Classical Mechanics: From Newtonian to Lagrangian Formulation

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Lecture – 31
Rigid body dynamics – 5

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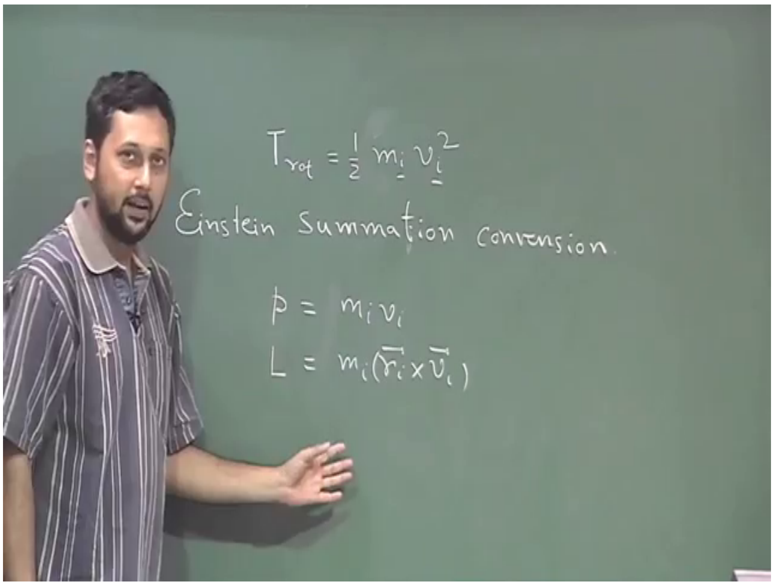

Kinetic energy of a rigid body is T we can write T rotation which is half m i v i square. So, now onwards we will be using a summation convention, which is called Einstein summation convention; that means, if in a summation if the index is one index is repeated twice that means, a summation is taking place over that particular index. So, for example, this one actually there is a summation over i, so total rotational kinetic energy is half m i v i square sum over all i, so that means it is the kinetic energy of one particle and then the summation is running over all i of the system. So, i essentially runs from 1 to n, n being the number of particle in the rigid body. But if we follow Einstein summation convention we do not need to write this summation explicitly instead what we can do it, we can just simply write it like this.

Now, in this assumption in this convention because the index i is repeated twice in this summation that means, there is a summation implemented over i anywhere. So, we will be following this summation convention for the rest of this discussion on rigid dynamics and also in the next section when we will be discussing Lagrangian dynamics we will be using the same summation convention. Now, please keep this in mind. So, if I will just give you an example. So, in this case for example, we wrote for a system of particle total linear momentum p was given as m i v i sum over i. So, in this convention, sorry p will be simply written as m i v i. Similarly, the angular momentum L, which was a summation over r cross p i, so that will be given as m i r i cross v i.

So, we are just getting rid of the explicit summation symbol using this convention and this can be a bit confusing at times, if necessary we will switch back to the you know conventional summation symbol by putting a summation here. If necessary because sometimes it can be if there are more than one indices in the system, it can get a bit you know bit confusing sometimes some of the some of the indices has not we do not need to assume a summation over some of the indices also those things I will explicitly mentioned. But typically when there is only one index, we will be following the summation convention.

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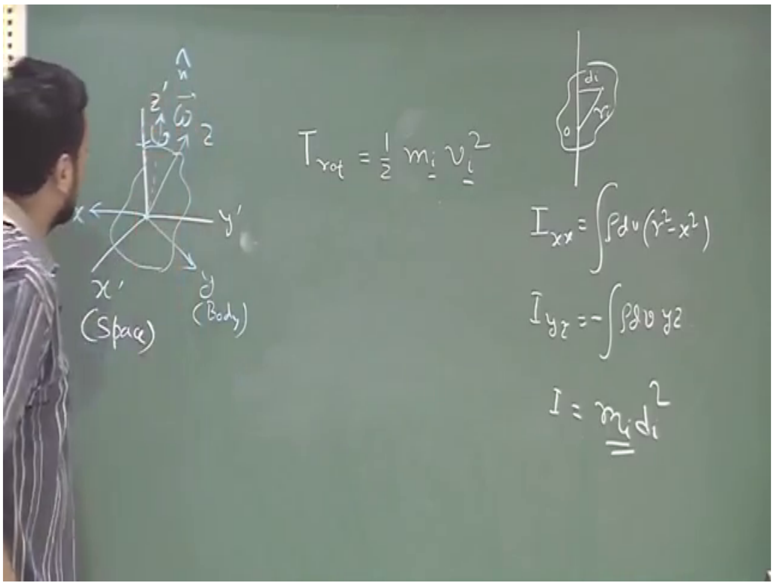

So, coming back to the dynamics of rigid body, so the situation was following. We have a rigid body; and there is a space set of axis, which is fixed in space. And there is a body set of axis which is moving with the rigid body. So, we have the x prime, y prime, z prime axis; and we have x, y, z axis which is moving along with the rigid body. And what we are doing here we are simply writing the rotational kinetic energy of this rigid body, right now we are writing it in terms of a fixed or rather the space set of coordinate system. So, just write it here. So, this is my space set of coordinate system and this is the blue one is my body set of coordinate system. Space means fixed in space; and body means rotating with the body.

Now, so actually what we were aiming to do in the last class, there were two definitions of moment of inertia one was this so called tensorial definition I let us say I x x which was given as rho dv r square minus x square and I y z for example, was given as minus rho dv y z. So, this is one definition and the other definition we already know from our you know school textbooks or early you know whatever classical mechanics we have studied so far we know that I the moment of inertia around an axis system can be given as m i di square. Where di is the physical distance of this particle i ith particle from the axis. So, let us say this is my rigid body and this is my axis, so then I have I hope you can see it, so this is my ith particle. So, this is my physical distance di right. And let us say this is my origin, so the position vector of ith particle is r i.

Now, how to correlate this once again we are using the summation convention, so it is not necessary to put the explicit summation symbol here. So, the question is how to correlate this? In order to do that what we are trying to do is we are just writing this this expression for rotational kinetic energy assuming that the body is rotating with an angular velocity omega around this fixed or does not matter if it is around this fixed direction, it could be any arbitrary direction n capped around which this rotation is taking place. The whole thing is body set of axis is moving alongside the body and space set of axis is fixed

Now, we are writing this expression for total rotational kinetic energy which will be half once again m i and if you recall v i is nothing but r i sorry omega cross r i right. Because when we look from the space set of axis, the only source of velocity linear velocity is the angular velocity. So, we can due to this angular velocity. So, if you remember how we rewrote that this is the expression which we can write.

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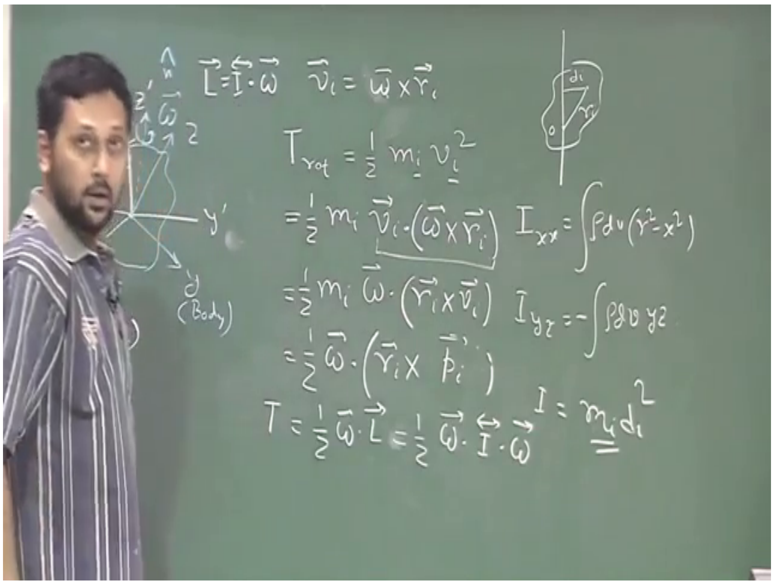

Now if i do this substitution, so it will be v i one we just we are just keeping like this into r i. Now, we can do a series of manipulation what we can do is essentially is, so this three if you recall this is a vector one dot product and one cross product between three vectors and we can do this cyclic permutation and with that we can bring r i. So, you can write it as r i dot v i cross omega then we can bring omega in here. So, after three such permutations, we can essentially write this as, so the final will be omega dot, so omega means omega comes here that means, r i has to go there r i cross v i. And you can slightly modify this again and you can write this as r i cross m i v i.

Now, if you now pay attention r i cross m i v i is nothing but your p i right that is the linear momentum of the ith particle; and r i cross pi is nothing but your angular momentum L right, so this essentially is omega dot L. So, T is equal to omega dot L and if you recall L can be written as I omega. So, we wrote this tensorial equation that L is equal to I omega, I being the moment of inertia tensor and omega is the angular velocity vector that is the equation we wrote in the last previous class. Now, if we substitute here then we get T is equal to half I dot sorry omega dot, omega dot I dot omega. So, this is the final form we can write. So, what we are doing is we will just move it here removing this I hope you can understand these steps. So, T is simply half omega dot I dot omega.

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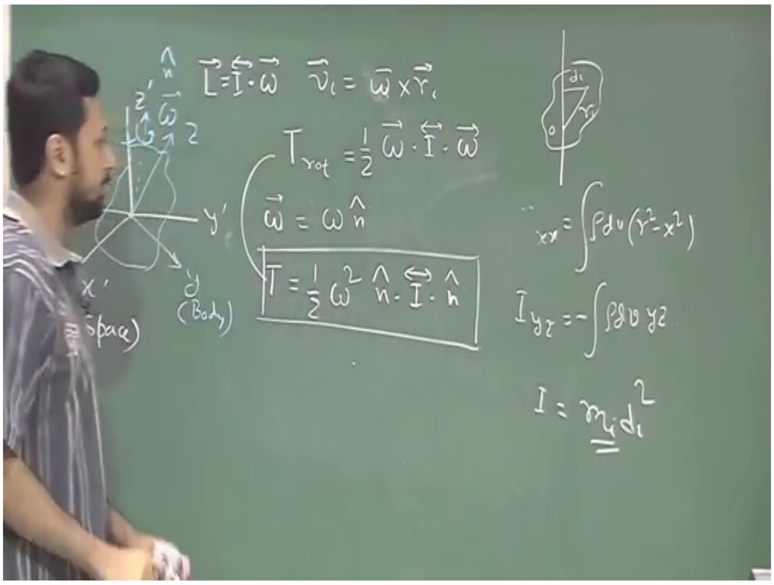

Now, omega is the angular velocity around arbitrary axis of rotation. So, omega can be expressed as omega n capped, where n capped is the direction of this arbitrary axis of rotation. So, it is a unit vector in the direction of an arbitrary axis. So, using this the expression for T can be written as half omega square n capped dot I dot n capped right. Now, keep this in mind, so T is equal to this. So, this is one expression we have to keep in mind. Now, if I go by this definition of moment of inertia, then what do we get. So, I have to remove something let us remove this part, because we will need this figure.

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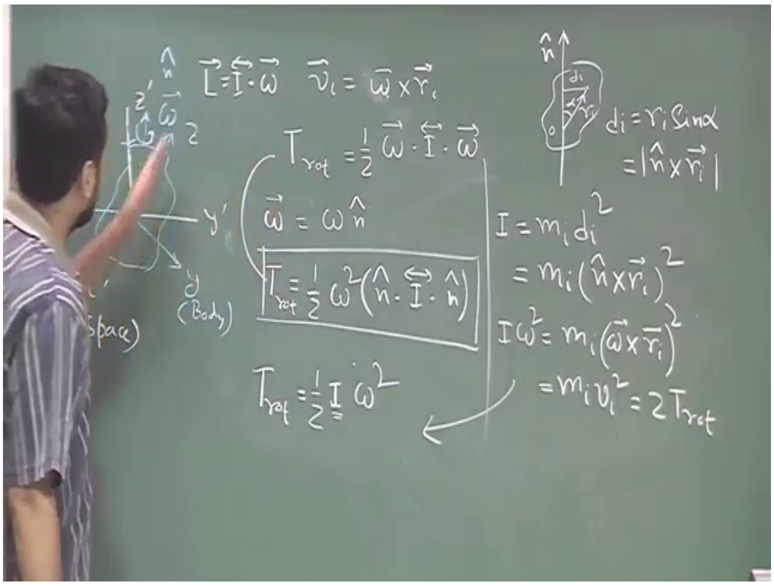

See di is the distance the perpendicular distance between this and this. And let us say this if this angle is some angle alpha, then we see from this diagram that d i is equal to r i sin alpha. Now, when we specify an axis along which the angular momentum has to be computed, that means the rotation is taking place around this particular axis. So, this is our n capped direction in this particular case. Now, di is r i sin alpha; r i is pointed in a particular direction and the angle between n capped and r is also alpha. So, this can be written as, so di is, so this can be written as n capped cross r i mode. So, we can do that. Instead of writing r i sin alpha we can just write it as n capped cross r i which will the magnitude of n capped cross r i which will be once again r i sin alpha.

So, I by definition which is m i di square please remember that there is a summation convention that means, summation is running over this index i, which is m i di square is n capped cross r i square. Now, if I compute I omega square which will be multiplying this thing with omega square it will be m i, why we are doing it we will know in a moment just be patience, be patient for a while. Now, once again we can use the relation that omega is equal to omega n capped. So, it is actually we can take this two this omega square inside and we can write this as omega cross r i whole square. Now, omega cross r i is nothing but your v i, so that is another relation that we already have seen. So, this is your m i v i square which is nothing but from the initial relation of T rot what we have got T rot was half summation over I m i v i square right. So, this is nothing but your two T rot

So, once again we got another expression for T rot. So, this is one expression. And from here we got another expression for T rot, which is half I omega square. So, once we compare these two expressions, what do we find we have a quantity here which is n dot I tenser dot n and we have a quantity here this is simply I. Assuming that these two n caps are the same. So, the n cap in this direction in this figure picture or in this in this calculation whichever we have taken is the same n we can easily compare this two expression and we can write what we can write I will just remove this part. So, comparing these two expressions, what we can write is I is equal to n capped dot I tensor dot n capped. Now, this is a very, very, very important result.

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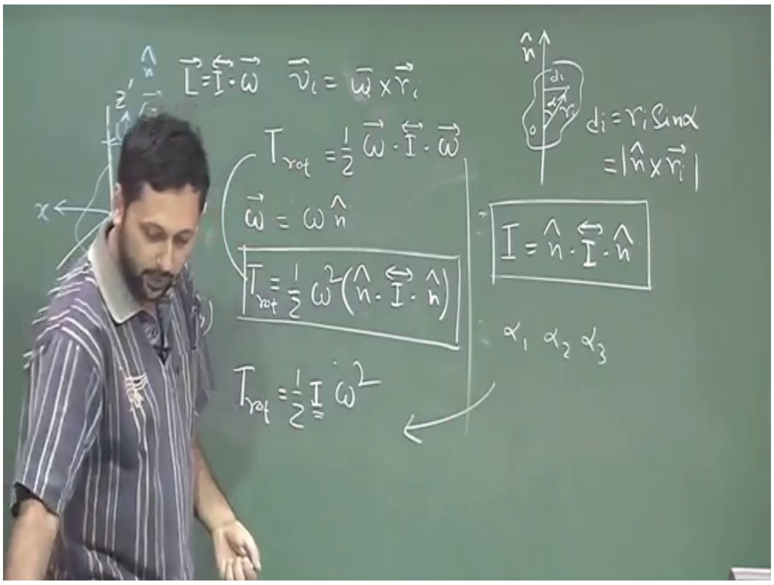

What does it signify see what I told you in the last class I told you that a tensor essentially contains all the information it need to have we need to know about a particular property. If it is a polarization tensor then it tells you everything about the polarization of that particular system we want to know. If it is a resistivity tensor then it tells us everything about the resistivity we want to know of a particular system. Similarly, if it is a inertial tensor measured with respect to one fixed point this is one thing I am pressing once over and over again. Because once we move this origin to another point this I tensor will completely be different it will be a new set of I tensor. If we move it along one particular, you know if we move it in a systematic manner maybe there is some very specific relation between two sets of I tensor, but in general it will be different from each other right.

So, given the origin remains fixed, the two sets a one set of I tensor the moment of inertia tensor contains all the information about the moment of inertia of the system of that rigid body we want to know. And if we want to know the value of moment of inertia in a particular direction which is given by this n capped, all we need to do is we need to project the tensor please understand that we need to project the tensor in that particular direction and that is exactly what we are doing here.

Now, if you look in terms of the matrix operation this operation see what is n capped n capped is a direction vector, it gives you an unit vector in some specific direction.

Typically, an unit vector in some particular direction is represented by a system of direction cosines; we are all familiar with the concept of direction cosine. So, we write three angles alpha 1, alpha 2, alpha 3 which are direction cosines with respect to some fixed x, y and z I mean we have of course, we have an axis system we cannot have an arbitrary direction right. So, with respect to that axis system we define the cos, so the angle between that particular direction and x axis is alpha 1, y axis is alpha 2, and z axis is alpha 3. So, these are my direction cosines.

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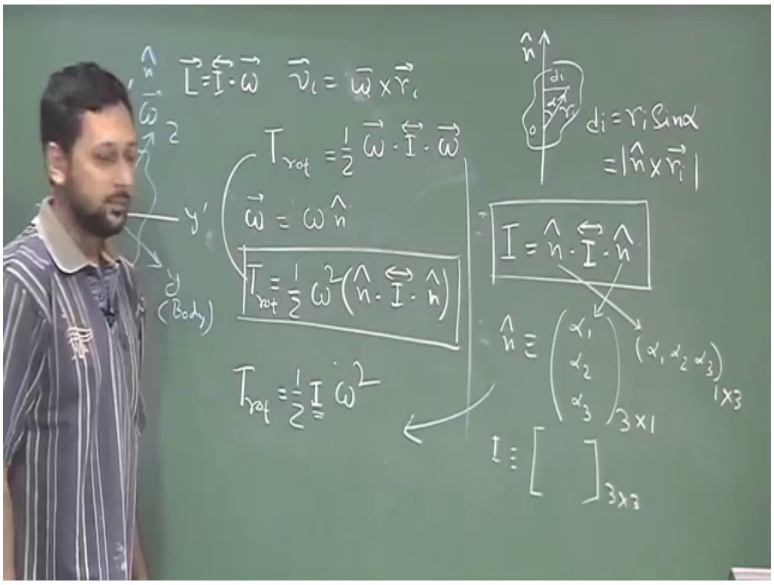

So, your n vector typically this in the vectorial representation n capped is represented by this column vector. So, this is a 3 cross 1 vector, 3 cross 1 matrix. A vector is a 3 cross 1 matrix. Now, when we operate a vector from left hand side of and I is a as we know it is a 3 cross 3 matrix we will come we will discuss about specific properties of the 3 cross 3, 3 cross 3 matrix very soon. And n when we operate from the left hand side. So, it becomes a not a 3 cross 1 matrix in this form, but a matrix in this particular form. So, it will be a 1 cross 3 matrix if we operate it from the left hand side. So, this one is this and this one is this. Now, assume this look at this you have this 3 cross 3 matrix, we do not need those things anymore.

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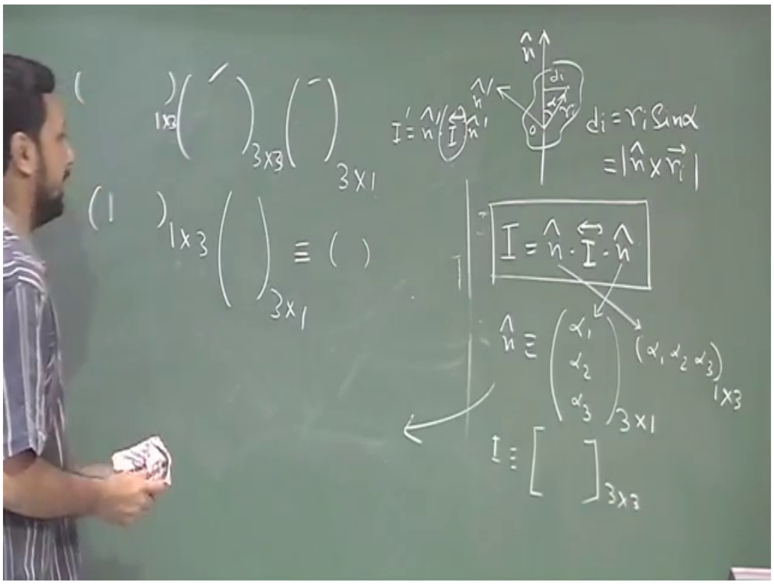

So, we have this 3 cross 3 matrix which you multiply from left from this side, you operate it with a 3 cross 1 matrix; and with from this side you operate it with the 1 cross 3 matrix, it is a 1 cross 3. Now, the operation between this and this, what will you get. So, sorry it is a 3 cross 3, it is a 3 cross 3 matrix, and then you operate it with the 3 cross 1 matrix. So, this from the rules of matrix multiplication you know that the this operation will give you once again a 3 cross 1 matrix. If you are not familiar with this type of operation, I would suggest that you please go back to your textbook, where matrix multiplication has been described. And you will see that this operation will give you 3 cross 1 matrix.

And a multiplication of on 1 cross 3 matrix to a 3 cross 1 matrix will essentially give you a single number which is 1 cross 1 that means, a single number and that number the final answer is, so final answer of this product is a number which is the moment of inertia along that particular direction. You want to know moment of inertia to with respect to another direction which is given by some arbitrary you know n capped just calculate I dot dash will which will be equal to n dashed I n dashed. So, this tensor I contains all the information you need to know about the moment of inertia with respect to this fixed point when the rigid body is I mean executing rotation with this fixed point this matrix essentially contains all the information you want to know. So, this is essentially the idea of a tensor.

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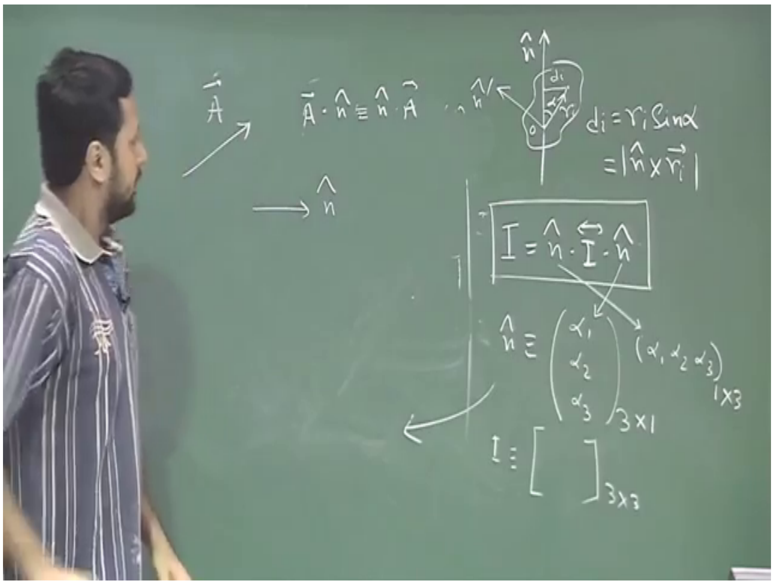

Now if you want to compare it, so this is how we project a tensor, if you recall how do you project a vector again once again we have a direction cosine. Let us say this is a vector, some vector A. And we have a direction cosine given which points to this particular direction n. How do you do this, how do this how do you project A in this particular direction? We simply take A dot n or n dot A dot product does not matter which we take it both are you know both are vector. So, if it is n cross A or A cross n, we can might as might as well write n cross A. So, this is how we project a vector. Now, in case of tensor we have just seen that we should have an operation in this symmetric form it will be n dot I dot n. So, we see that this is the projection of a vector.

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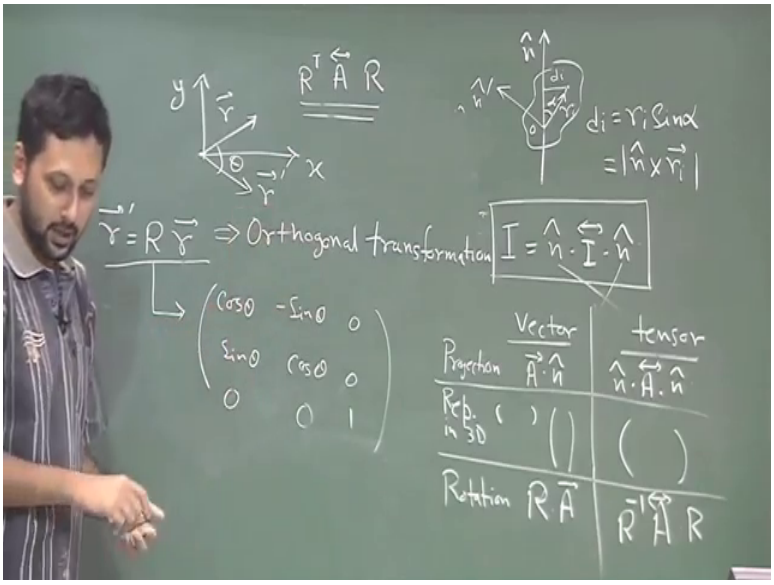

So, when we compare vector and tensor. So, in terms of projection vector is simply A dot n is it feasible, and tensor is simply n dot I dot n or it could be any tensor actually I mean it does not matter, if it has to be the you know we can just take some tensor A. So, here a is A vector, here is A tensor, tensor symbol is this and vector symbol is arrow in one direction tensors symbol is a closed array, a closed arrow sorry whatever. So, and in terms of representation in I will just write rep, rep for representation; representation is either a row mat row vector or a column vector for in 3D actually in the 3D world we know of and here the representation is a 3 cross 3 matrix.

Now, there is another very important property we need to consider is the rotation. So, this is one property, this is another and third is rotation. Quickly, let us look at it we have x, y, we have a vector let say some vector r in this x, y plane I am just taking 2D, because it is easy to demonstrate. Now, we want to rotate it by some angle theta in let say clockwise direction or anti clockwise direction, does not matter. So, now, this new vector is called r dot it is just the same vector rotated by an angle theta. How do we write it if you recall we write it with the help of, so we write an equation of the form r dot is equal to R times r.

Now this capital R is a rotation vector which in this particular case 2D case in a for 2D rotation has the following form cos theta minus sin theta minus sin theta cross x square sorry sin theta 0 0 1, 0 0, this 0 0 1 means the z axis remains invariant under this rotation because it is in the 2D plane. So, the rotation is around z axis, z axis is it is pointing out of this board, so that is why we have 1 here and the modification. So, the components of r prime and R they are related by this matrix transformatiom. So, essentially because we are just taking 2D we can just get rid of this column also anyway.

So, this is how a vector is rotated and this set of transformation vector transformation, there is a very specific name for it, it is a class of transform group of transformation which are called the orthogonal transformation. So, this is orthogonal transformation. So, we see that for vector this is how we apply the orthogonal transformation, the general form of orthogonal transformation for a vector is r dot is equal to R times r. So, capital R times r vector, where capital R is a rotation matrix of this general form. So, it can have many different forms also and there is there are very specific sets of property for an orthogonal transformation matrix which we are not discussing in this course.

But what is important is if we want to rotate a tensor in a 3D plane how do we go by I mean we cannot represent a tensor with an arrow like this we cannot do that. But you have to trust me on this that a tensor in 3D plane in is rotated by this following transformation. If A is the tensor then this is the transformation that rotates a tensor in a 3D yeah in a 3D coordinate system. So, for rotation, we have R times A; and for rotation of tensor, we have R inverse A R. Now, this R inverse can also be R transpose because of there are certain properties of orthogonal transformation which we are not discussing here. Now, what is important here what we what we need to understand is there are certain things which can be proved for an orthogonal transformation.

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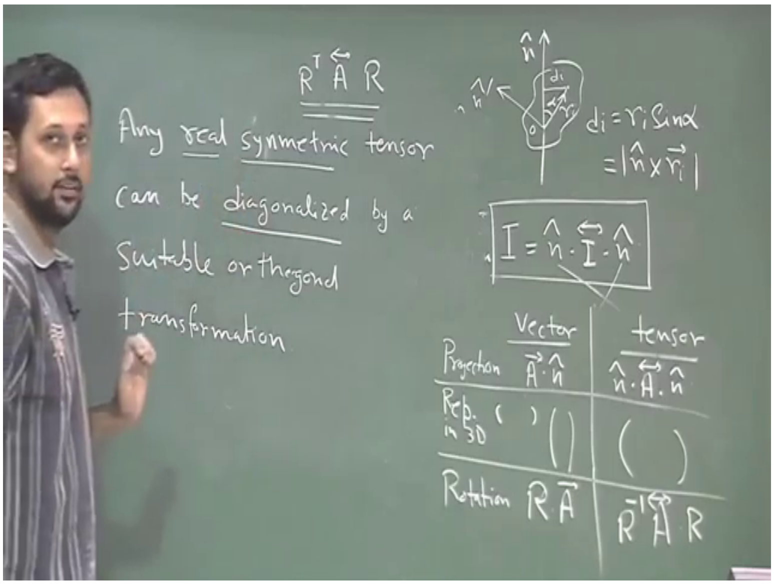

The very important aspect, which we will be considering is any real symmetric tensor can be diagonalized by a suitable orthogonal transformation. There are two things which we need to focus on one is the real symmetric tensor and the second word is diagonalized. Now, let us look at the first word, what is real symmetric tensor? Symmetric tensor is when we have an upper half. So, in the matrix representation, if we have if I draw a diagonal in this matrix and if we have an upper half which is exactly identical to the lower half then it is called a symmetric tensor. And real means when all the elements of this matrix are real numbers nothing imaginary here then it is called a real symmetric tensor. Now, we will take it up in the next class and we will show you that our inertia matrix is a real symmetric tensor; and by this result, it can be diagonalized by a suitable orthogonal transformation; also we will discuss what is the meaning of diagonalization.

Thank you.