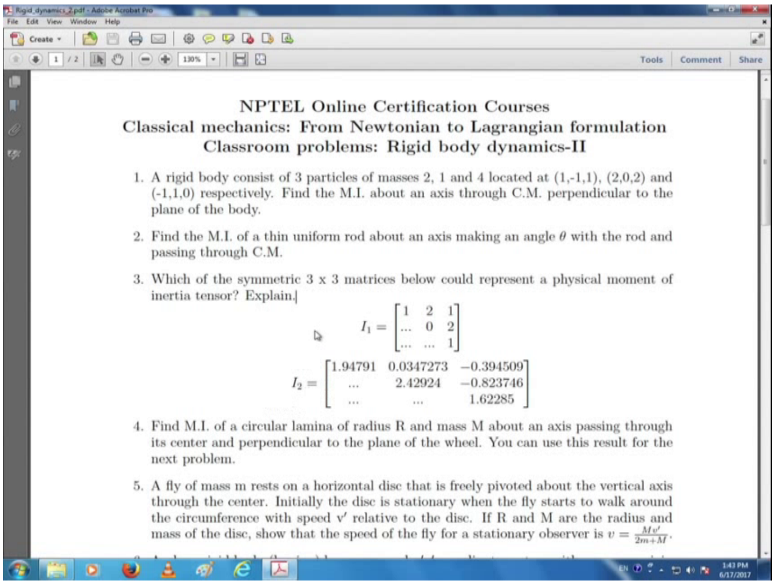
Classical Mechanics: From Newtonian to Lagrangian Formulation

Prof. Debmalya Banerjee
Department of Physics
Indian Institute of Technology, Kharagpur

Lecture – 34
Rigid body dynamics – 8

In this lecture, we will focus mostly on solving problems. So, let us start with the problem set.

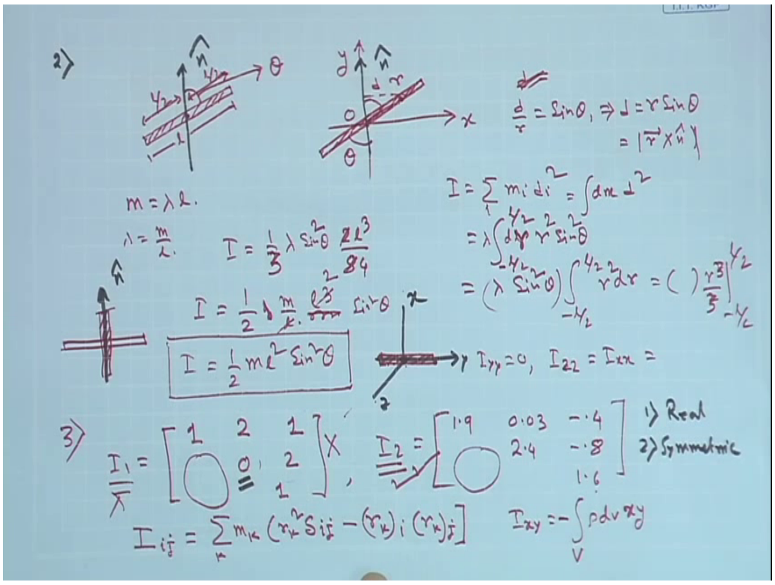
(Refer Slide Time: 00:27)



Now, in yesterday's class, we discussed the first problem, which was problem of finding moment of inertia about an axis, which is perpendicular to the plane of this plane of the object, and perpendicular to the ah, going through center of mass. Now this one we did, I mean we did not solve it exactly, but I gave you enough hint, I guess you can solve it yourself, that will be a good practice for you as well.

Now, the next problem is, find the moment of inertia of a thin uniform rod about an axis, making an angle theta, with the rod and passing through center of mass, this one is also an easy problem.

(Refer Slide Time: 01:07)



In order to solve this; so, the situation is this, let us say this is the axis, which is like this, and let us assume that, this is the rod in question. So, according to the problem passing through center of mass; now if it is an uniform rod, the center of mass by the symmetry of, it will be the midpoint. So; that means, if l is the length of the rod, then at each side we have l by 2, and this side we have l by two, and the angle, this angle is theta right. So, we have to find out the moment of inertia for this particular tilt angle theta.

Now, recall now. So, let me draw it again, just for better clarity, because I have marked so many things on it. So, this is my axis n capped, this is my, this is a rod in question right. This angle is theta. Similarly this angle is also theta. Now, so let us say assume, let us take this as origin. So, we have, if I draw an axis here. So, my x axis will be like this, and y axis will be passing through this axis of rotation right.

Now, for any point, let us say any random point at a distance R right, what we need to find is this distance d. Now this distance, we know that it will be d equal to. So, from the symmetry of the problem, it is d by R is equal to sin theta, which gives d equal to R sin theta. This is also a familiar expression. So, this is essentially mod of R cross n n capped that we have seen earlier. So, moment of inertia according to the standard formula is mi di square, which in integral notation is lambda dl. So, this will be dm d square which will be dm. We can write lambda d l lambda, being the length of this. Sorry lambda being the mass per unit length for this rod. So, the total mass m can be written as lambda l, right. So, lambda is equal to m by l.

So, lambda d l and d, or we can take a length element as dr. For example, not dl, then we have R sin theta. So, sin theta is a constant here. So, the integration limit is minus l by 2 to l by 2. So, it will be an integration minus l by 2 l by 2 R dr. So, which will give you an I, which will be half lambda sin theta, and integrating this, there is a constant here, which will be R square by two. So, we have taken this half here, minus l by 2 to l by 2. Now if you put this limit it will be. No sorry I think we, I think I have made a mistake, because this actually vanishes. So, let me check.

No, sorry, it is R square sin square theta. So, ds R sin theta, yeah that makes sense. So, R square, this is sin squared theta. So, it will be R cubed by 3, yeah it has to be this otherwise. So, 3 sin squared theta, and this will be 2 l cubed by eight. So, finally, I this will be four, I will be 1 by twelfth. Now if I can substitute lambda equal to m by l l cubed by yeah. So, we have already taken this 4 into account, which will be I equal to square half m l square. Also there is a sin squared theta here sin square. So, this is the final answer.

Now, what happens, if we change theta, if we put theta equal to 90 degree, we simply have half ml square, which is a well-known result for you know. So, if the rod is like this, and the axis is perpendicular to the rod like this, then we have half ml square, and if I put theta equal to 0 degree; that means, if the rod is along the axis, then of course, its an one dimensional object around its own axis. This moment of inertia manages. So, the answer is this, half ml square sin square theta, right.

Now, let us move to the next problem; problem 3. Now this problem in the problem set, it is a very conceptual problem, it is not much of calculation involved here, but we have to understand what we are doing . So, we have, what is there, we have 2 symmetric matrices; one is I 1 which is one some numbers of course, only the 6 matrix elements are given, and this 3 will be identical to this 3. Similarly here some numbers are given, the first matrix looks nice and tidy, it has you know very nice numbers, very you know easily understandable numbers. Second matrix is bit ugly, but also it has some ugly, because just, because it has some numbers which are you know bit long and all, but it is also matrix, both of them are real symmetric matrix, matrices symmetric we can see, because only one half is given and real matrices, because all the elements are real.

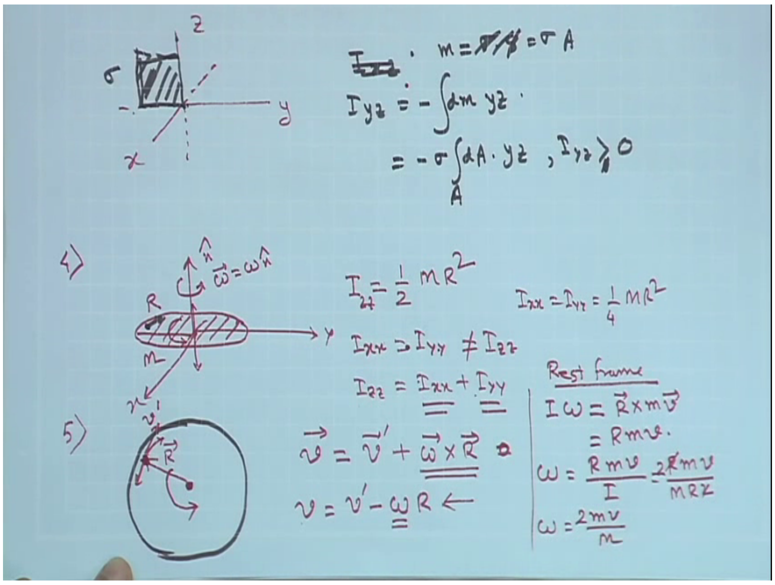
Now, which one represents a physical moment of inertia tensor; that is the question one of this is a physical moment of inertia. It represents a physical moment of inertia tensor. The second one, the other one does not. To understand this we have to realize that, we have to go through the properties of, not only the properties of moment of inertia and. So, we know that moment of inertia is a real symmetric tensor; that is both of them satisfy this. So, we have to go beyond the properties, and look more you know we have to basically try to see, which one represents a physical system, and which one does not represent.

Now let us look into this matrices carefully, we have I one. So, let us focus on I 1 first, we have 1 2 1 0 2 1, and of course, we have I 2 which I am not writing the explicit form, but just write, let us say 1.90 0.03 minus 0.4. Let say I mean does not matter really, because this numbers can be approximated minus 0.81 0.6, and it is the repetition. This half will be the repetition of this half. Similarly this part will be the repetition of this part.

Now one, and if we recall the, in a moment of inertia tensor, we have you know this diagonal elements at the moments of inertia, and up diagonal elements at the products of inertia, and the general form is I ij is equal to sum over k mk rk square delta ij minus rk I rk j, right. So, this is a general form. So, we have seen that moment of inertias are inherently positive numbers. sorry move the dia moment of inertias at the diagonal elements, and products of inertias are up diagonal elements, which could be negative as well, because the product of inertia, this term will vanish uniformly, and then will have a minus in the integration form the, for, our expression of product of inertia will be minus rho dv. Let us say Ixy will be minus rho dv xy integration over the entire volume in question.

Now, you can ask the first one does not have any negative numbers in the product of inertia tensor, and second one has some negative number is that the reason . So, product of inertia in principle we have a negative sign in the integration does not necessarily mean that all the products of inertia will be negative.

(Refer Slide Time: 11:10)



Let us take an example, let us say we have one axis system x, y and z . So, we have in the negative direction also negative x negative y and negative z. Now let us say we have a mass distribution in this particular axis system, which spreads along negative of y and positive of z. So, let us say we have a rectangular sheet standing like this.

So, now if I calculate the product of inertia I x z. Sorry Iyz negative of yiz. So, Iyz which will be minus rho. So, it is actually dm dmy z, right. So, dm in this case will be sigma dv, sigma is the lesser density of, when the surface density of this sheet sigma. So, we have minus sigma dv not dv ds. So, total mass m will be equal to sigma times s. The total area right, or sigma times a, if you do not like s. So, we have da yz. Now look at this mass distribution, it is spread over negative of y axis and positive of z axis. So, this integration is over the entire surface, entire area of this disc. This integration will always give you positive value, because one of this 2 quantities y and z is always negative, and the other one is always positive. So, resultant Iyz will be greater than 0, or could be equal to 0. Also in the case, actually it will not be equal to 0. So, it will be greater than 0 right.

So, that is why not having a negative value in the product of inertia, does not make a tensor invalid. So, that is also not the line of thinking. So, we have 3 properties; one is or rather we have 2 properties; one is real, both of them are real, second is symmetric, which both of them are symmetric, and third one is, we should have negative values in the product of. I mean we have, we can have negative values in the product of inertia term, which I have proved that it. I mean the integration has a negative side does not mean that, it will have negative values. Now what? Then what will tell us, whether this one is a proper inertia tensor or this one is a proper inertia tensor or both.

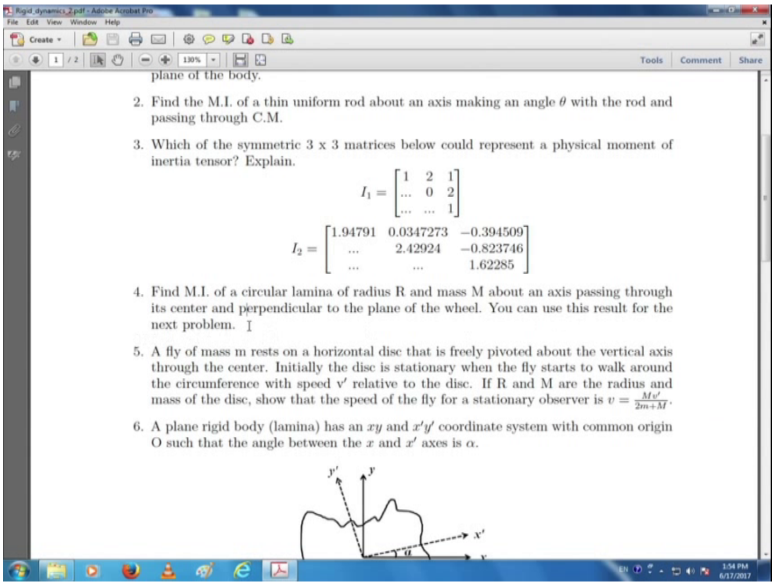
Now let us look into this carefully. So, one; we have I xx equal to 1 Izz equal to 1 and Iyy equal to 0. Now there comes the hint, now if I yy equal to 0; what we have seen here, if we put in this example, for this is a one dimensional system rod. If we put theta equal to 0; that means, the raw orientation of the rod is along the axis of rotation, because it is an one dimensional object, then we immediately see that the moment of inertia vanishes. So, in this case if I extend this picture a bit, or if I just slightly modify this picture; let us say I have my y axis, x axis and z axis. So, this is my y, this is my z, this is my x, and the rod in question, the one dimensional rod in question, is lying along y axis. Then what will happen, if this happens, then your moment of inertia around y will be equal to 0 right, and that will give you equal values of moment of inertia around z and around x, because it is a one dimensional mass distribution perpendicular to both the axis, this origin is passing through center of mass.

So, in this case we will have I y. So, we are doing it for this problem, but anyway I y y equal to 0 I z z equal to I x x equal to some number whatever. So, we can have equal numbers in this, but if so; that means, we are in this tens in this particular matrix. It describes the moment of inertia tensor, which is a one dimensional mass distribution around z. There is only around y. Now if this happens, can we have, you know products of inertia, all nonzero. Of course, not. So, product of inertia which includes, so that you see the mass distribution is only along y right.

So, whenever we have a product of inertia which is along x or along z, this must vanish, it has to vanish. So, all the products of inertia terms sum of the product of inertia term has to be equal to 0, think of it. So, this one cannot represent a physical moment of inertia tensor, because if we have a diagonal term 0, then we must have some of the octagonal terms 0, which ones. I am just leaving on to you, think of it some of the terms has to be equal to 0, this one is free from this problem. We have non0 diagonal terms, non 0 up diagonal terms, we have some negative, some positive, this is, this can happen, I have just shown you that negative integration can give you positive numbers, given that the mass distribution is in the negative direction. So, this is and this is not. So, I 2 is the physical moment of inertia tensor, I 1 is not a physical moment of inertia tensor.

So, this problem is a very conceptual problem in essence. There is not much of calculation involved except for that understanding part. We have to understand what the moment of inertia, that this tensor. These numbers are actually trying to tell us. So, this is why I included this into your problem set. I hope you and student you enjoyed solving this problem with me. Now let us move on to problem number 4 and 5, actually 4 and 5 are very much related with each other.

(Refer Slide Time: 18:04)



So, let us look at 4 quickly find the moment of inertia of a circular lamina of radius R and mass m about an axis passing through its center and perpendicular to the plane of the wheel. You can use this result for the next problem.

So, problem number 4 is very simple. You must have already done it a million times in your plus 2 standard itself. So, let us say this is my axis of rotation, and this is my lamella, which is lying in the xy plane, and the circular lamina has a radius R and mass m, and we all know that moment of inertia around this axis will be half m R square. So, I think we are all very familiar with this result. So, I am not doing it for you. What I can suggest is, you can try one thing, you can try to calculate the, so I z actually. So, you see that by the symmetry of the problem we discussed. So, this problem has a cylindrical symmetry.

So, I x x will be equal to I y y, which will not be equal to I z z, but from parallel axis theorem, we know that I z z .Sorry perpendicular axis theorem, we know that I z z equal to Ixx plus Iyy, because it is a 2 d mass distribution. So, what is suggest is, you calculate Ixx and Iyy supplement one of the Ixx and Iyy, and see if you get Ixx equal to Iyy equal to one fourth mr square. So, of course, you will get it. It has to be, because this 2 combining should give you half mr square, and they are contributing equally. So, both of them should be equal to one 4 Mr square.

So, that is just a sight thing, probably your all familiar with it already, because this is such a common problem, but let us focus on the next problem, which is what we target, actually the problem is a fly of mass m small m rests on a horizontal disc; that is freely pivoted about the vertical axis through its center initially. The disc is stationary, when the fly starts to walk around the circumference, circumference with speed V prime relative to the disc if R and m are the R and M are the radius, and mass of the bigger disc, then we have to show that the speed of fly with respect to a stationary observer is capital. Sorry V is equal to mv prime divided by 2 m plus V.

First let us try to understand this problem. So, this is the disc in question, we have a fly sitting here, very close to the edge. So, it starts walking. Now if I draw it, draw the top view of the problem. So, this is my disc in question, I hope you can see this, you see can this is my disc in question, which is pivoted at the center. So, the axis of rotation is just perpendicular to the plane. So, it is up here, as I have drawn. So, we have our fly in question is here. So, at t equal to 0, it starts walk around the rim with a speed V prime, which is measured with respect to this, this disc right, and as you see what happens, because it is a frictionless pivot, what happens moment, this just fly starts working in the. Let us say clockwise direction, this disc starts rotating in the anti-clockwise direction giving rise to a rotational velocity omega.

Now if this happens, this becomes a non-inertial frame of reference. So, what is given here is, the speed of the fly with respect to the disk, which is a non-inertial frame of reference, and what we need to find out is, a speed V which is measured from outside, that is an inertial frame of reference, writing it in vector notation, recall this R being the radial vector, which directed outwards, right. You recall this, this is the standard expression of velocity, when observed from an inertial frame V is the velocity of an inertial frame, also from an inertial frame, which is the stationary frame, V prime is the velocity measured in the rotating frame. This one and omega cross R is what we need to find out. So, our job is to find out an expression for omega cross r. Now do one more step, now let us see.

Let us assume that the fly is moving in the clockwise direction. So, the disc starts rotating in the anti-clockwise direction, anti-clockwise rotation is positive, I think we have discussed it in the initial part of our lecture, that anti clockwise rotation is positive; that means, the direction, let us say if this is how the fly is moving, then the disc is rotating in this direction, anti-clockwise; that means, it is giving rise to an omega, which is equal to omega n capped upward direction. So, it is the omega is playing, you know omega is pointing upwards right. So, we have this.

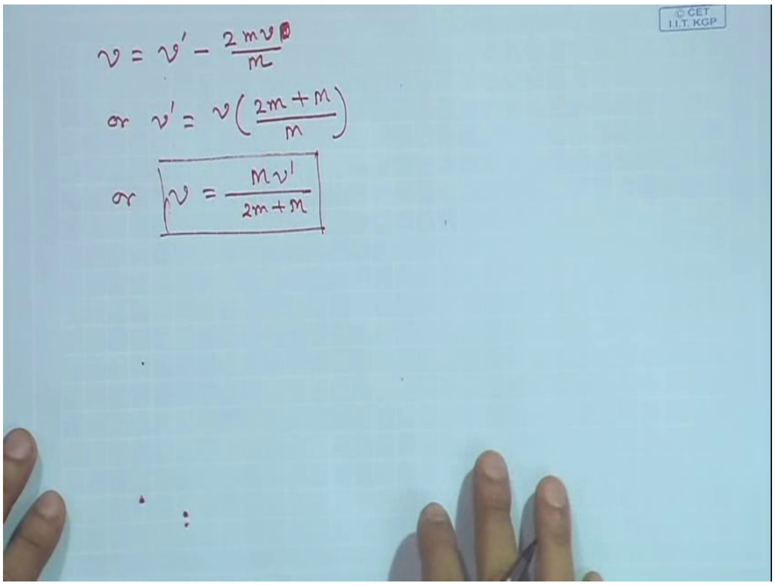
Now, you take a cross product of that omega, which is pointing out of the surface of this paper with r, and you will see that omega cross R is. So, at any point instantaneous position, I mean in the velocity V prime is tangential to the instantaneous position of this co position vector r. sorry position of this of this fly; that means, V prime is perpendicular to R and you will see that omega cross R will always be in the opposite direction.

So, if I now bring it down to scalar. So, V will be V prime minus omega r, and as omega and R they are perpendicular to each other, omega cross R will be simply minus omega r. Try thinking of Ith of it, in this way. Let us assume that fly is moving in anti clockwise direction; that means, the disc is rotating in clockwise direction; that means, omega will be pointing downwards in that case. Once again you calculate omega cross r, you will see that if fly the flies movement is in this direction, then omega cross R is in this direction. So, whatever you do, you will get V equal to V prime minus omega cross r. This is the scalar equation right. So, this is one thing still we have to figure out what is omega.

Now, thankfully we have the conservation of angular momentum. Now according to the conservation of angular momentum, the total angular momentum of the disc rotating disc; sorry, yes right total angular momentum of the rotating disc will be I omega, which we have, which by conservation of angular momentum has to be equal to the angular momentum of the flay, which is moving with a velocity V prime with respect to this frame, or rather which is moving with a velocity V with respect to a rest frame, and all the conservation laws I am pressing it again. All the conservation laws are valid only in rest frame. So, if we see from rest frame, only then I can write I omega is equal to R cross mv right, and this will be, and this will once again give you R m V, right. So, omega is equal to R m V by I right.

Now put it into this scalar equation, and simplifying R m V by I is equal to half M R square. So, it will be 2 M R square right which will on simplification. This will cancel out 2 Mv by m. So, you put it back in here, V will be equal to. I will just write in the next page maybe.

(Refer Slide Time: 27:18)



So, V will be equal to V prime minus. Sorry V prime minus 2 m V R by M, or you change sides V prime will be equal to V times 2 m plus. No I think R. Sorry there is an omega r. So, right. So, it will be sorry small mistake here, sorry this will be R which will give you omega R is equal to 2 mv by right. So, it will be 2 m plus m divided by m. So, you get V is equal to m V prime by 2 m plus m. So, this is the result clear.

So, I hope this is clear to you all, that how to solve this problem, how we did solve this problem. I will just repeat it once again, what we did was, we initially wrote the equations. So, the velocity V which we need to find out is, the velocity is as seen from the rest frame right. So, we wrote this equation which gave us this, then we wrote the angular momentum conservation in rest frame, which gave us a relation between or expression of omega in terms of non-parameters, put it in here simplifying you get this, right.

So, there are other, there are 2 more problems in the problem set, which we will take up in the next class. And what we will do is, next class we will once again go back to the theory of moments of inertia. We will define something called ellipsoid of inertia, we will try to understand what it is, then solve some problems and move on to Euler's equation.

Thank you.