Classical Mechanics: From Newtonian to Lagrangian Formulation
Prof. Debmalya Banerjee
Department of Physics
Indian Institute of Technology, Kharagpur

Lecture – 30
Rigid body dynamics – 4

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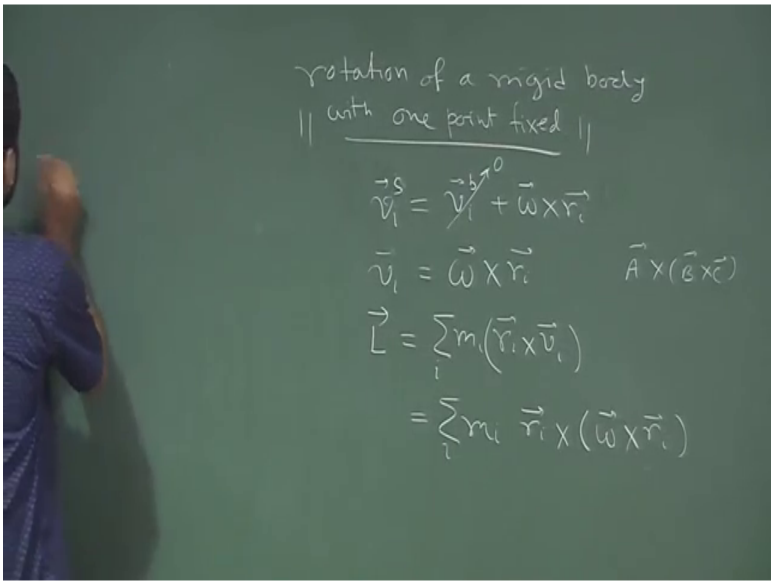

So, this expression once again is of the form A cross B cross C. And we know that this is an identity.

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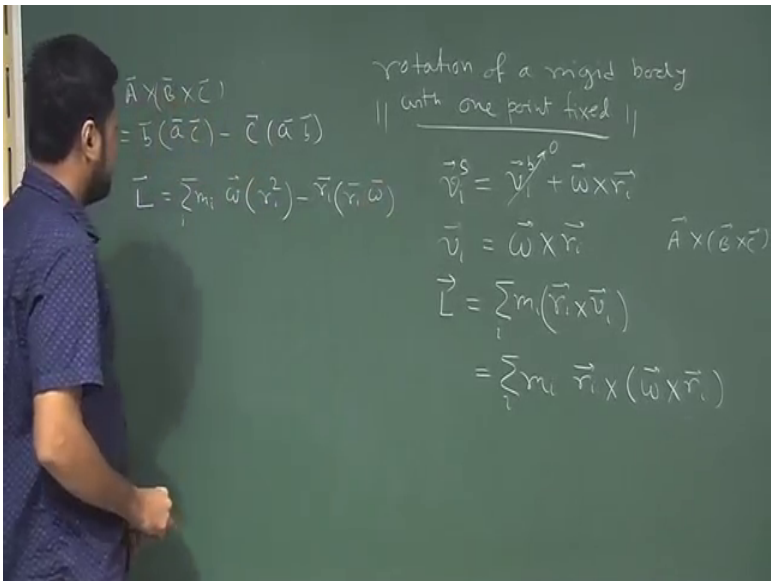

And if we you know write this it will be so A cross B cross C is equal to B times A dot C minus B times A dot C minus A times, right, sorry; C times A dot B C times A dot B. So, that that is an identity we all remember. So, using that identity we can put this and using this identity, we can write l is equal to sum over i mi. So, it will be omega times ri square minus C is equivalent to ri times ri dot omega, right.

So, this will be the expression. Now if we open this I mean. So, these are all vectors. So, there is a vector dot product and this is the vector here is a vector here. So, if we open it up.

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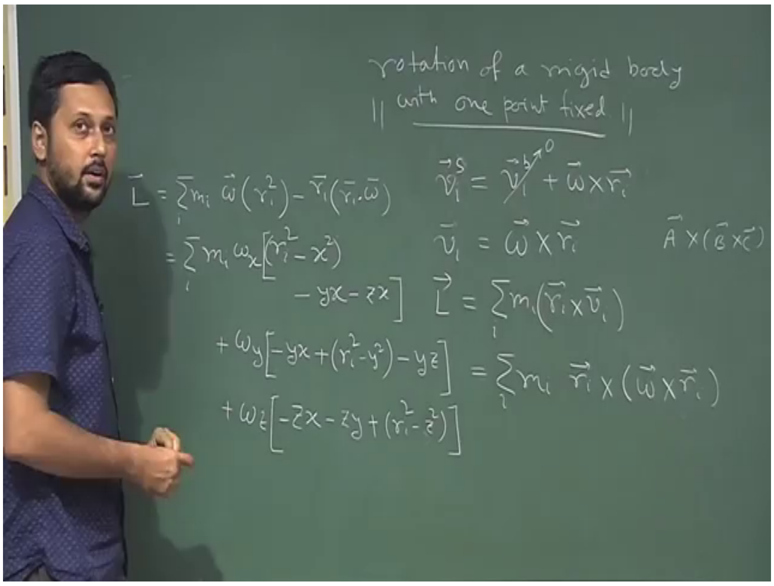

What we will find is sum over i mi. So, the first term will be omega x, if I take omega x common it will be ri square minus x square sorry ri square minus x square this will be one term then there will be additional term of minus y.

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Yx minus z x right and there will be additional term of y, there will be 2 more term of omega y give me a second it will be r x square right. So, we will have terms like minus y x plus ri square minus yi square minus y z plus omega z minus zx, no it will be x square right minus z y plus ri square minus z square right. So, omega x omega y omega z and we have an mi common here, right.

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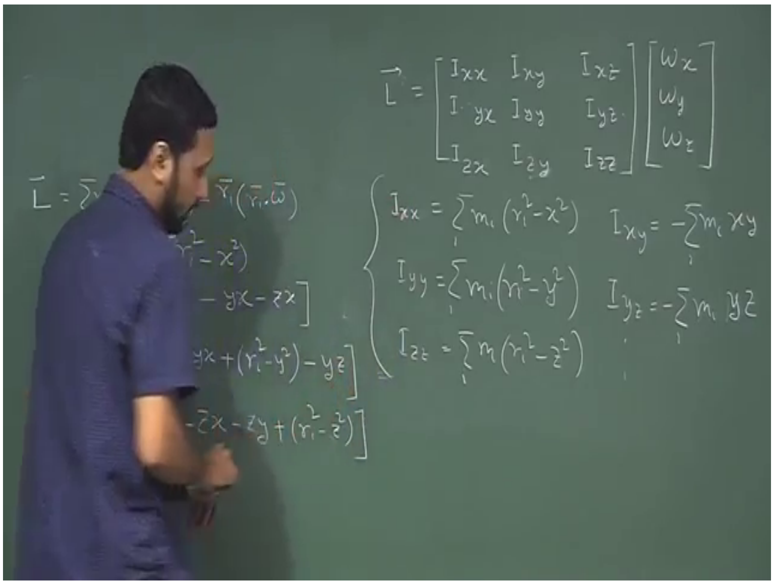

Now this form can be written in terms of a vector equation. So, we have an equation in l. So, we can write I x Ixy, Ixz, Ixy Iyy, Iyz and Ixz Ix, sorry Iyz, yz right Izz and omega x omega y omega z right. So, this particular equation vector equation, if I bracket into components it reduces to this. And we can write an vector equation or a matrix equation of this particular form where Ixx is equal to sum over i mi ri square minus x square Iyy is similarly ri square minus y square. And of course, we have terms like Ixy which is equal to sum over imi mi Iyz is equal to minus sum over i mi y z terms like that. So, just for completeness, I can write y square and I zz will be similarly some over i mi ri square minus z square.

So, I think this forms probably some of these are familiar to you, if not let me tell you these are called moments of inertia and terms like this which will be there will be many more terms i xy. So, it will be ideally speaking it is yx and it is xz and this is zx sorry this is zx this is zy and this is zz right. So, technically we should write not x y, but y x here and x y here.

So, these terms we see the diagonals diagonal term have a has a specific form that this is sum over i mi ri square minus x square ri square minus y square and z square, whereas, off diagonal terms also has a specific form it comes with a negative sign and has a form mi times x xy, right. So, this diagonal terms are called the moments of inertia, whereas, the off diagonal terms, they are called the products of inertia.

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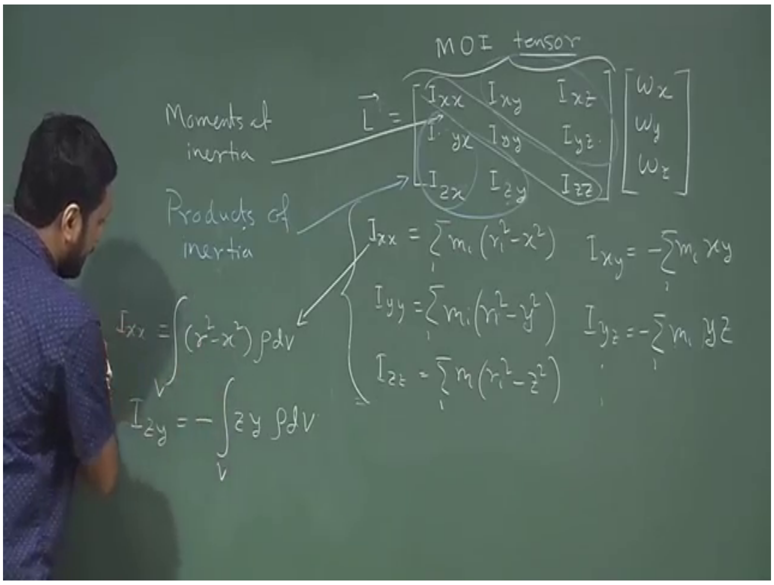

So, this 3 terms they are called moments of inertia and rest the off diagonal term; this 2 this 3 here and this 3 here, they are called products of inertia, right and this whole thing; the matrix as a whole is called a moment of inertia in short form MOI moments of inertia tensor this whole ok.

So, I think tensor is, but a tensor might be a familiar word to you some of you might be familiar with this term some of you might not have heard this term before does not matter we will have a very brief description of what is a tensor very brief because the framework of this course does not allow a detailed discussion on tensor which will be discussed in your mathematical methods courses also see that this can be written in an integral form for example, this one we can write this as r square minus x square t m and like before we can replace dm by rho dv. So, Ixx can be written like this similarly a product of inertia let us say I zy can be written as minus z times y rho dv and the integration is always over the entire volume v of this rigid body.

So, we will take up examples, on how to compute moments of inertia and product of inertia, but firstly, most importantly what does the word tenser means; what does it mean and what do we know what do we learn from it physically. So, tensor first of all it is a mathematical jargon for any measurable quantity vector is also a tensor vector is a tensor of rank one, a rank one tensor in a nth dimensional space. So, let us forget about it. Now let us talk about tensor a way for a while, then we will come back to the specific cases of moments of inertia tensor.

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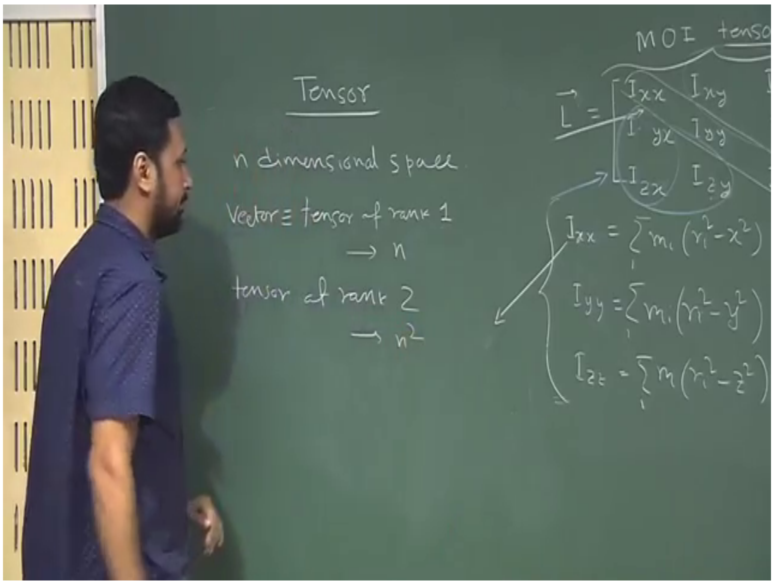

Let us say if we are if the if the in terms of mathematics, if we are working on a n dimensional space, then a vector which is a tensor of rank one has a dimension of n, and this the tensor, tensor what we should call it a tensor of this is actually a tensor of rank 2 should have a dimension of n square. Now as the world of rigid body is 3 dimensional; that means, the according to this law it will have a dementia tensors which represents some property of a rigid body will have a dimension of n square which is equal to 3 square equal to 9 which is exactly the case here we have 1, 2, 3, 4, 5, 6, 7, 8, 9 elements in that matrix and tensor in the tensor of rank 2 is represented by a matrix tensor of rank one, which are vector they are represented by row vector or column vector; that means, a matrix of this form is a vector which is a tensor of rank one a matrix of square matrix in the world v or in the in the in the understanding. We have, for example, in the domain of understanding a square matrix is a representative of a tensor. Please understand that all square matrices square matrix might or might not be tensor a tensor is represented as a square matrix not the other way a rounded square matrix is not always a tensor.

So, in this case a moment of inertia tensor is represented by a square matrix. Similarly, the angular velocity of a rigid body represented by a column vector, right. Similarly, if we have for example, if we want to you know determine, if we want to represent a position vector, we can represent it as x, y, z. Similarly, we can represent the angular momentum in omega in terms of omega x omega y and omega z right. So, this is our r.

Now, what is what does a tensor physically mean a tensor, if we take the examples of tensors which is associated of with some measurable physical property that essentially means that, if we consider a tensorial representation of a physically measurable property; that means, within that representation all the information content is stored, what I mean is let us take an example of a block. Let us say this is a metal box block for an x for a for a moment. Let us assume that it is a metal block. Now we can define the axis as, this is the x axis this is my y x or this is my y axis and this is my z axis.

Now if I apply a certain amount of current in along the x direction of this box. So, I am flowing a current in this direction, and if I measure the voltage across this along this direction we will get a voltage. Similarly, for this same current if I measure a voltage between these 2 surfaces or these 2 surfaces I will also get some measurable value right.

Now, if I, if I now represent this, if I now put the current let us say in this direction say the z direction and I measured the voltage in x direction and y direction also in z direction will get some number. So, what we can do is we can represent; this measurements by this equation.

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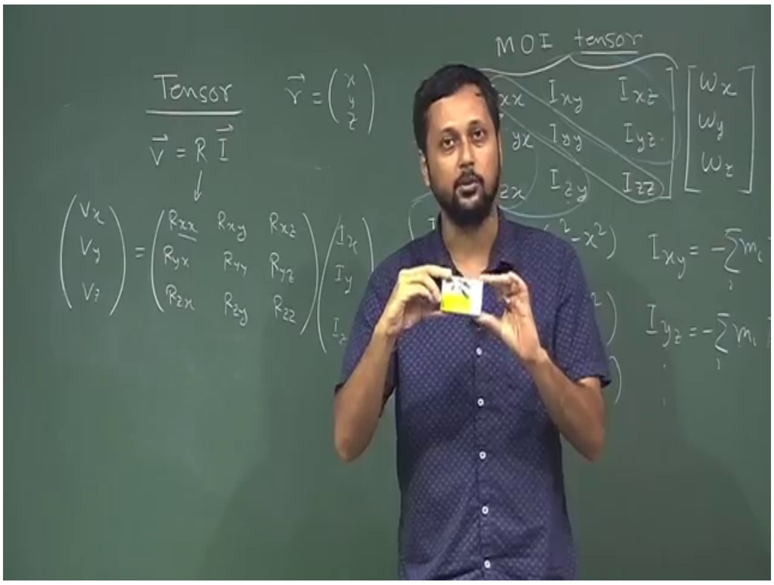

If we know that v is equal to v equal to IR, that is our ohms law, i being the current v being the voltage and r being the registers. Now what we can do is we can have a vectorial equivalent of this equation in this particular form, where the current in each of this direction x y and z will be represented by I x, Iy, Iz and voltages which are measured along each of this direction will be represented by Vx, Vy, Vz.

Now, because if we have a current in x direction and we can still measure a voltage in y or actually we can think of it this way we apply a voltage in x direction, we can still measure will be able to measure a current along y direction. So, if we that is we will have a resistance not only along x direction, but also along y direction and z direction. So, we can represent this equation as vx sorry just up to straighten it a bit Vx, Vy, Vz, which is given by Rxx, Rxy, Rxz, Ryx, Ryy, Ryz, Tzx, Rzy, Rzz. And we will have I x, I y, I z, why I have taken the example of current. And voltage because that is something which we will understand more intuitively that is something we all most of us are familiar with right.

So, this Rxx which represents the resistance when we apply a voltage along x direction and measure the current along x direction, that that gives you some resistance value and that is represented by Rxx, similarly Rxz represents the case when you are applying of current let us say along x direction. And measure a voltage along z direction that particular registers corresponding resistance will be represented by Rxz.

Now, what is the advantage of such a representation? The advantage is now once we have this full set of information; that means, if we know all these 9 parameters we know the resistance behavior of this particular assume that it is a metal block. So, we will unders; we will know all the information, we need in order to know the resistance of our information related to the resistance of this piece of metal right. Then you can put any arbitrary value of Vx, Vy and Vz; that means, you can app you can give me any arbitrary direction of voltage which will be applied across it, then I can give you what will be the direction of current or vice versa. So, you can apply a current which will be in you know any direction through this medium, if we know the particular direction with respect to this axis system I can immediately tell you it what voltage we will you measure along which direction.

So; that means, this 3 dimensional representation or this matrix representation of resistance essentially contains all the information you need to know. Similarly coming back to this the angular momentum is represented by a tensor equation or basically a matrix equation we have a matrix which is a moment of inertia matrix or moment of inertia tensor and we have this angular momentum.

Now, this tells us, if I know this moment of inertia tensor this field this kind tell me what will be the angular momentum for any given orientation of the angle the omega vector.

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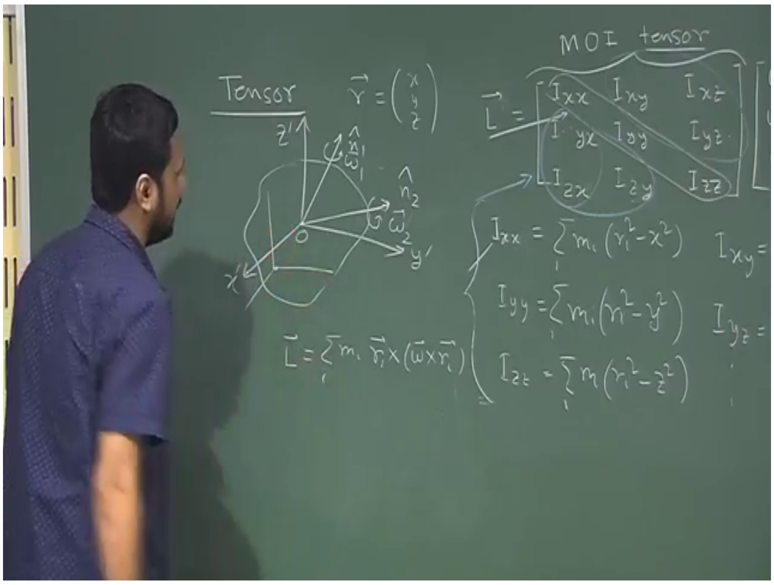

So, we have an object sorry we have an arbitrary object, we have of course, the fixed point has to be fixed. So, let us say we have yeah we have some axis system associated with it and we have a direction of rotation. So, this direction of rotation will correspond when this angular velocity, which it can be decomposed into some axis system which is associated with or associated with the rigid body or so called the body set of axis.

So, omega will have components in the body set of axis or you can think of it as if. So, right now we are talking about an axis system, which is fixed to this we can also think of axis system which is which is fixed in space. So, let us assume that for now let us assume that we have x prime y prime z prime which is also a valid description because if you recall we started with an equation l which was given as imi r cross omega cross Ry right. So, that was the starting point where from which we got to this equation.

Now, if you recall omega as. So, Ri does not matter if we represent it in a in body set of axis or space set of axis because the origin o is the same. And omega is the velocity which is measured with respect to space set of axes. So, actually l in this equation is represented with this set of axes only now for any arbitrary rotation omega we can always decompose omega x, omega y and omega z in terms of the body, set of a sorry space set of axis system. And if we know this tensor I can exactly tell you what will be the value of l or what will be the direction of l vector for this value of omega, you change omega 2 or you change the change your axis here. And you have another value of omega which is omega prime let us say do not.

So, let will just call it omega one and omega 2 just to avoid any confusion. And now also if I can decompose omega 2 into x, y and x prime y prime and del z prime system. And I can immediately give you the knowing the moment of inertia tensor, I can immediately tell you what will be the value of mu l right. So, this is the advantage of having the tensor of course, getting the entire tonsorial information; that means, knowing all 9 components of the moment of inertia is it is a tedious process, we might need it we might not need it at some point depending on the geometry of the problem depending on the symmetry of the problem. We have to decide whether we have to go for the all we have to go for calculating all 9 components or we can just use a reduced z, but what is important is all this moment of inertia tensor which essentially contains all the information you need to know about the rotation about this fixed point might will not be valid, if I now move the shift point here we sorry the fixed point here.

So, moment we shift from this fixed point to another set of fixed point we have to recalculate the entire moment of inertia tensor. So, moment of inertia tensor of an object is defined with respect to a fixed origin it is not a general description please keep this in mind, but. So, ideally the similar thing should happen for resistance or for example, polarization. When I polarize polarizen object or polarized 3 d mass u by external externally applied electric field the polarization also is a tensorial quantity, what I mean is if I apply a electric field in along this particular direction.

And the body is polarized there is a good chance that we will get a polarization along a direction which is perpendicular to the applied field direction as well, but please understand that none of these are specific to a origin, but moment of inertia tensor is that is what the difference comes because moment of inertia this entire construction is valid for one particular origin moment, I shift the origin shift my origin my total the description of Ris; the values of Ris are changing so; that means, in this integration my you know in this integration my rr square minus x square, r square minus y square this quantities or xy quantities are also changing, right.

So, moment you shift your origin you have to redefine your moment of inertia tensor. Thankfully we can have there are theorems which can be applied in order to shift the origin for a certain given that there is a specific symmetry in the object we can always use that theorem in order to calculate the new moment of inertia tensor will come to that slowly, but for now, let us move on from here. Problem is when if you remember we have this learnt this definition of moment of inertia.

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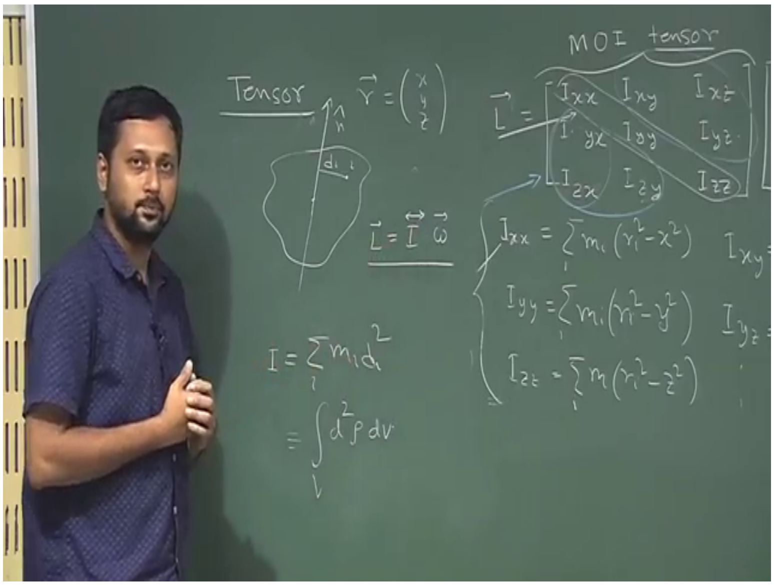

I let us say we have a rigid body we want to calculate the moment of inertia around this particular axis which goes through the body.

So, let us see if this is the position of ith particle, what we did was we calculated the distance di and we have defined moment of inertia as I some over mi di square that that is what we have learned. And so in integral form it will give you d rho dv d will be some function of some function of x and y and z. So, it is a distance sorry d square, right. So, this is what we have learned. And now we are getting moment of inertia in terms of this complicated 3 dimensional matrix, how to correlate this too is there in a way where, how can I correlate this definition with that definition. So, in one hand we have an equation that l is equal to i times omega. And in the other hand we have a definition, I this is a notation both sided arrow it is a notation of a tensor right.

So, this is the tensor equation that is how we define moment of inertia and here the this is a simple equation which is there is no tensor your symbol i is just a number because we know that in terms of finally, what we calculate is just a number as a certain dimension mass times distance of square, but it is a number how do you correlate this to in turns out that a correlation of this 2 is also possible, but in order to do that.

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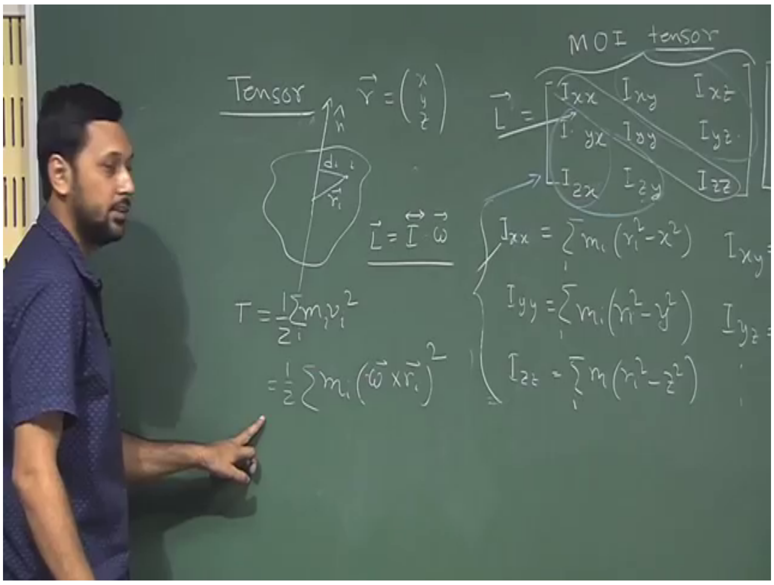

What we need to do is we need to calculate the kinetic energy of rotation, which is once again if I consider for this fixed point it, will be mi vi square sum over all i let us say this is the position vector of this, if the velocity of this particle is v i. So, it will be mi vi square and this will be half mi ri cross sorry omega cross r omega cross r i square right. So, this is one way of this is a way of defining the total kinetic energy of the system. And this will help us in correlating this definition with the standard definition, but what we have learnt during our school days. So, will take it up in the next class, and start from here.

Thank you.