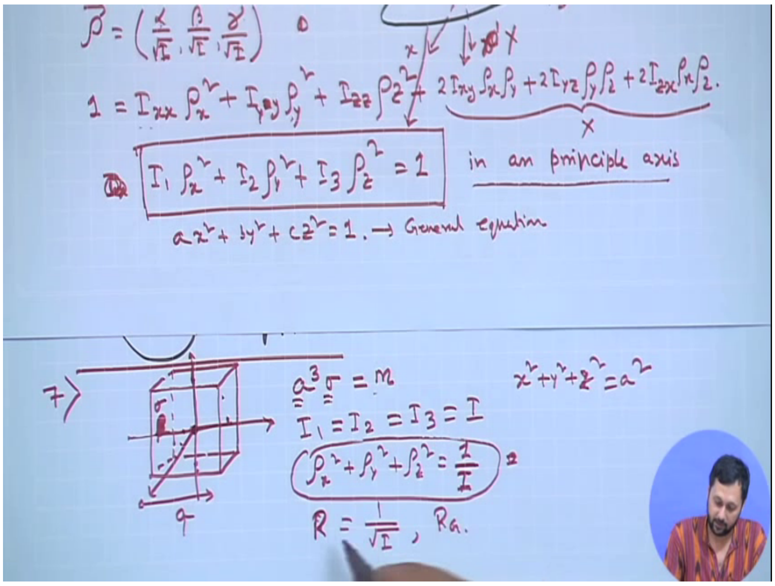

Also there is a important concept which is important in terms of a for many you know practical engineering calculations, which is called the radius of gyration radius of gyration which is defined as I is equal to M R G square. Let us take an example. If you recall in the last class or one of the last classes, we found out or we discussed about the moment of inertia of this circular uniform circular disk of radius r along an axis, which is passing through its center and perpendicular to its plane. And we have found out that I for this particular axis is given by half M R square.

Now, radius of gyration is hypothetical radius for which we can calculate for any object if for any given axis. If we know that I, so if I compare these two now if I set this equal to is M R G square, then we get R G see M M immediately cancels out then we get R G equal to R y root 2. So, this essentially means if we find out a length which is given by R G by root 2. Now, root 2 is 1.41. So, it will be something which is shorter than R G. So, if we find out this particular radius, and which is roughly these. And if an object with mass M which is the mass of this whole system rotates around the same axis with a radius of R G it will produce the exact same amount of moment of inertia.

So, radius of gyration is essentially a point mass approximation it gives a good point mass approximation for any system. For any system we can try to calculate an R G which physically means if the mass of the whole system is reduced to a point and let say this is my object, and this is my axis of rotation. So, we find out some value of I for this particular axis either we can calculate it for this specific axis or we can take the tensor for this fixed origin let say this is my origin and try to project. I try to project this tensor along this particular direction whichever way we can find out some I value. Now, this I can be equated with M R G square and we can find out some R G, which is equal to I by M whole root. Now, this what it means is if the mass of the object is M, and we represent this whole system with a point mass which is at a distance R G and rotating around the same axis it will produce exact same amount of moment of inertia. This is a very important aspect of engineering design and we will take up maybe some examples or I will give it to you in your assignment.

Now, let us move to the last problem of this set. Show that for a homogeneous cube, the ellipsoid of inertia at the center of the cube is a sphere and calculate the radius of the sphere in terms of the density and dimension of the cube. Now, we have two parts into this problem; one is we have to show that the cube is the radius ellipsoid of inertia is a sphere and calculate the radius of the sphere I mean radius of ellipsoid of inertia in terms of the density and dimensions of the cube. The first part I think we have already discussed. Now, so we are doing problem number 7. We have a homogeneous cube which is let say this one and of course, we have depth of I mean it is a 3D object. So, we go to the center of the cube. And we set our axis system there.

Now because it is a homogeneous cube with some ok, so let if I the write density as rho it will be bit confusing. So, let us call it sigma, which is typically used for surface density, but it is ok for one example. So, if each of the dimension of this cube is a then total mass of the cube is a cubed sorry total volume of the cube is a cubed and then we multiplied by sigma to get the total mass M. Now, from the symmetry of the problem, we immediately see that I 1 if I set these three axis I mean of course, from the symmetry of the problem we know that this three axis which is one actually goes through the middle of, so this is not a very good drawing.

So, one axis will pass through the center of this surface, the second axis will pass through the center of this surface, and third axis will pass through the center of this surface. So, basically one axis goes through the center of these two opposite planes this will go through the center of these two opposite planes and this one will go through the center of this to opposite plane. So, if I calculate the moment of inertia of each of this axis, it will be I 1 equal to I 2 equal to I 3 equal to sum I. What is that I, I am not going into the details of that right now. Now, if this is the case then this equation reduces to rho x square plus rho y square plus rho z square is equal to 1 by I. So, we this is the equation of a sphere right. So, the first part of the problem is we have already proved, so that the ellipsoid of inertia of this uniform cube when measured around I mean when we set the origin at the center of the cube we see that it is a sphere that is proved.

Now, second part is we have to calculate the radius of the sphere. Now, if you remember the radius sorry the equation of a sphere is x square plus y square plus z squared is equal to a square. So, this is the radius of the sphere for this particular spherical surface will be R equal to 1 by root I. So, our only job is we have to calculate I in terms of sigma and a that is M essentially. So, I am just leaving it to you to calculate you can take any of this I mean you can calculate I for any of this axis, it will be the same. Just calculate the value of I put it in here then you will find R equal to this. Also I request you to calculate the radius of gyration for this I values.

So, I will be equal for all three directions just calculate the radius of initially inertial ellipsoid and calculate the radius of gyration, you try to see if there is any relation between these two. So, with this we conclude the discussion on this part of rigid dynamics. Now, it is time that we will move to the next part which is Euler's equation, but before that we need little bit of consideration in terms of we have to write some equations you need to understand some physical properties.

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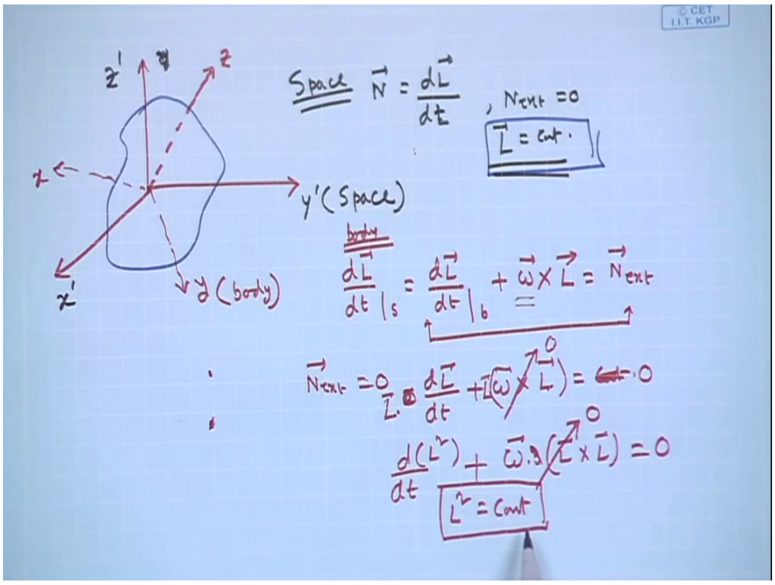

So, let us do that. So, first thing is right. So, far we have examined the rigid body sitting on the with respect to the space frame as in all the dynamics let say this is our this is the rigid body in (Refer Time: 12:44). So, this is the rigid body in question. And we have found out what is the dynamics or we have wrote equations of the form n equal to dL dt. Well, strictly speaking, we have not done much with this equations right now, but we have seen that we can write equation that total external torque n is equal to rate of change of angular momentum and that equation is strictly valid from the space set of axis. So, we will go back to our original notation, we will put x prime, y prime sorry it is z, z prime for this space set of axis. And what we do is we attach with I think we already discussed it. We attach this another set of axis with this system, which rotates along with x along with the body. So, we have z, x and y right.

Now let us try to understand what will the what will be the form of this particular equation in the reference frame which is rotating along with the body so that means, we try to gain I mean try to write this equation in the body set of coordinate. Now, dL dt space will be equal to dL dt body plus omega cross l. So, and this will be equal to n or we can also write n external total n external torque. So, this will be the equation if we are measuring things measuring angular momentum from a coordinate, which is rotating with the body with the rigid body with a velocity omega.

Now if we set n external equal to 0 that means, external torque equal to 0. Now, if we are in the space set of system, so the for space set of system this is the equation; for body set of system this is the equation. So, if we set n external equal to 0, the first equation tells you that if n external equal to 0 then l is equal to constant. But for the second set of equation, if I set n external equal to 0, what do we get, we get dL dt plus omega cross l equal to constant. Now, with what we can do is sorry equal to 0. Now, we can take dot product sorry if we I can take dot product with l dot with l from left hand side. So, the first term becomes d dt of L square and the second term actually gives you a 0, because in you can just do one step of cyclic permutation and you can write this term as it will be L omega sorry we have to do two steps of cyclic permutation. So, omega goes here, so we can write this as omega dot L cross L, which will uniformly vanish.

So, this gives you value of set n external equal to 0 sitting at the at the body frame of reference you get l square equal to constant. So, if you are were observing this from the space set of frame reference frame, then you get total angular momentum is a conserved quantity. But when you are observing it from the body set of axis system; that means, if you are as being an observer rotating along with the rigid body, it is when you are if you are also moving with the rigid body then you do not get angular momentum as constant, but the magnitude of angular momentum is your conserved quantity. So, this in this case, it is the vector that is conserved in this case this is a scalar that is conserved.

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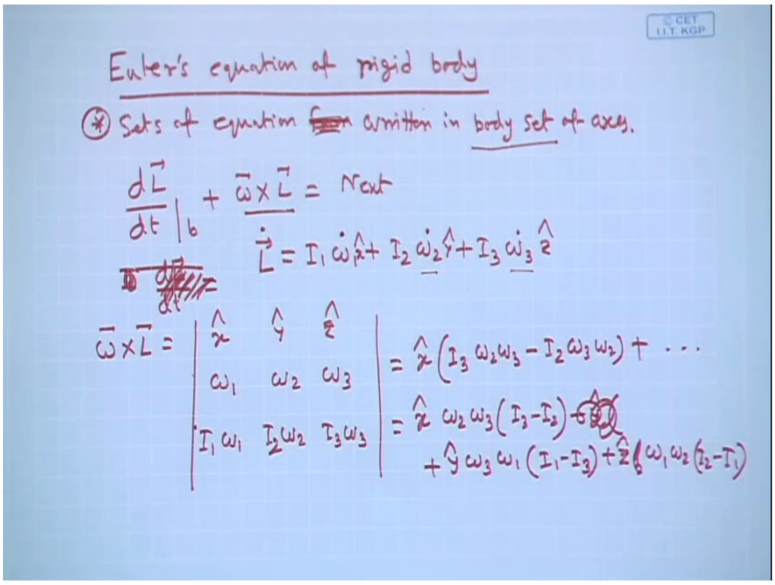

Now with this knowledge we ex basically extend this equation slightly and write three sets of equation which are called the Euler's equations of rigid body. So, what are these equations. Please note that these are sets of equations written in body set of axes. And there are ways of connecting this with the because finally we want to integrate and see what are the dynamics for certain cases of course, we are happy with sitting in the body set of axes, but certain times we would like to set ourselves in the space set of axes and see the dynamics from there. So, of course, the master equation is dL dt body plus omega cross L is equal to n external. Of course, we can stop writing body because explicitly Euler's equation is in the body set. So, we can simply try to reduce this in components.

Now, if we are going so because we are in the body set of axes and the principle set of axes will also be a part of I mean one of the orientations of the body set of axes. If we write this set of equation in principle axes system we will have certain advantage. So, let us go to principle axis system in which I or rather in which dL dt will be of rather I would start from beginning. What happens in principle axes system, we have already seen that l is equal to I 1 omega 1 plus I 2 omega 2 plus I 3 omega 3, so it will be let us call it x cap yeah there will be x cap, y cap z cap. So, that means, the angular momentum for each of this orientation will be yeah. So, x component of angular momentum is a purely function of the x rotation, y component is purely a function of y rotation, and z component is purely function of z rotation. So, that means, we are killing all the cross terms in this calculation.

So, once we are in this particular reference frame then we can write omega cross L as x cap, y cap, z cap, omega 1, omega 2, omega 3, I 1 omega 1, I 2 omega 3, I 3 omega 3 which you can find out is I 3 omega 2 omega 3 minus I 2 omega 3 omega 2 plus terms like this. So, which can be written as x prime omega 2 omega 3 I 3 minus I 2 plus y capped. So, just without calculating just by symmetry of it I can write that it will be omega 3 omega 1 I 3 minus I 2, so it will be I 1 minus I 3 plus z capped omega 1 omega 2 I 2 minus I 1. So, these are the three components of this term. And the first term will simply give you, so if I put a dot here so that means, dL dt it will be this one, this one and this one because I 1, I 2, I 3 these are constants right. So, that is why we can simply put dots on the components of omega. Now, of course, there is a vector sign here which I missed.

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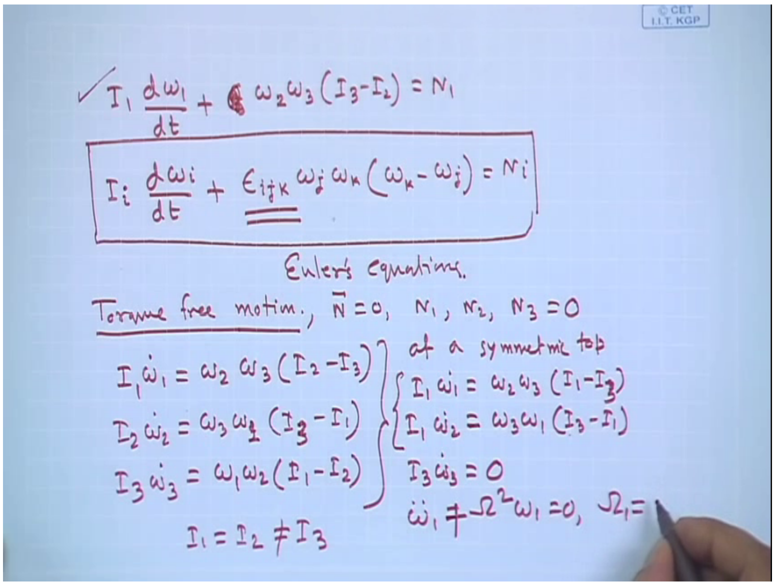

Now decomposing this into three components we can write this as I 1 d omega 1 dt plus epsilon sorry, if I decompose it will be omega 2 omega 3 I 3 minus I 2 equal to n 1. Similarly, we can write two more equations. In the general form I i just writing in terms of indices omega I dt plus epsilon I j k omega j omega k omega k minus omega j equal to ni So, this is a more general form of it. So, essentially we have three equations we have written the equation for I 1 and epsilon ijk is the standard legacy beta symbol which gives you 1, if ijk are in cyclic permutation. So, 1 2 3, 2 3 1, 3 1 2, if they are not in the this particular order this gives you minus 1; and if any two of the indices are equal that gives you 0. So, I think you are all familiar with the legacy beta function and we all know that the cross product can be written in terms of legacy beta function. So, these are my actually, so this equation is not a single equation this is set of three equations, one I have written explicitly here, this is the first equation, there will be the second and third equation, these are called Euler's equations.

Now, once we have this Euler's equation. So, we can also write it out in full and also we can take out the take a take up the case of torque free motion. Torque free motion means N equal to 0 that means, N 1 N 2 N 3 all equal to 0. Now, if we are dealing with the torque free motion and we are in the principal frame, please remember this equation is written out in principle axes system. If we are deviating from the principal axes system, then of course, there will be additional terms into this equation, and it will be not as simple as this one.

Now, let us simplify it even more; let us take up the case of torque free motion and write it out. So, in torque free motion it will be simply I 1 omega 1 dot equal to omega 2 omega 3 I 2 minus I 3 I 2 omega 2 dot is equal to omega 3 omega 2 I 3 minus I 1 I omega 3 omega 1 sorry. And I 3 omega 3 dot is equal to omega 1 omega 2 I 1 minus I 2 right. So, these three are the equations, we have in hand that is for torque free motion and principle axes system.

Now, we will make it even more specific really simplify it even more, we will take up the torque free motion for a symmetric top. Now, in symmetric top, what happens I 1 is equal to I 2, which is not equal to I 3. So, instead of writing I 1, I 2 and I 3 we can simply write I 1, I 1 and I 3. Now, if we simplify this equations even more and you immediately see that the last term we will have a 0 in the right hand side. Now, if we simplify these equations little more for torque free motion of a symmetric top then we see the equation becomes I 1 omega 1 dot is equal to omega 2 omega 3 I 1 minus I 3, because I 1 and I 2 are identical. So, I 1 omega 2 dot so I will just write out, I will replace all the I 2s with I 3s and omega 1 omega 2 I 1 omega 2 dot will be omega 3 omega 1, this will be I 3 minus I 1. And the third equation simply gives you I 3 omega 3 dot is equal to 0. And also the first two equation you see that the right hand side is identical with there is of course, there is a negative sign which is not placed here I mean. So, I can just write this as omega 3 omega 2 omega 1 into I 1 minus I 3 instead of I 3 minus I 1, so it comes with a negative sign.

So, if I reduce it, so I mean if I slightly modify this, so what can I get is we can reduce this equation 2 from the first 2 equation, I can write omega 1 double dot is equal to sum capital omega square or plus capital omega square omega 1 equal to 0. Where this capital omega 1 is given by capital omega will be let us do it systematically that will be easier.

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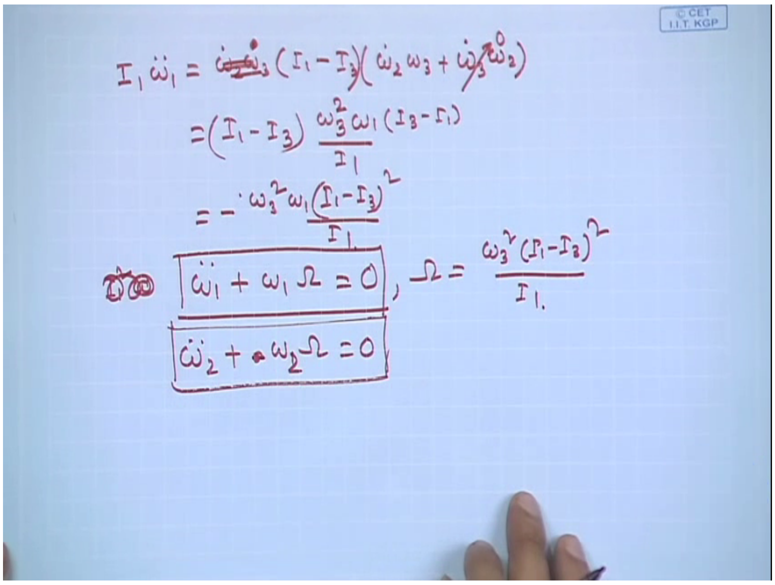

So, I can take the first equation and I can write I 1 omega 1 double dot is omega 2 dot omega 3 dot I 1 sorry omega 2 dot omega 3 I 3 omega 3 dot omega 2. Now, omega 3 dot from the third equation gives you 0. So, we can set this equal to zero. So, this will be I 1 minus I 3. Now, omega 2 dot is omega 3 omega 1 I 3 minus I 1 divided by I 1; and of course, there is a I 3, so it will be I 3 square. Now, this can be written as minus omega 3 square omega 1 I 1 minus I 3 whole square by I 1. So, take this I 1 to the other side and you immediately get I 1 square or rather I 1 to this side. So, you immediately get omega 1 double dot plus omega 1 times omega equal to 0, where capital omega is this constant which is given by omega 3 square I 1 minus I 3 whole square divided by I 1.

And also please see that this equation directly gives you is omega 3 is equal to some constant and we can set that equal to k. So, this is my constant and my equation is this. Similar equation can be written for omega 2 double dot also, so it will be omega 1 omega 2 omega which is equal to 0. So, we will have this right, these two sets of equation to solve and get a physical interpretation. I will do that in the next lecture.

Thank you.