Classical Mechanics: From Newtonian to Lagrangian Formulation

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Lecture – 33
Rigid body dynamics - 7

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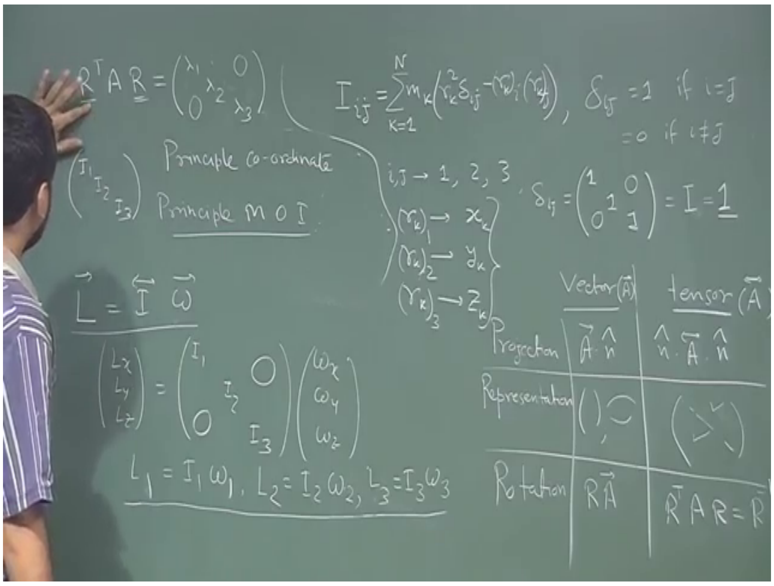

So, we are back here and we are discussing principle moment of inertia right. So, principle moments of inertia essentially are the Eigen values of this particular tensorial representation of moments of inertia right. So, what are the advantages of finding out principle moments of inertia we will see later, but before that we will just keep it in mind that what do we need to. What is the take back message from this, this discussion that the moment of inertia can be represented in this tensorial form number one.

Number 2 is any tensor in any real symmetric tensor can be. So, there is a theorem which I am, which can be proved mathematically. I am not doing it, but you have to trust me that any real symmetric matrix can be taken into this diagonal form by a suitable similarity transformation, where the diagonal elements are the Eigen values.

So, similarly following the same line, because its a real symmetric tensor, it can also be taken into this diagonal form, where the diagonal elements are the principle moments of inertia, which are the Eigen values of this inertia tensor. And in this particular frame, if I write this equation, we immediately see that if I have Eigen. Sorry if I have angular velocity in along 1 of this principle direction principle coordinates. So, that along if I have angular velocity along, Let us say the first axis, then we have moment of inertia only along the first axis. We can also write 1, 2, 3 here, right, and similarly for second axis and third axis. So, this is what we learned from this right.

Now, let us deviate slightly from this discussion of vectors and tensors. So, these are the things we need to keep in mind, and also 1 more thing, this rotational matrix what we have derived here is nothing, but the combination of the orthonormal Eigen vectors. When we put together the orthonormal Eigen vectors, we get lambda 1 lambda 2 and lambda 3.

So, this is not a result which is limited to the discussion of classical mechanics by the way this is a, this is the similar, in a similar manner, we can diagonalize and Hamiltonian in quantum mechanics right, or any operator in quantum mechanics for example, right. So, this is a procedure we generally follow. So, any diagonalization of a tensor tensorial quantity in general and n in an n dimensional space, n could be 3 and could be any higher dimension we follow the same procedure.

And, but only thing is some of the Eigen value, I mean some of the tensors, whether it is, whether it can be brought into a diagonal form or not, that it depends on many other condition, but when we are discussing moments of inertia, and later on we will be discussing small oscillation. There also we will be dealing with real symmetric tensor, if the tensor is real and symmetric, then we are always good, then we can always find a suitable frame of reference, which where the tensor becomes diagonal.

Physically what it means, I will just try to draw a picture and give you an idea.

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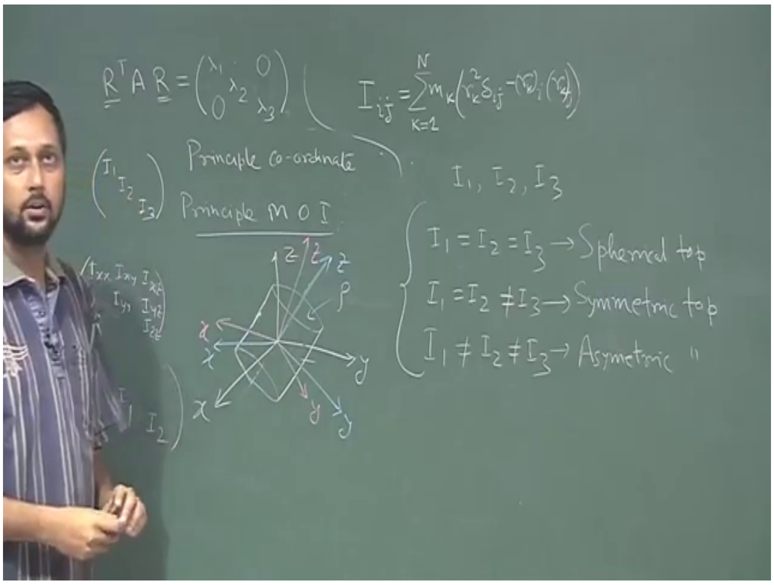

Let us assume we have a cylinder, let us draw it in a slightly different way. This is our cylinder. Now typically what happens is, this principle coordinates especially for moments, when we are discussing moments of inertia, which is a property associated with the distribution of mass, principle moments of inertia in most of the cases are related to the symmetry of the object.

So, let us look at this example, let us say this is the initial coordinate system, we have x and y z. Here we are talking about the frame which is, let us say we have this axis. So, this will not work. Just give me a second, this will not work, this will not work; again. Also keep in mind that moment of inertia tensor depends a lot on the origin, if I shift my origin my moment of inertia tensor will change.

So, let us say at the bottom or in the middle, I have a point; that is the center of this. By symmetry I can always say that, this will be the center of mass, I have a axis system which goes through the center of mass, but which is tilted, which is not following the symmetry of this object. Now if I; and it is a cylinder with some uniform density rho.

Now, if I calculate the moments of inertia of this particular cylinder in this axis system, I can assure you that I will get all the components Ixx, Ixy, Iz, Ixz. Actually we need not write the lower half itself, because I will tell you why, because it is a real symmetric tensor. So, this half and this half will be identical.

So, what we can do is, for real symmetric tensor, we can simply write upper half and lower half will be identical to the upper half. So, we can write essentially 1, 2, 3, 4, 5, 6, 6 elements. So, 6 elements is sufficient. We do not have to write all 9. Now if I start rotating this coordinate system, the. Now what happens is, moment we start rotating the coordinate system, every time we are deviating, let us say slightly or do not have to be slightly, let us say I go to a coordinate system which is this. So, this is my new x, this is my new y, and this is my new z.

So, what will happen, this tensor, the values will be changing, definitely this values will be changing for sure, because every time we are moving, this coordinate keeping this point intact, that is a very important thing. You have to keep in mind, this point should not change, I am just rotating this by some angle, some angle theta right, this will change.

Now, finally, if I go to a coordinate system, where the z runs along the axis of the cylinder, and I have an x in this direction, and y in that direction, what will happen. Just by the symmetry of this problem, I can tell you that in this particular, this blue frame of reference, this tensor will take this particular form. So, I 3 will be Iz in this case, and also I can tell you something, because it is a uniform, the density is uniform, and it is a proper cylinder. This 2 will be I 1 and I 1. So, I 1 and I 2 will be equal.

So, I am not writing I 2 here, because I 1 and I 2 will be equal, we can just call it Ix Ix or what, what is more convenient is, we just call it I 1, I 1 and I 2, this I can just I, I can tell you just by looking at the symmetry of the system. If I take a bottle for example, which has a cylindrical symmetry I can immediately tell you that.

If I take an axis which runs, where the z, if I take an axis system, where does that axis runs, through the axis of this bottle, that will be 1 axis of symmetry, and I can choose x and y arbitrarily, because we have spherical. Sorry cylindrical symmetry in this direction, does not matter where I choose it, I just need 2 perpendicular axis, and once z axis that runs along the bottle that will be the symmetric axis.

So, typically for moment of inertia, because it is a property which is strongly dependent on the mass distribution of the system, the symmetry; the geometrical symmetry axis are the principle moments of inertia axis, but if, but typically, I mean typically that might not be the case we can have; for example, I gave you the example of resistivity tensor, resistivity tensor of this, this so called imaginary metallic block.

Now, if it is a proper like crystalline cut, what I mean is, if I take a piece of material and start cutting it by the cleavage planes. Cleavage planes are the planes which follows the crystalline symmetry, then the real x axis, the physical x axis might coincide with the x axis, where which is also an axis of symmetry, and then I might have for some physical property, I might have that x axis as a principle axis.

But typically when I am talking about physical, my other physical properties then moment of inertia, then we might not have geometrical symmetry axis, and the principle axis coinciding, but for moments of inertia, because it is a physical property which depends strongly on mass distribution, and some. I mean in mass distribution al1. So, it for uniform bodies, typically the symmetry axis is the principle axis, right.

Now, we can also classify systems depending on the symmetry of it, as I said any how we do not need this anymore. As I said we can always have systems, we can always diagonalize the moments of inertia tensor; always there will be definitely some orientation along which the moment of inertia has to be delivered. So, we can always find I 1 and I 2 and I 3 for any object.

But please again keep in mind that moment we changed the origin, your I 1, I 2, I 3 will also be changing right. Even here if I am shifting my origin from here to here or some arbitrary position, even along this axis my I 1 I 2 I mean I 1 I 1 and I 2. These values might change, not might definitely it will change, it has to change right. So, whenever we are talking about principle axis, we have to, have a fixed point along which is a fixed point I mean. So, the rigid body is rotating keeping 1 fixed point in it and that fixed point is our origin, all the time please keep that in mind.

Now, for a system if we have I 1 equal to I 2 equal to I 3, then we call it a spherical top. Spherical top that typically happens for sphere, only sphere or some system which has this very strong physical symmetry; for example, cube, cube is also a system, if I take a cube, uniform cube has to be uniform, and if I set my x, y, z such that its exactly in the middle of the sphere.

So, all 3 moments of principle moments of inertia, just by observing the symmetry of the system, I can tell you if I set the origin at the center; that means, origin is my fixed. I mean the center of it is in the fixed point, along which it is rotating then and my. So, sorry actually the x axis has to run exactly through this corner, if that happens. Sorry not this corner.

So, it might, it has to run exactly through the middle of this phase. Similarly y axis middle of this phase. So, in that case this will happen. So, this is also an example of spherical top, only if it is the extremely symmetric object. Second example is, I 1 is equal to I 2 not equal to I 3, then its which is an example we are dealing with here, which is a cylinder uniform cylinder, where the fixed point is anywhere on this axis.

It could be here of course, changing this fixed point will give you different values of I 1 and I 2 and I mean I 1 and I 3, I 1 and I 2 are exactly equal, but the just by following the symmetry of the object I can tell you, does not matter if we are staying at this point on the axis, or that point will definitely have I 1 equal to I 2 which is different from I 1 equal to I 2 which will be different from I 3.

So, this set of systems is called the symmetric top, and if I have a system which is very regular in shape, where I 1 is not equal to I 3, not equal to I 3, we call it an asymmetric top, symmetric top, spherical top, symmetric top and asymmetric top. So, motion of this we will be discussing shortly; asymmetric top actually not much to discuss symmetric top a spherical top also. It is too easy to understand the motion everywhere, it is symmetric anyway.

This is the portion, this is the topic symmetric top on which we have to spend a considerable amount of time. There will be 2 types of motion discussed; 1 is top free motion, 1 is with top motion. So, this there will be long calculations in this chapter, this particular topic. So, be prepared for it, before we go into all this, we need to quickly discuss few other things; 1 is the, theorems 1 is the parallel axis theorem, and next will be parallel axis theorem.

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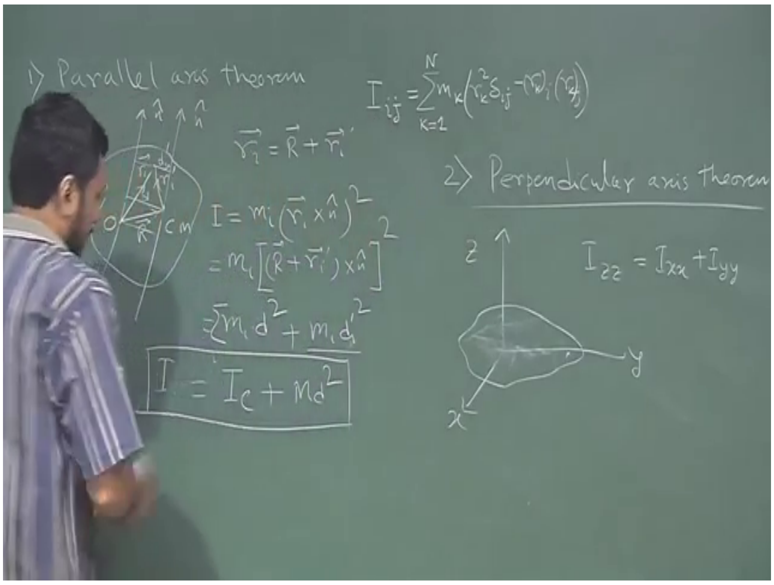

Let us assume we have this arbitrary rigid body, and let us say we have 1 axis of rotation given by some direction n running through this, yeah this is with this origin right o, and let us assume this is the center of mass of the system right, and let us assume that another axis which is exactly parallel to n parallel to this axis, also runs through the center of mass.

Now, we want to see, is there any relation between the moments of inertia is calculated around this axis, and that axis for that. Let us take 1 point, which has a position vector ri and ri prime, right, and this is r, this construction looks familiar, right. We did that when we discussed general. We generally discuss the motion of a system of particle. So, r i is equal to r plus ri prime.

So, I which is the moment of inertia measured around this axis, which will be given by sum over i. Again I am not writing the sum, because it is we are following Einstein summation convention, di is the distance right, which is ri cross n square right, breaking it into this by putting ri equal to this. So, it will be r plus ri prime cross n capped mi r cross n capped.

Now, r cross n capped will be the distance d between this axis, and center of mass right. So, I am just substituting. Sorry there will be a square here, right. So, my first term will be r cross n capped square, which will be D square. Sorry this small d square and second term will be mi r, this one will be the distance.

So, second term will be ri prime cross n whole square, which will be the distance of this particular mass from this axis, which goes through center of mass. So, let us call it di prime. So, this will be simply di right di prime square, and there will be an additional term which will be 2. So, it will be 2 r right mi r cross n into dot ri prime cross, right. So, there will be 3 terms.

Now, what we are going to do is, once again we are simply take this mi inside. I think you all understand by now, what we have to do, and this term is again what mi ri dashed, which is ri ri prime, which is the moment measured about center of mass, and for a system of particle we have already shown that this moment uniformly vanishes. So, the third term will not contribute anything. So, the third term is equal to 0.

So, I will be, and again what is the second term. Second term is the moment of inertia measured around this particular axis, which is going through center of mass right. So, what we can do is, we can simply call it I c center of moment of inertia, measured around an axis around the parallel axis, which goes through center of mass plus. What is this sum of there is a, there is a sum over I implemented on the whole thing. Now sum over I means it is the total mass. So, it will be m d square, right.

So, that gives us the. So, called parallel axis theorem, which says I about any arbitrary axis is the moment of inertia, about any arbitrary axis is a combination of moment of inertia measured through an axis, which is parallel to that axis, what going to center of mass plus the mass times distance square, between that to these 2 sets of axis basically the perpendicular distance between these 2 axes, right.

So, this is our theorem, which is, and now we move to the next theorem. We do not, we have to remove this part here, move next theorem is the perpendicular axis theorem. perpendicular axis theorem says, what let us say we have a laminar object, which is lying in the x and y plane.

And. So, we can measure the moment of inertia around x, around x axis around y axis and around z axis, then we can write Izz which is the moment of inertia measured around z axis, which will be Ixx plus I yy. I am not proving it. So, simple you can find it in any textbook. You can also do it yourself. Please remember you just remember that the mass is in the xy plane, there is no mass distribution along the z axis.

Just put that into this, into I mean write an expression for Ixx and Iyy, put it into. I mean sum it up, and then you will immediately get Izz. So, I am not proving it for you. Now coming back to this parallel axis theorem, using this if I go back to our earlier example of a cylinder.

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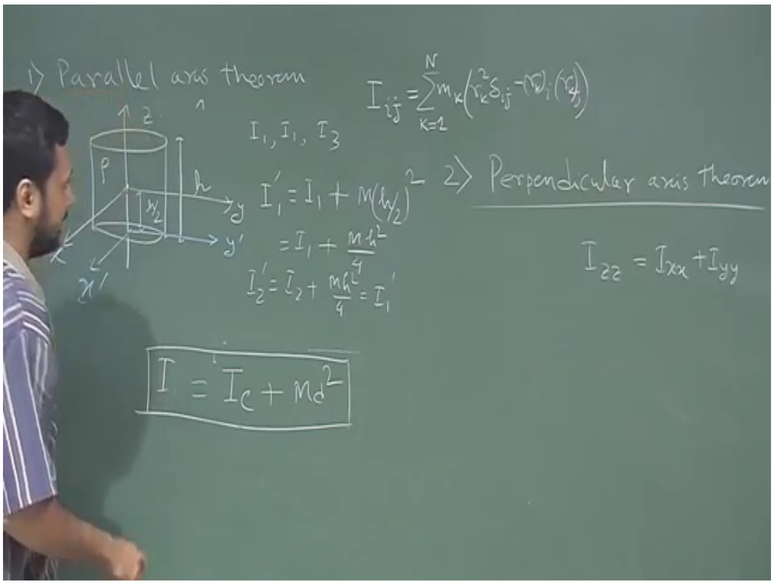

Let us assume that we have this cylinder, and this is the axis of the cylinder, and let us assume that initially the height of the cylinder is; let us say h. Initially we have an axis system, where it is exactly in the middle. So, it is at a distance h by 2 from the top or the bottle. So, this is my x and y and z. So, let us assume in this particular construction, I have I 1, I 1, and i. Let us call it I 3, just to clarify I mean just to make indistinct.

So, I 1 and I 1 at the moments of inertia, around x and y and I 3 is along z. Now let us assume, and also please remember that by symmetry of the problem, this is also the center of mass of the system, because it is a symmetric system with uniform density rho, the middle point of the axis has to be the center of mass right.

So, now if I shift my axis system down here; so, let us say this is my new set of y and x axis, z axis remains the same. Can you use the parallel axis theorem to check what will be the new sets of moment of inertia around, or I mean of course, it will give us the principle moment of inertia, because we are still staying on this axis? What will be the new values of I 1; I 1 and I 3. Of course, we can do that. So, the shift is by h by 2 right.

Now, if I try to calculate it for I 1. So, I 1 initially it was i. Please understand that initial I 1 is this ic for x axis. So, I 1 prime, let us call this I 1 prime, let us call this the primed axis. Please do not confuse it with the previous prime notation we have used am, just taking a specific example. So, 1 prime is equal to I 1 plus m, let us m, let us say m be the mass m h by 2 whole square. So, I 1 prime will be I 1 plus m h square by four.

Similarly, I 2 prime will be I 2 plus m h square by 4 I 2, which will be equal to I i 1 prime, and I 3 what will happen to I 3. See we have not shifted along the z, I mean sorry the shifting is along z. So, there is. So, we have kept our self on the same z axis, z has not shifted. So, I 3 prime will be equal to I 3. So, we have shifts in the first 2 components or the x x and y component iz component remains unaltered. So, this is a result we might use later on.

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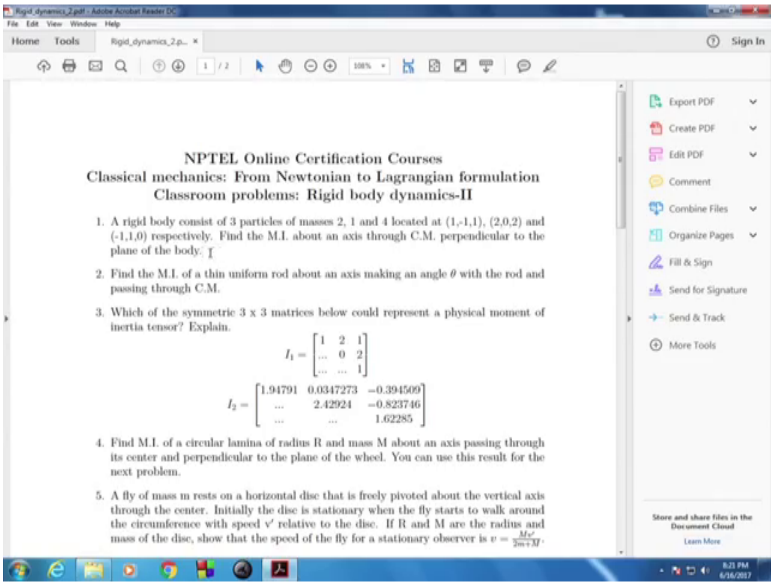

Now, we will end the class with a very small problem. The first problem of this problem is that we have many, actually I have just kept it for till the end, so that we can finish it all together. So, we have a rigid body. Once again this is a familiar rigid body, we have 3 particles at 3 different positions. We have to find out the moment of inertia about an axis, which is going through center of mass and perpendicular to the plane of the body.

Now, once again I am not solving it for you, but I will give you a sufficient hint, so that you can do it.

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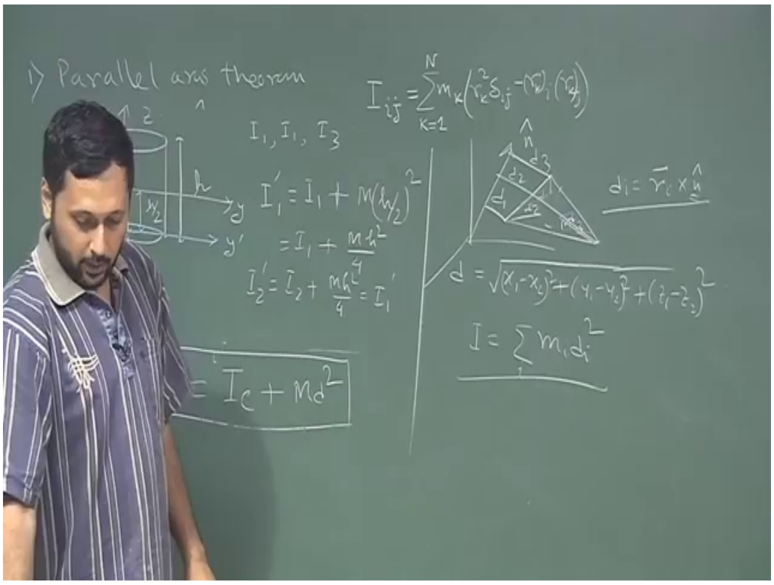

Now, let us say this 3 has the mass points. So, we can always form a triangle through these 3 masses, and by symmetry of this problem the center of mass. See I frankly I do not recall what was the values we have determined for center of mass, but you can find it out once again, it is very easy the center of mass will be somewhere on this plane by the symmetry of the problem.

Now, we have to find out the moment of inertia about an axis, which is perpendicular to this plane, plane of the triangle, and going through center of mass. Ideally we have to find out the equation of, or we have to find out this vector direction, and have to take all the distance from this vector by using di is equal to ri cross n capped, but in this case, this is not necessary, because we already have a point, which will be falling on this axis, and also it is in this plane which is the center of mass.

So, what we can do is, once we know the coordinate of the center of mass, which we calculated in the previous class, we can just take mutual distance of these 2 points, and we know that mutual distance between 2, any 2 points, d is given by root over x 1 minus x 2 whole square plus y 1 minus y 2 whole square plus z 1 minus z 2 whole square.

So, similarly what we can do is, we can calculate d 1 for d 1 and d 2 and d 3 between these 3 points, and we can compute I to be equal to sum over I mi di square. So, you can finish it yourself, its 1 way of looking at it. Now that was, we were, that was easy, because we already specified that the axis is going through center of mass.

Let us assume if I give you any arbitrary axis to work with it is; which is not even going through center of mass, or let us say I give you an axis, which is going through the origin in some arbitrary direction, then we just have to follow this construction. We have to construct a unit vector, which is, if I give you a direction; that means, I am giving you the unit vector, or if I am giving you a vector direction, you just have to divide it by the magnitude, and you get an unit vector. And from this you just calculate the mutual distances of 3 points, does not matter if it is falling inside the rigid body or outside, does not matter, really moment of inertia can be calculated.

And then you have to just, you know in that case this will be your d 1 d 2 and d 3, and then you go back to, go back and put your values in this formula, and you get an value of moment of inertia. So, we can always do that right. So, we end here and next class we will do some problems. And also we will go forward in terms of, in discussion of civil moments of inertia, we will define something called ellipsoid of inertia, and then we will move on to Euler’s equations.

Thank you.