Classical Mechanics: From Newtonian to Lagrangian Formulation

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Lecture – 32
Rigid body dynamics – 6

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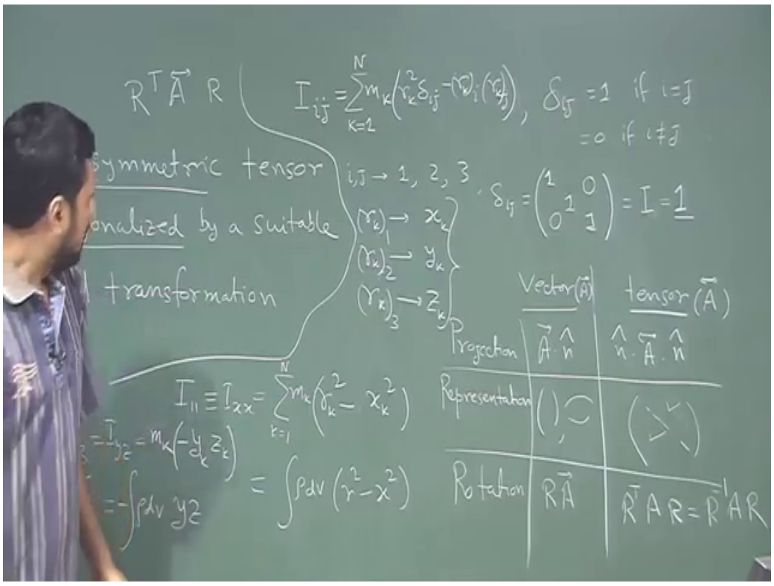

So, we were talking about diagonalization of a real symmetric tensor. Now first of all when I say real and symmetric; we have already discussed that a moment of inertia tensor is both real and symmetric real because in a tensor if it is all the elements all the 9 elements in a 3 d representation a tensor has 9 elements; if all 9 elements are real, then it is a real tensor.

Now, if I write the moment of inertia in a slightly different manner, if I write a particular in a particular notation which is I of i j i and j being 2 indices is sum over K mK and it will be right rK square delta i j minus ri rj right, r of i and r of j. So, this is the form. So, this is a way we can represent a moment of inertia tensor.

Let me explain i and j these are 2 indices which. So, i and j; I will just put a line here i j in this case can take values of 1, 2 and 3; I 1 or when I say for example, r i and if let us say I is equal to 1. So, it will be r 1. So, r 1 means X r 2 means y and r 3 means z.

So, this is just another notation another type of representation we can use. So, the tensorial form is I of j ij; that means, it would be you can read it as I xx; I xx will be I 1 1 I yy will be I 2 2, I zz will be I 33 and I xy will be I 1 2, I yz will be I 2 3, I zy will be I 32 and mK. So, there is another index which is running index over this particular sum.

So, mK corresponds to the mass of kth particle and K can take values once again K can take values of 1, 2 and 3. So, let us say sorry, sorry, sorry, sorry, sorry; I have my mistake. So, K sorry, sorry, K can run over yeah rK; sorry. So, here rki right; this will be rki and rK; K can run over one to n n being the number of particles in the rigid body. So, when I say when we are you know. So, this particular term actually measures the it is a metric which measures the distance of ith object ith particle or kth particle from some fixed axis.

Now, if I; let us take an example, then it will be clear if you take i equal to 1 and j equal to 1, then this will be I 11 which is equivalent to I xx which will be equal to sum over K equal to one to n mK rK square minus r. So, rK 1 will be rK 1, rK 2, rK 3 rK 3 will be xk, yk and zk.

So, rK 3 will be zk. So, kth; so, this means the z component of the kth position vector. So, rki will be rx rK; sorry r K of X and rK j will also be rkx, right. So, it will be just kx am sorry, sorry, just made made a mistake. So, it will be xk into xk. So, xk square right and if I just replace this summation by integration this will be nothing, but integration rho dv r square minus X square, right.

So, similarly if I write it for I 23 which is nothing, but I yz. Now see what happens; now this summation, I am not writing the summation explicitly because the summation convention is in action. So, it will be m K. Now I is not equal to j. So, then what happens this delta ij which is the kronecker delta; I I hope you are familiar with kronecker delta; if not; I will just briefly tell you. So, you do not have any space I can remove this I think I do not need it any more, alright.

So, delta ij is equal to one if I equal j and equal to 0 if i not equal j. So, if I take I 2 3; that means, I is not equal to j. So, this means this term will vanish and rK 2 will be xy sorry yk sorry yk and this will be yz sorry zk. So, this will be simply minus y K z K of course, there is a summation convention in action and in integral form this will be simply minus rho dv yz the standard form for I yz. So, this is a in serial notation for the moment of inertia, right.

So far, we have wrote this expressions or this expression I am putting everything together into single expression which is this. So, keep this in mind it looks a bit complicated in the beginning, but if you just look at it for some time if you try to realize what this individual terms are saying, then you will immediately realize this term will come only if we are calculating the diagonal elements so; that means, when we are going along this only then delta ij will survive because delta ij also can be represented, it is also a tensor and this can be represented by this matrix 1 1 1 0 0.

So, this is my delta ij in the standard 3 dimensional space, we are working with which is exactly equivalent to the identity matrix which in some book people write with I or in some time you will find this particular symbol also of one with an underscore. So, delta ij is nothing, but the identity matrix. So, it is also a tensor of rank 2 and there we have a 2 two terms which are which can be represented this combination can also be represented shown to be a tensor of rank 2. So, this whole thing is a tensor.

Now, once again there is no imaginary object here none of this integration will lead to imaginary values it has to be real and most importantly it is symmetric because if you change the indices i and j if you. So, right now we are calculating I 2 3; I 2 3 is equal to Iyz and this is exactly equal to I 3 2 which is Izy. So, Izy and Iyz is equal. So, the upper half of this tensor is equal to the lower half. So, these 2 halves are identical. So, that is why it is a symmetric tensor. So, the tensor we are dealing with is real symmetric tensor.

Now, I think we are we are convinced that it is a real symmetric tensor and we have already seen; how does it transform under rotation we have we have seen it here that rotation in vector is r times a and rotation in tensor is r transpose a r or which is exactly equal to r inverse a r, right. So, we can either write transpose or we can either write inverse which are same because we are dealing with a set of orthogonal transformation.

Now, if you are really. So, I will just focus on the diagonalization part. Now what does it mean by now just a side note in case you are interested to know more about orthogonal transformation what it is how it works then I suggest you I strongly suggest the book of Professor A K Raychaudhuri; it is a wonderful book; there are only few pages on orthogonal transformation, but it is beautifully summarized in this in this course we will not have enough time to go into the details of orthogonal transformation, but right now we are going to discuss some aspect of it.

Now, this transformation what we wrote here it is or we have it here it is an orthogonal transformation because this rotation matrices they correspond to a certain group of matrix they compose; they form a group which is called orthogonal transform orthogonal group. So, this is also called a similarity transformation similarity transformation.

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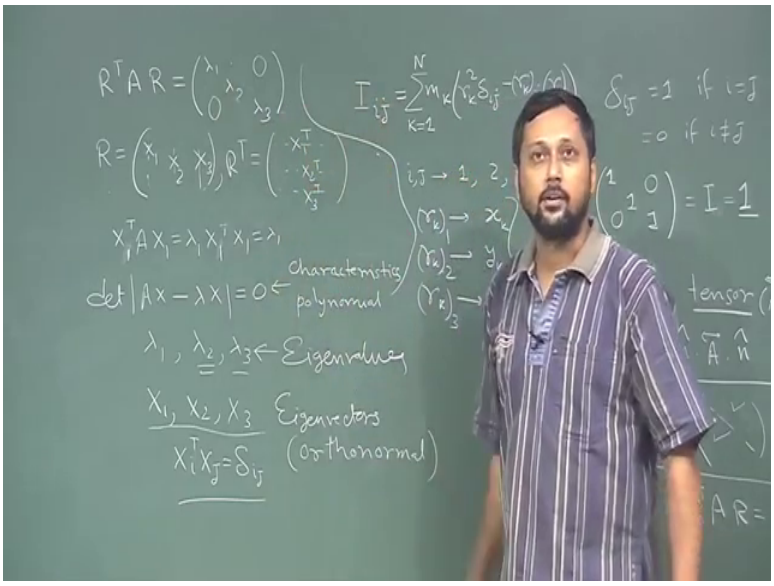

Because the you know it is the matrices; they are working the very similar mattresses work from left and right of course, they are in the transpose side, but this type of transformation there is a mathematical name for it is similarity transformation.

Now, let us come to diagonalization; we are all familiar with the diagonalization of matrices does not matter if its orthogonal matrix or symmetric matrix real matrix does not matter we all know that diagonalization essentially means if we have a matrix a; then we can write a what was it yeah ax equal to lambda X right. So, this type of equations we are familiar with and the next step is we write ax minus lambda X determinant equal to 0.

And this gives you a set of a polynomial which is called the characteristics polynomial a is the typically what we take is a; a is a symmetric matrix or square matrix which is either 2 by 2 or 3 by 3 matrix X is a if it is a 3 by 3 matrix, then it is a 3 cross one column vector if it is a 2 by 2 matrix, then it is 2 cross 1 column vector. So, most importantly it is the square matrix, this is a column vector; lambda is a constant which are which is the Eigen value and this is the same X once again.

So, we are familiar with this now where comes where is the rotation coming into it the thing is this type of operation is also a rotation I cannot go into the mathematical details of it which is very nicely worked out in Professor A K Raychaudhuri’s book are notes on classical mechanics, but what I can tell you what do we do what are the steps.

We first solve this polynomial which is called the characteristics and we get for let us say it is a 3 cross 3 matrix let us say assume that a is 3 cross 3 then we have 3 distinct Eigen values by solving this equation lambda one lambda 2 and lambda 3 that that is what we have right, this is a very standard procedure we are all familiar with next what do we do we take lambda one put it into here.

So, we now come back to this original equation we set a X 1 equal to lambda X lambda 1 X 1 and calculate X 1 find out X 1 and then we put lambda 2 and then we try to find out an X 2 which is if you recall perpendicular to X 1. So, X 1 essentially means it is a vector direction in the 3 d space we are working with it is a vector direction, right.

Now, in that particular direction, we try to find out another direction which is perpendicular to the X 1 direction that is what we do and then we put X 3 lambda 3. So, we put X 3 equal to lambda 3 times X 3 and we try to find out another vector direction X 3. So, these are called the; so, these are Eigen values and we have corresponding Eigen vectors which are X 1, X 2, X 3 and we try to calculate X 1, X 2, X 3 such that the Eigen vectors are orthogonal to each other.

Orthogonal means what are the conditions the condition is once again X 1 dot if I write it in vector notation X 1 dot X 2 equal to 0 that is or X 2 dot X 3 equal to 0 X 1 dot X 3 equal to 0 or if we write in a general you know tensor notation what we try to find out is Xi Xj once again is delta ij where delta is a chronicle delta.

So, this is the condition we impose. So, it is not only orthogonal, but when we put X X X 1 square equal to one X 2 square equal to one it remains it remains ortho normal. So, not only orthogonal, but normalized to one the magnitude normalized to one that is exactly what we do in a matrix diagonalization exercise. Now we have done that we have done this many times without realizing this 3 sets; I mean probably some of you have found it or you know it already.

But if you are not familiar let me tell you that now what happens is if I set. So, see X 1, X 2, X 3 these 3 are sets of orthonormal Eigen vectors; they have in the standard notation we follow they have this particular form of either yeah we take it as column vectors typically now what happens if I form a matrix by placing X 1, X 2, X 3 side by side; that means, I have see in X 1, X 2, X 3, I have 3 column vectors, right.

Now, what I do is I form a matrix which where I put 3 column vectors side by side and make a 3 cross 3 matrix that that will happen. So, if I put 3 column vectors one after the other. So, we have 3 elements in each and we have 3 such vectors side by side what we will have is a essentially a 3 cross 3 matrix call it r, let us call it r. Now do this operation on the original vector a what was it ax equal to lambda X right that was the original equation we had, right.

Now, let us assume let us take this equation ax one equal to lambda one X 1 this is also true because lambda one is an Eigen value corresponding Eigen vector is X 1 multiply this with X 2 transpose from left hand side and multiply. So, this will come here as well X 2 transpose X 1 actually this condition because 2 vectors cannot be multiplied unless and until we take transpose of first one.

So, this condition will be xi transpose Xj equal to delta ij use this condition here what we will get is a 0, right because X 2 transpose X 1 according to this particular condition is will give you 0, but if I change it to X 1 according to this condition; what will happen if I multiply this with X 1 transpose from the left hand side the right hand side will give you lambda 1 only, right.

Now, in this r; the first column is X 1 second column is X 2 and the third column is X 3 right take transpose of r what will happen the first column transpose means you are just interchanging rows and columns. So, here we will have X transpose in the first row X 1 transpose in the first row, second row will be X 2 transpose and third row will be X 3 transpose is not it right.

So, first column is X 1. So, the after transpose the first row will be X 1 transpose second column is X 2; X 2 transpose and X 3 transpose; now do this operation multiply this from left hand side take the original vector or original tensor or matrix a in the middle multiply r from the right hand side what do we get think of it now do it do this for any vector your familiar with or very I mean for take any you know simple 3 cross 3 real matrix do this operation matrix the condition is there should be a set of orthonormal Eigen values do this operation.

And you will see rt transpose a r will be nothing, but lambda 1, lambda 2 and lambda 3 just going by the same logic because if the multiplication will be such that only the diagonal elements will survive diagonal elements means you will be multiplying X 1 with X 1 transpose X 2 with X 2 transpose and X 3 with X 3 transpose it will give you lambda one lambda 2 lambda 3 here will have 0 0 and here 0 0 0 and 0 everywhere else is 0.

So, this is essentially a diagonal form where we have Eigen values in the diagonal and 0s otherwise and let me tell you going by this construction we can generalize it for any dimension of matrix we are just showing it for X 3 dimensional case where we have X 1 X 2 X 3, but if we are working in an n dimensional space where we have n number of distinct Eigen values and n number of orthonormal Eigenvectors this construction will always work. So, in this case the final matrix will be an n cross n matrix, right.

So, if we start with an n cross n a matrix then essentially we end up with an n cross n diagonal matrix only thing is only similarity is the diagonal matrix that the final diagonal matrix will have the Eigen values along its diagonal and everywhere else it will be 0 right and that is where. So, you see as I said already this is a rotation this type of operation is essentially a rotation of a tensor.

So, if a tensor is represented by a matrix in 3 d space and if we are doing this operation; that means, when we are choosing a suitable Eigen sorry suitable rotation matrix in terms of r which is happen to be the combination of the orthonormal Eigenvectors, then we are essentially diagonalizing it. Now physically what does it mean? Physically it means now let us go back to our original example of resistance we took this object, right.

So, this is our imaginary metal block what we did was we applied voltage across it we measured current or let us say we applied a current in the X direction we measured voltage drop across it in the x, y and z direction applied a current in the y direction or z direction measure the voltage across all 3 x, y, z direction and constructed the resistance or resistivity tensor which has all these components.

Now, if let us say this material once again, it is an imaginary metallic cube, it is not a metal, it is a cardboard box, it is a box which has chalks in it. So, if it is a; let us see if it is a metallic box and it has certain symmetric I mean molecular symmetry inside, it everything every metal has some crystalline ordering in it right now along certain order direction it might so, happen that if you apply current in that particular direction that will have that will affect only that direction; that means, you will have voltage only along that particular direction not in any other direction and that is reflected in the diagonal form.

See what happens in diagonal form is a matrix goes into this particular form right.

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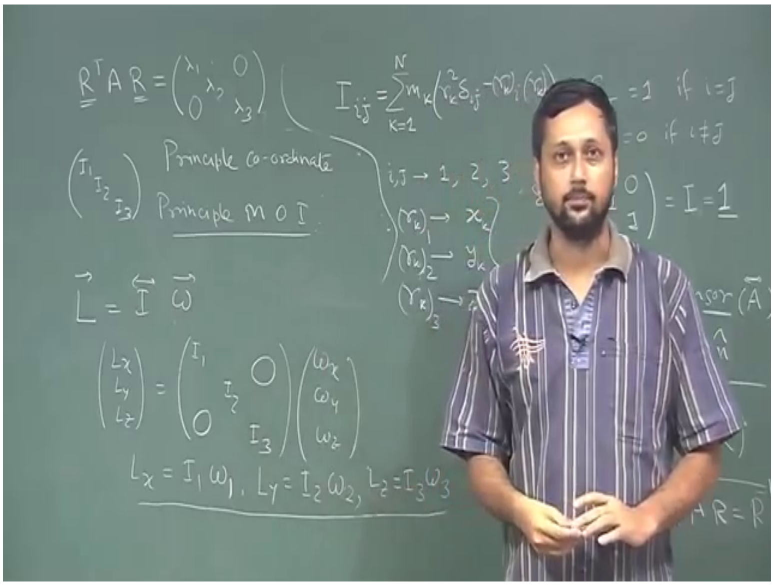

So, in general if we have an n cross n matrix we have lambda 1, lambda 2 up to lambda n right. Now you can write an equation; now let us take the case of; so, moment of inertia tensor. So, moment of inertia tensor let us assume there is some certain rotation for which it becomes I 1 I 2 n and I 3.

Now, remember this equation L equal to I omega what does it physically mean physically means if I break it into components I have Lx, Ly, Lz and I have I xx I xy blah, blah, blah and I sorry I yy I zz. So, there are 2 two other components here and we have omega X omega y omega z.

So; that means, if I have. So, so this was the reduction of this equation to begin with the equation was L X equal to I xx omega X plus I xy omega y plus I zz omega z similarly we had equations for Ly and Lz; that means, if I have a velocity angular velocity component in the y direction that will produce a angular momentum in the X direction if I have an angular velocity component in the z direction that will produce a angular momentum in the X direction, right.

Now, what happens in diagonal form let us assume that we have somehow brought it into this particular form and in this particular form if I have only I 1, I 2 and I 3 and 0 everywhere then my Lx will be I 1 times omega X Ly simply I 2 times omega y and I Lz simply I 3 times omega z so; that means, we are bringing in to or sorry.

Now, rotation means we are moving the coordinate axis right that that is what we mean by this we are moving the coordinate system by certain amount and then we are bringing the I mean as we move on as we are changing the coordinate our the tensorial representation the values in this tensorial representation also shifts right and so, essentially, we can bring we can find out some coordinate axis in which it comes in a diagonal form; that means, in that particular coordinate system if I have a rotational velocity component in omega x direction or x direction that will produce an angular momentum only in x direction not in y or z direction.

Similarly, if I have a omega y component or if I rotate this object in that particular frame particular x, y, z frame, if I rotate it only along y direction that will produce an angular momentum only along y direction right, no other direction. So, that is why for this we can also I mean we can also write omega one instead of omega X omega 2 instead of omega y and omega 3 instead of omega z.

So, these are my; this is a coordinate system in which we have this particular diagonal form of the moment of inertia tensor; that means, the angular momentum depends only on one component of angular velocity and no other components and this particular set of coordinate is called the principle coordinate and these 3 numbers of the these 3 numbers of this I 1, I 2, I 3 which will be essentially the Eigen values of the moment of inertia tensor are called the principal moments of inertia. So, with this we end this particular lecture we will continue from here in the next lecture.

Thank you.