Classical Mechanics: From Newtonian to Lagrangian Formulation
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Lecture - 35
Rigid body dynamics - 9

Hello and welcome back. So, we continue with our problems; we like solving problems in rigid dynamics.

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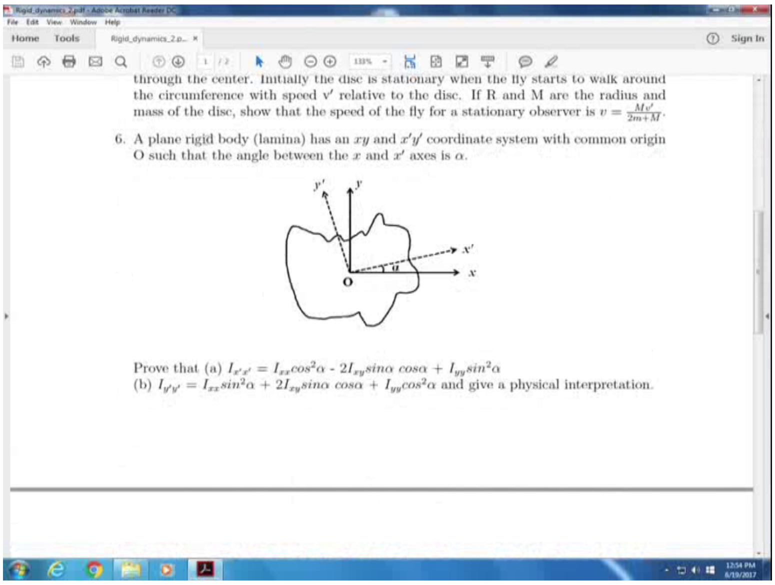

We have already done first 5 problems of this set; now let us move to the sixth problem. This problem describes a situation where we have a 2D lamina, lamina of which is lying in the xy plane. Now, there is a rotation of the axis system given with respect around the z direction; z direction is pointing up from the plane of this page right, we are giving a rotation to the axis system from x and y we are going to x prime and y prime, slightly different convention typically we keep the fixed axis at x prime y prime and rotated axis as xy, but it should not be a problem. We have to prove first that I xx and I yy's are given by this expression.

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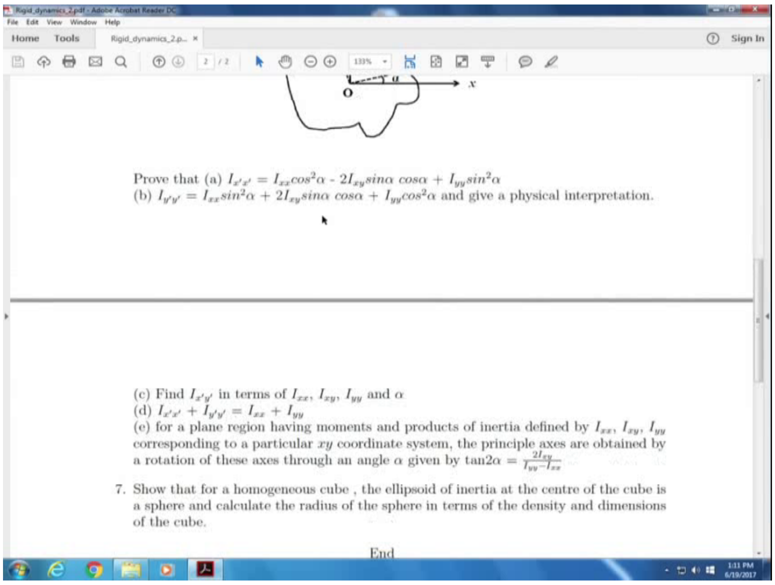

And we have to find an expression for I x prime y prime in terms of the known parameter we have to prove e and d. What is given here essentially is let us assume that we know I xx, I yy and I xy right. If we start writing this situation in terms of tensorial notation and mathematics, what do we get?

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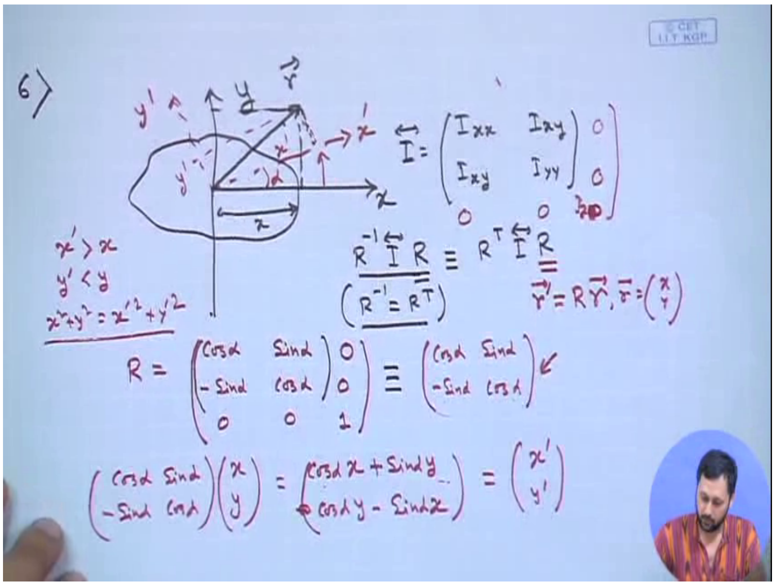

So, this is problem number 6 of the problem set. We have some 2D mass distribution sorry this is my x this is my y. Now, what do we do, we give a rotation by some angle alpha x prime y prime. The moment of inertia tensor in the xy system let us assume it was I xx I xy I yx which will be equal to I xy again and I yy. So, this was the tensorial form of the moment of inertia tensor i.

Now, we have to find out new expression for I in the new coordinate system and if you remember, we discussed this in the class and I have given you an idea on how to do this. So, a rotation can, I mean, the tensor is rotated by this following transformation; R inverse, I and R. We have to find out a suitable R, then we have to execute R inverse which is equivalent to R transpose, I, R for a group of rotation, a group of transformation called the orthogonal rotation which in this case, it is a simple coordinate rotation. In this case R inverse is equal to R transpose. For any rotation matrix, these things we are not proving neither of these or that. So, this relation for example, is something that we are not proving this is something that we are not proving, but you have to trust me on this these are all this can be easily proved by orthogonal from the properties of orthogonal transformation now.

So, basically the problem reduces to this, if we can find out a suitable expression for R, our problem is solved. Now, I am telling you R in this case can be given as cos alpha sin alpha 0 minus sin alpha cos alpha 0 and 0 1, which means the z direction remains invariant and the rotation is around z axis; that means, the z axis remains invariant and the x and y axis they are changing. Similarly, please keep in mind that this moment of inertia tensor we are just reducing using a subset of the full tensor there is a 0 0 0 0 0 term here, sorry not 0 your it will be an I zz term.

So, we are just using the small subset in a similar manner we will be using only this part of the rotation matrix also. Essentially, it is a 3D problem, because the mass distribution is in 2D, we are reducing it we are just ignoring this part of the tensor. Similarly, we can ignore this part of the rotation matrix and we can write down the final expression for rotation matrix as cos alpha sin alpha minus sin alpha cos alpha.

Now, how do you prove that this is the proper rotation matrix? Let us look at it this way; if we have a vector in these coordinate systems, for example: the vector was this, we have some vector, position vector r let us say. Now in the new coordinate, we have a projection which is x, we have a projection which is y. What happens in the new coordinate system? We have a new projection along the new x axis, which is given by x prime and we have a new projection along new y axis, which is given by y prime. Please note that x prime is greater than x and y prime is less than y. That is just from the geometry of the problem, there is no you know; you do not have to, you just have to understand that this is a length which is called x and now this length is slightly longer than this length because of this I mean direction of rotation. If the rotation was in the opposite direction, rotation of coordinate it would have been opposite and of course, x square plus y square is equal to x prime square plus y prime square. So, this is something which is which means that the length of the vector remains invariant.

Now, if I apply this or this rotation vector on r what do we get; cos alpha, please remember that like this is the rotation of a tensor, a vector is reduced affected by a rotation by this simple relation. So, r prime will be the rotation vector times r vector and r vector is nothing, but a combination of simple column of x and y. We have cos theta cos alpha sin alpha minus sin alpha cos alpha x y and then it will give us cos alpha x plus sin alpha y and minus or rather this will be cos alpha y minus sin alpha x and this is also a column vector which is my new x prime y prime right.

You see that x prime has increased in length and y prime has reduced in length. This is a proper that actually proves that this is a proper transformation or rotation matrix, so this is good.

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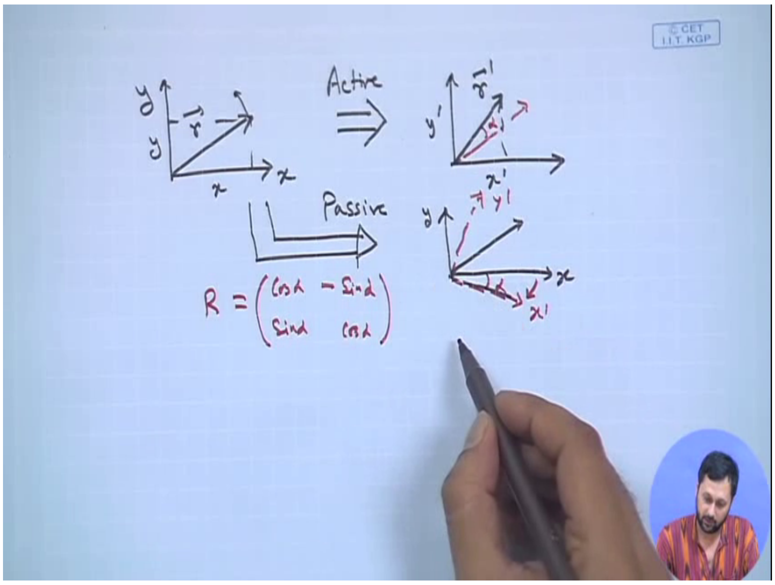

Now, also this is a site talk, but I will also like to discuss a situation. Let us assume that we have a situation where we start with this xy system with this coordinate vector or some position vector r which has x and y projection. Now we can have 2 types of rotation; one is the active type of rotation, let us say in the clockwise or anti clockwise sense, this one active rotation means the actual vector rotates the coordinate axis remains invariant. So, this is my new r prime which will have projection of x prime and y prime. It was initially in this direction and let us say, this is my alpha, so this is an active rotation.

Also we can have a situation, where instead of having an active rotation, we have a passive rotation. What is that? Passive rotation means the vector remains as it is, nothing happens to the vector, but the coordinate axis rotates in the opposite direction here the vector was rotating in the anti clockwise direction and the coordinate axis rotates. Now, we have a new set of coordinate, which is rotated in clockwise direction by exact the same amount alpha with which the vector was rotated in the first case. So, this is a passive rotation. We have, initially we had x y or let say and now we have x prime. So, the situation here is a case of passive rotation, now this will in both case it will give you the this will I mean this particular rotation matrix will serve the purpose.

Because, right now we are dealing with the case of passive rotation with which has. So, in this case, in the actual problem the coordinate axis was rotated in the anti clockwise direction. In this case we have drawn it as if the coordinate axis are rotated in the clockwise direction. So, my R in this case will be cos alpha minus sin alpha sin alpha cos alpha and what we have here we have cos alpha minus sorry sin alpha minus sin alpha cos alpha and this change in sign in the crossed of diagonal term only tells you that the rotation in this case is in clock. I mean let us call talk about passive rotation; the passive rotation in this case is in the clockwise direction.

Whereas, in the previous case, the case we are discussing currently the passive rotation is in the anti clockwise direction. So, I just wanted to give you an quick overview of active and passive rotation which probably. When we will be discussing you will be you know you will have your classes courses on mathematical physics you will be you will have a lot detailed discussion, but just a hint.

Similarly, now like we considered the vector, I mean rotation of a vector in it could be an active rotation, where the vector itself is rotating and in passive rotation where the vector is fixed and this coordinate axis is rotating. We can have a similar situation here either the coordinate frame is rotating or this entire object can rotate in the opposite direction. Both will result in the exact same you know rotation vector, result final result will be the same, but the mode of rotation will be different. Anyway now, once we have the radius vector now let us keep this aside for future.

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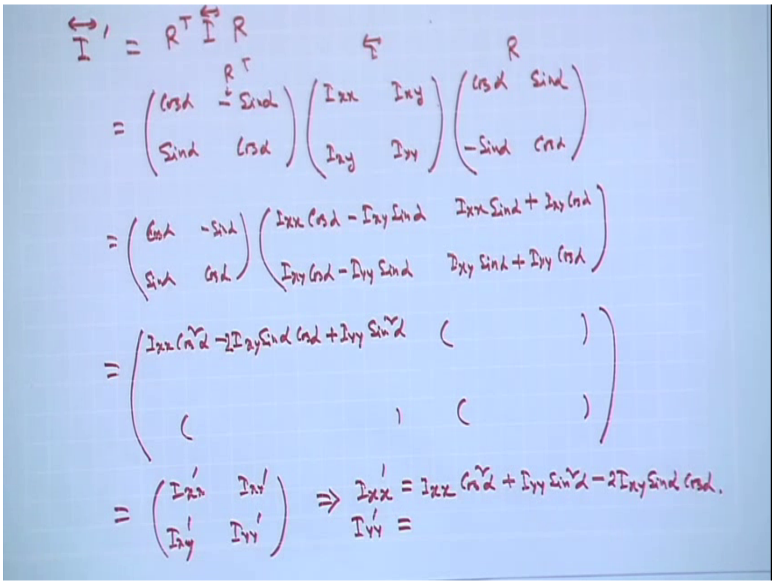

As we have the radius vector we can calculate I prime in the new coordinate system which will be simply given by R transpose I and R. Now, if I write explicitly the form of R transpose will be cos alpha minus sin alpha sin alpha cos alpha and I is I xx, I xy, I xy, I yy and we have R which is cos alpha sin alpha minus sin alpha cos alpha. So, we have R transpose this one we have I and we have R. Let us execute it. We keep this as it is, the second term will be I xx cos alpha minus I xy sin alpha I xx sin alpha mine plus I xy cos alpha. You also do the calculation along with me because I might make mistakes in the long calculation, so, you should do the correct thing minus I yy sin alpha and I xy sin alpha plus I yy cos alpha and of course, here we have cos alpha minus sin alpha sin alpha and cos alpha.

Once again we have to execute this calculation. So, it will be, you have to multiply this with cos alpha and this with minus sin alpha and add, it will be I xx cos square alpha minus I xy sin alpha cos alpha, the second term will also be I xy sin alpha cos alpha minus we will have a 2 here and then we have I yy sin square alpha. Similarly we will have one term here, one term here and one term here and this will be my new I xx prime

I xy prime I xy prime I yy prime. Comparing, we see that I xx prime is nothing, but I xx cos square alpha plus I yy sin square alpha minus 2 I xy sin alpha cos alpha.

So, which is what we need to prove similarly we can we can execute each term individually and we can write an expression for I yy prime which is the second thing we need to prove, let us come back to the question. We have the first thing, which we proved already this expression. I have we can also get an expression for I yy which is also very easy, also we have to get an expression from the for the cross term, that is I x prime y prime in terms of this which is once we execute this tensor everything is absolutely ok.

Also, this fourth one it will come out automatically, just to add this I xx the expression of I xx and I yy and what you will see is it is the 2 I xy sin alpha cos alpha terms cancels out from this 2 expression which leaves us with I xx cos square alpha sin square alpha plus I yy cos square alpha plus sin square alpha which both will give you one the sin square alpha plus cos square alpha so; that means, d I mean once you start you can have the proper expression for a and b, d is very easy.

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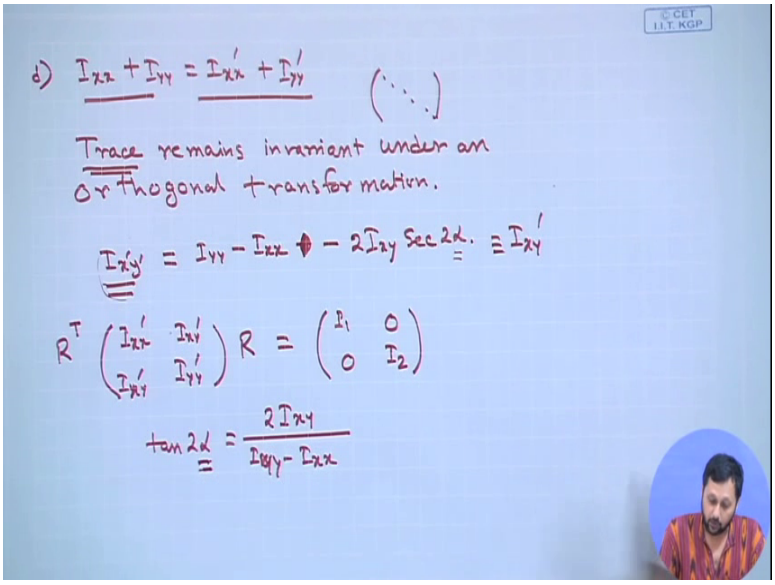

And this is also a very important result because, what it tells you that in d what we what we can prove is I xx I yy is equal to I xx alpha plus I yy alpha. This tells you that the trace; we are all familiar with the trace, for any tensor or any matrix the sum of all the diagonal elements are called the trace of the matrix. For a tensor if that matrix is representing a tensor, then the sum of all the diagonal elements are called the trace of a tensor. In this case, this is the trace of the tensor and what we see here is, the trace remains invariant under an orthogonal transformation. Trace of a tensor remains invariant under an orthogonal transformation and that is also a very important result in terms of tensor calculus, tensor algebra, which again we are not proving formally, but we are at least proving it for this specific example we are dealing with.

So, because we have not gone into the details of tensor algebra and orthogonal rotation, we cannot prove it formally. It is also very easy to prove, anyway. I told you can go through the book of Professor A. K. Ray Choudhuri, the lecture notes on classical mechanics, where there are beautiful discussions on orthogonal transformation, but here we have proved it at least for this particular case.

Now, if we go back to the question once more, the last part we have for the plane region having moment and products of inertia given by this I xx, I xy and I yy correspond to a particular xy coordinate system, the principal axes are obtained by a rotation of these axes through an angle alpha which is given by this. So, I will tell you what happens. If you calculate the expression for I x prime y prime you will find that it will give you something like I yy minus I xx just a minute I yy I xx plus it will be sorry it will be minus 2 I xy sec 2 alpha. This will be the expression what you will get if you compute part c of the problem right.

Now, what we are dealing with in part e, what is asked is that for a particular coordinate orientation; that means, a particular value of alpha, the principal axes system is obtained. I told you in the class already that a rotation; we can always find a proper set of rotation in which the matrix will be orthogonal in nature. What it means is in some particular rotation, under some particular rotation or a similarity transformation I mean we of course, for each rotation we will have a rotation matrix. Now, if we transform or apply a similarity transformation on moment of inertia tensor, using that particular rotation matrix, it will be diagonal. Now what happens in diagonal; in diagonal the cross term, if you look at it carefully, this type of terms will vanish, because after transformation the matrix is I xx prime I yy prime I xy prime I xy prime.

I actually whatever we have written here, this is actually is equal to I xy prime. I have noted it with I x prime y prime, it is the same thing actually. So, it is equivalent. What happens is, we can always find a particular R, for which R prime, R I R, will give you I

1, I 2, 0 and 0. Once we have the expression for I x prime y prime, what we can do is, we can set that equal to 0 and if we do that we will find this particular relation that tan 2 alpha is equal to 2 I xy divided by I yy minus I xx. For this particular value of alpha, the I xy term I sorry I x prime y prime term vanishes; so, that means, for this particular value of alpha the tensor becomes diagonal. For this particular rotation, we will go to the diagonal frame of this particular object. This is the physical interpretation of this result. So, this is, by the way the part of the problem you can find in the theoretical mechanics by Spiegel, you can go through that problem once more, I mean some, that is given as an exercise.

I will suggest that, you go through this problem all by yourself once more, to understand different aspects of orthogonal transformation. If you follow this problem carefully you will realize the trace remains invariant; first of all you will realize how a tensor actually transforms under orthogonal transformation, then you will know that the trace remains invariant and then you will know that there is a particular orientation or particular value of alpha, for which you will always get an orthogonal or diagonal form of the matrix. So, now, we have finished this sixth problem of the problem set. Now for the 7th problem before we could proceed to the 7th problem, we need to develop another concept which is called the inertial ellipsoid.

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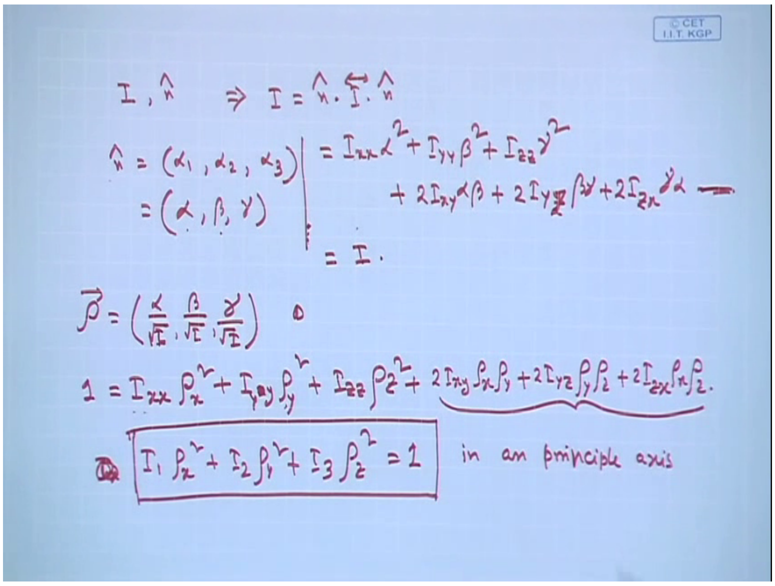

I moment of inertia for any given direction n capped, as I think this is also been discussed in the class, is given by I is equal to n capped, I tensor n capped. Now n capped, in terms of the direction cosines can be given as, alpha 1, alpha 2 and alpha 3 where alphas are the direction cosines measured or actually in more convenient notation, we write it as alpha, beta and gamma; where alpha is the direction cosine with respect to the of that particular direction with respect to x axis, beta is with respect to y axis and gamma is with respect to z axis.

Now, if we execute this, if we break it into components and execute this sum what we will get is; I request you to go through it carefully and do it yourself, I will just write the final expression for you. It will be I xx alpha square plus I yy beta square plus I zz gamma square and there will be lots of cross term; 2 I xy alpha beta plus 2 I yz sorry y z beta gamma and 2 I zx gamma alpha and, these 3 terms will be there, anything you know, that is it. So, this is equal to I, now if I construct, now what I do what I can do is, I can construct a new set of vector which is given by rho which has components of alpha by root I beta by root I and gamma by root I.

So, if I define such a vector then what I can do is, I can essentially take this I to the other hand or divide the this side by I and we can write, I xx rho x square plus I xy or I yy rho y square plus I zz rho z square plus 2 I xy rho x rho y plus 2 I yz rho y rho z plus 2 I zx rho x rho z. This we can write and this whole thing is equal to 1, because we have taken this I in the other hand. Now, look at this equation, this equation is an equation of an ellipsoid. Ellipsoid is also called the ellipsoid of evolution. So, what it is actually? It represents this 3D surface, where in each of the, sorry, not ellipsoid of evolution that is something different sorry. So, each of this dimension if I take you know the axis as rho x rho y I sorry if I in the axis system, this each of this term, if I take an axis system where the axis is marked by this vector rho instead of I xy x y z we have an axis system which is scaled by one over root I, then in that particular axis system, this one represents a ellipsoid.

Now, if that axis somehow coincides with the principal axis system, if alpha, beta, gamma represents an axis which is one of the principal axis system, that is why all the cross terms will be equal to 0 and the equation will reduce to I. So, we will just write it at I 1 rho x square plus I 2 rho y square plus I 3 rho z square equal to 1 in an principle axis. This is also an alternative definition of ellipsoid of inertia.

So, I will start from here in the next lecture. And we will continue with the discussion of inertial ellipsoid, and I will try to give you a small short physical description of this. So, that is that all in the next lecture.

Thank you.