Classical Mechanics: From Newtonian to Lagrangian Formulation

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Lecture – 29
Rigid body dynamics – 3

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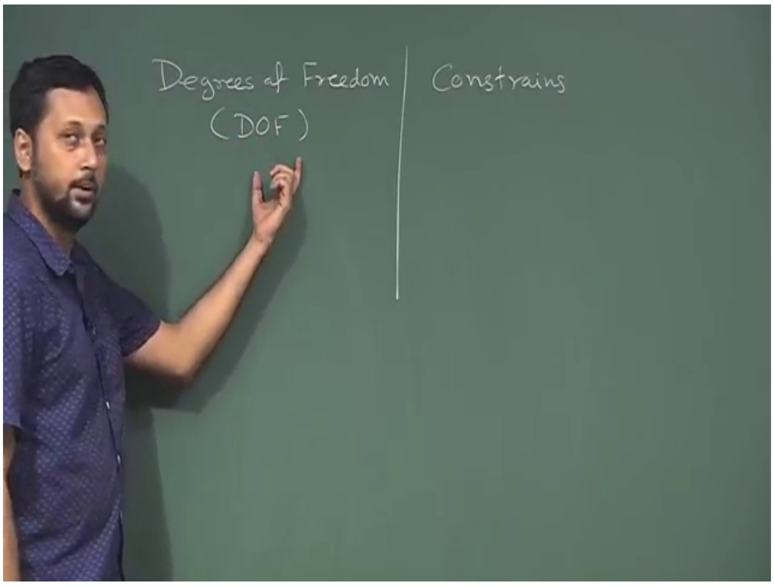

Two terms degrees of freedom we call it DOF and constrains. Degree of freedom means the number of independent, coordinates required to describe a system constrain are the number of any condition that reduces the number of degree of freedom. Basically degree of freedom gives you an idea of the dynamics of the system and constrain is the forces that restricts the dynamics of a system.

Strictly speaking these terms are not restricted to the domain of rigid body dynamics, but because we are discussing rigid body dynamics here, we will be taking care of the degrees of freedom of a rigid body, constrain is something that we are not going to discuss right now, but later on when we will be discussing Lagrangian dynamics, we will have a very detailed discussion of types of constrains and how to remove the constraint forces so on and so forth.

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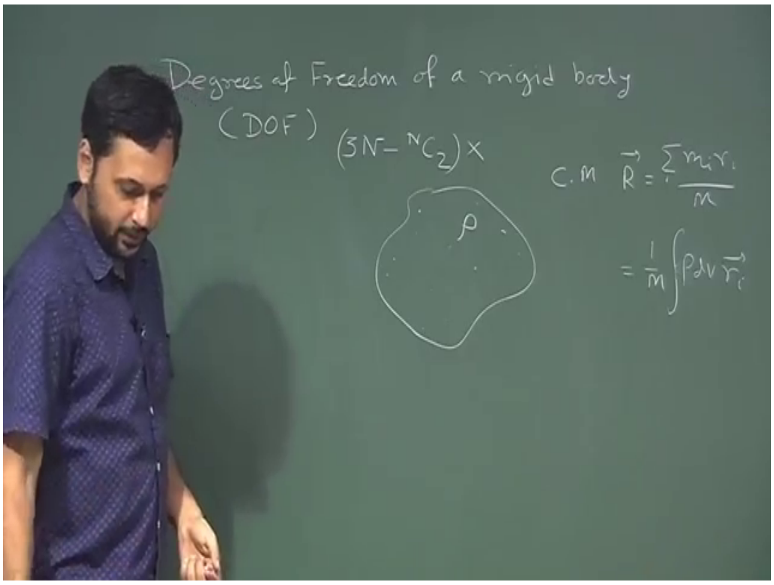

So, right now, let us focus on the dynamics of rigid body.

So, degrees of freedom of a rigid body; this is what is our point of focus correctly. Now let us assume there is a rigid body, rigid body is essentially a system of particle which has which will have let us say many many number of particles in it. So, many that we can in most of the cases we can we do not have to count the number of particles and we do not have to write this sum for different for example, center of mass is center of mass the position of center of mass is defined as sum over i mi ri divided by m.

Now, because this rigid body is a system of continuous masses, we can instead of writing equations for discrete particles, we can take a density function and we can replace this integration by dm ri, as we have seen that and dm we can simply write rho dv, right. So, this is our rigid body, now if this is a rigid body and we want to calculate the number of degrees of freedom of this rigid body, if we go by the standard construction that each point needs three coordinates. So, N points if we have rigid N points in this rigid body, which will need 3N number of coordinates, then it is wrong we have discussed it already and we cannot also use the number of constrains which are mutual distances between any 2 points in this rigid body which is a fixed number.

So, we cannot really use that to reduce this number of degrees of freedom. So, this particular construction of 3N minus N C 2, N C 2 being the number of such constrain conditions in a rigid body with n particles, this does not give the number of DOF. So, this is not right instead in order to get the degree of freedom let us do one thing.

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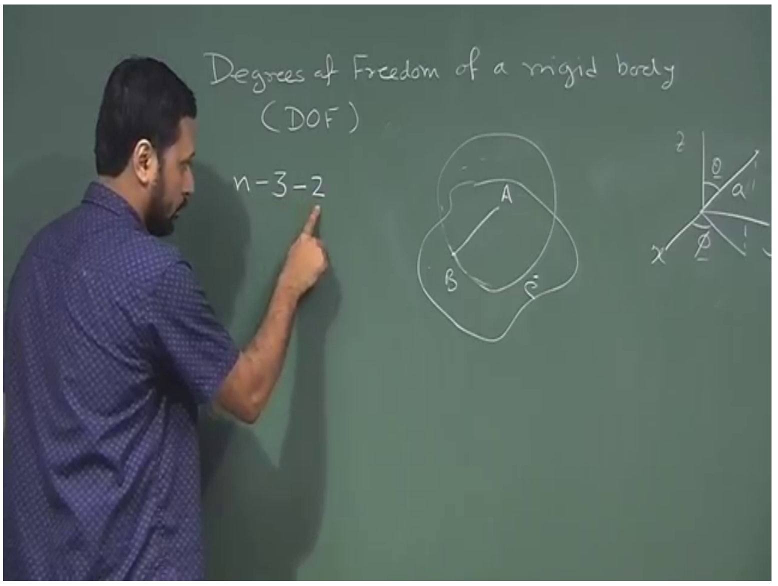

Let us assume let us take three points in this rigid body. So, there are many points we will just take three points A, B and C which are not lying on a same line that is very important, because it has to have you know if we take three points or as many as points on a say on a straight line this purpose will not served.

Now, let us first look into the motion of A, assuming that A is the; A is a I mean the entire rigid body is free; that means, A is free to begin with what are the number of degree of freedoms degrees of freedom in A. As we have discussed when we consider when we consider point masses, it does not have any rotational degree of freedom. So, there is no specific shape to it is just a point mass. So, we need three coordinates to specify its position.

So, the degree of freedom of A is essentially 3. Now if we fix this point in space; that means, if we somehow pivot the rigid body by pivoting by putting an pivot at point A how many degrees of freedom do we take from this whole system? Because we are fixing A and A has a degree of freedom 3. So, we take three degrees of freedom out of the system. Now if the rigid body to begin with has n degrees of freedom n is unknown at present then moment we fix A, we are taking out 3 degrees of freedom out of it.

Now, after we fix this particular point, what could be the possible motion of a rigid body it can. So, it can move with one point fixed, now let us consider the motion of B, initially it was free it, initially it was free to move anywhere before we pivoted A, but right now after we fixed point A, it can move anywhere on this on a sphere of radius AB. So, we can draw sphere, sorry, my drawings are not very good, but anyway I think you understand once A is fixed B can be B can move anywhere on the surface of this sphere.

Now, in order to move order to specify a point on a spherical surface, how many coordinates to we need? We need 2 coordinates because what happens in a in the spherical polar representation we have let us say this is my x, y and z, for any point location we need r, we need theta and we need phi. So, this is a general representation in spherical polar coordinate system, if we know that the point will stay on the surface of sphere of radius a, then we can instead of we can write r equal to A and we need only theta and phi that is generally true.

So, if we even if we do not engage a spherical polar coordinate system we can go away with 2 numbers of coordinates. So, moment we fix point sorry. So, the degree of freedom of b is now 2. So, moment we fix this point as well then we take some more degree of freedom away from the rigid body. Now what is that once we fix B how many degrees of freedom we take away? We take away 2 degrees of freedom. So, total number reduces to n minus 3 minus 2.

Now, once we fix this point B now this rigid body can move along. So, this sphere was an imaginary I mean it was a imaginary sphere. So, we can get rid of this, now we have an axis AB, now this entire rigid body it can move around that axis for example, if this is my rigid body if I put an axis I mean if I fix 2 points for example, this point here and this point here, now it is free to rotate around this axis only, right.

So; that means, we have taken out five degrees of freedom already and it can move around this axis.

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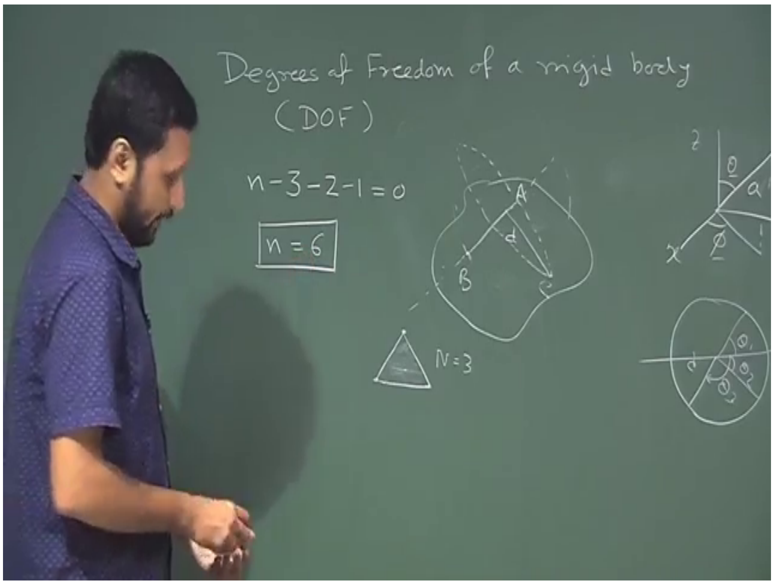

Now, consider the motion of point C, point C can move on a circle which is perpendicular to this axis AB and has a distance D some fix distance D from this axis. Now in order to specify the motion of C how many coordinates to in it? Consider this we have a circle I hope we understand that particles C can move only on this circle.

So, one party has circle of fixed radius that is also very important, this radius d the small d whatever this is fixed. So, we have a particle moving on a circle of fixed radius how many coordinates do we need to specify its position at any instantaneous location? Let us say here or here we just need one parameter that is theta this is theta 1 for this one this is theta 2 for this position this is theta 3. So, one coordinate or one parameter is sufficient to specify the position of C.

So, now if we fix C as well how many degrees of freedom do I take away from the rigid body? One more and now consider the case in this rigid body we have three fixed point can the rigid body move anymore no right. If you fix three points in a rigid body it cannot move any more you take this one for example, you fix this point it can move, if we you fix the diagonally opposite point it can move around this axis, now you move a third point that is it cannot go anywhere; that means, once we fix point C the total number of degree of freedom reduces to 0 and this equation tells us n is equal to 6.

So, total number of degrees of freedom of a rigid body in general, it is a I am not considering any special shape of the rigid body I am not considering any special property of the rigid body, for any general rigid body of any arbitrary shape degree of freedom is 6 and also there is another very important consideration of minimum number of particles in a rigid body minimum number of particle that can constitute a rigid body is 3. By rigid body I mean it is not an one dimensional object, I have 2 point masses suspended by a thread or mass less stick. So, that is kind of a small dumble, but it is a one dimensional object it is not a three dimensional object.

But if we have a third point let us say we have 2 masses, then this is a one dimensional object. We have a third point which is not in this line we have a lamina right we have three point masses connected by mass less threads or mass less sorry not threat, but mass less rods, then this constitutes a rigid body. Reduce the number of particle from three it will not work.

So, minimum number of N is equal to 3 to be considered as a rigid body and that rigid body will have a degree of freedom 6 right. Now with these 2 considerations, let us consider the motion general motion of a rigid time rigid body. Now general motion of a rigid body can have 2 components; we can have translation, we can have rotation or we can have we can. So, we can have pure translation or pure rotation or we can have a combination of both.

Now, there are couple of theorems that suggest that translational motion sorry a rigid general motion of a rigid body can be considered as a combination of translation and rotation, but let us first focus on the case when we have a rigid body with one point fixed in space. Now let us look into this for a second, we have a very famous name on screen this is the name of Leonhard Euler Swiss mathematical genius.

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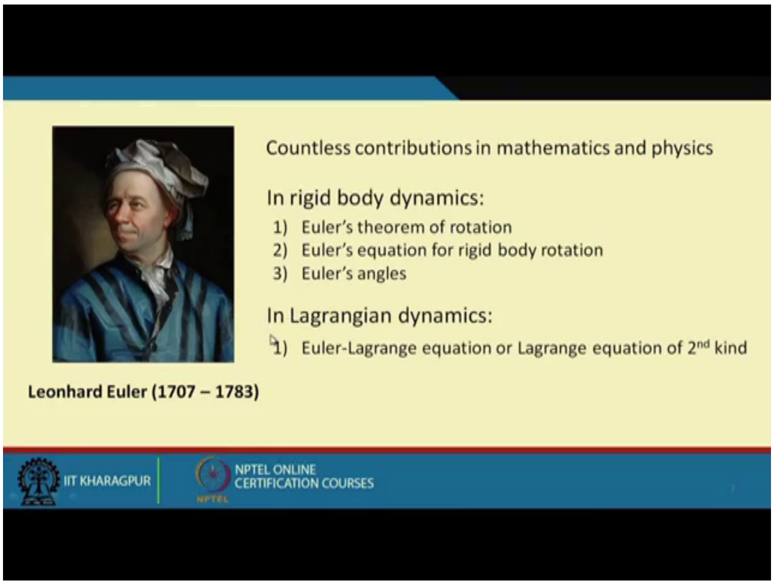

So, he was a mathematician to begin with which he had published enormous number of papers in different branches of mathematics. So, we have Euler's equations and Euler's theorem in almost every branch of analytic and analytical mathematics and also in physics.

So, right now; so, we are not I mean it is not possible to discuss the contribution of Euler in this in the framework of this course, but will be referring to him many at times specially when will be discussing rigid body dynamics, we will have Euler's theorem of rotation which will take up very soon. Later on we will see that there are Euler's equation for rigid body rotation and we will also have Euler's angles.

Now, later on when will move on to Lagrangian dynamics we will have Euler there is a set of equation which are called Euler Lagrange equation or Lagrange equation of second kind which are also very very very important thing.

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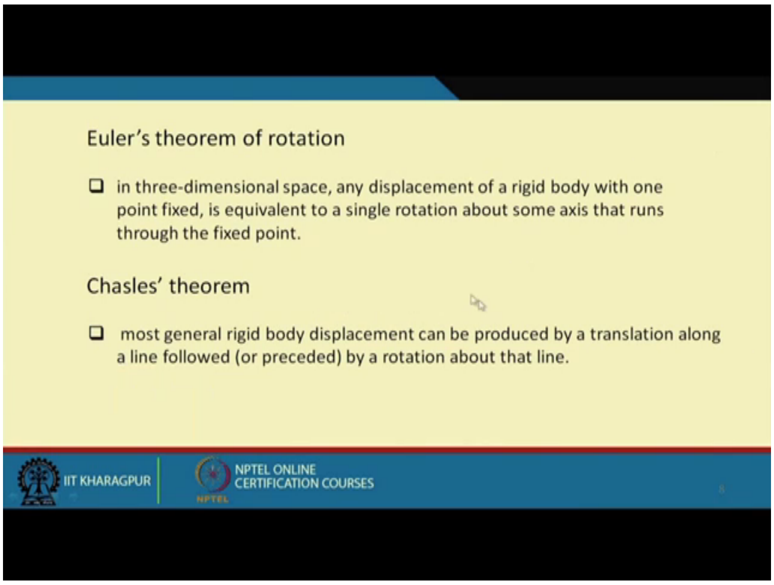

So, right now we will be focusing mostly on Euler's theorem of rotation, which says in three dimensional space any displacement of a rigid body with one point fixed is equivalent to a single rotation about some axis, that runs through the fixed point. Also there is a there is another theorem which is also very describes the dynamics of rigid body very well that is Chasel's theorem that says the most general rigid body displacement in space again we are talking about 3D space can be produced by a translation along a line followed by a rotation about that line.

So, when we are talking about these 2 theorems, we are strictly speaking we are not going to prove this theorems. Especially the second theorem we are not I mean second theorem was we can have a proof short proof, but still we are not going to have it. In this class if you if you have the textbook by professor A K Roy Chaudari there is a beautiful proof of Chasel's theorem there in the beginning pages of rigid body dynamics, which you can always go and have a look at its a very very very nice proof few lines only, but that gives you a very good description. But what we can what we do have we will do have is a brief description of what these theorems essentially tells us ok.

Chasel's theorem tells you that general motion of a rigid body, general motion means I mean both rotation and translation.

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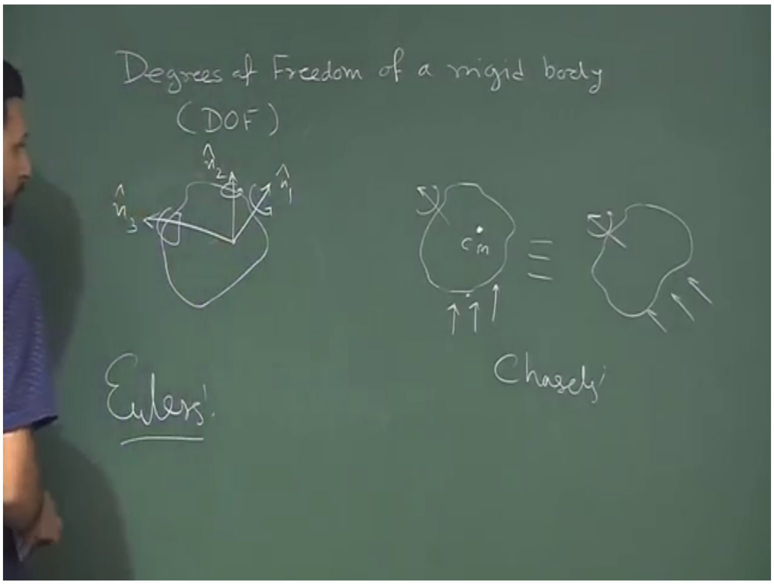

So, let us say the bodies have some kind of a twisting motion and also it is moving in a particular direction. So, Chasel's theorem states that this movement the general motion can always be reduced. So, let us say it is moving under some kind of an axis in this direction. So, it is evolving around this direction and moving in that direction.

So, Chasel's theorem states that you can always find an equivalent picture where the body is essentially moving in the same direction as the rotation axis and executing a rotation and of course, there will be a velocity component in the direction of this the original direction of displacement, which can be treated separately. If you recall the translation of a rigid body can be treated as if the entire mass is concentrated at some let us say this is my center of mass.

So, what is the meaning of this is, we can take this as the displacement of a rigid body and the remaining motion which will be a motion in some direction towards this, can be taken as a pure displacement and we do not need to consider the properties of a rigid body into it, we can just take it as if it is a point mass moving in some direction. So, it might be slightly confusing because the motions they are kind of perpendicular to each other it looks, but. So, let me draw it like this. So, let us say this is the general direction of motion and this is the general direction of rotation axis. So, this is the statement of Chasel's theorem.

Now, Euler's theorem on the other hand does not talk about translation at all, it talks about the situation where a rigid body one point in rigid body is already fixed and it says that any general motion with one point fixed can be described as a rotation around an axis, arbitrary axis passing through that particular point that means.

Let us say this is my rigid body at some point, please do not confuse it with the center of mass this fixed point need not be the center of mass, it could be center of mass as well, but it did not be the center of mass.

Now, this is the statement of Chasel's theorem, now Euler's theorem states that if this point is fixed and the rigid body is executing random dynamics, that the according to this theorem that random dynamics at any instantaneous for time point, can be represented by a rotation around some axis which is going through this fixed point. Now this axis is the direction of that axis can change very next moment. What I mean to say is let us at time t 1 this is the direction, and if the motion is totally opposite in a random then in the very next moment we will have an axis which is in this particular direction, but the rotation will be around that axis. In any other any of the following moments we can have an axis instantaneous axis of rotation moving pointing towards this direction, but still we will have a rotation around that.

So, this is the statement of Euler's theorem, now there is a very nice mathematical proof which is also possible at your level, but for that we need to go into the details of orthogonal the orthogonal tensors and their properties. So, we are not going to do that during this class, but you can find proof of Euler's theorem, in if you are interested you can go through the details of orthogonal transformation in bold stain and there is a proof of Euler's theorem.

So, this essentially means this theorem means any arbitrary rotation of a rigid body will always leave one direction invariant. So, what it means is for example, once again they let us say this is my rigid body its it has one fixed point let us assume it has one fixed point and it is executing some random dynamics, random reorientation. Now that reorientation according to this theorem means at least leave one point un unaltered one axis in that rigid body unaltered. Anyway let us not go into that any further right now maybe will come back later.

Now, what is important is we can have we can always represent the dynamics of rigid body in terms of some sort of rotation around some sort of fixed axis, that is very important and for the rest of this discussion we will be focusing on the rotational dynamics of rigid body only. Now total degree of freedom is 6 total and out of that three goes in order to describe the translation.

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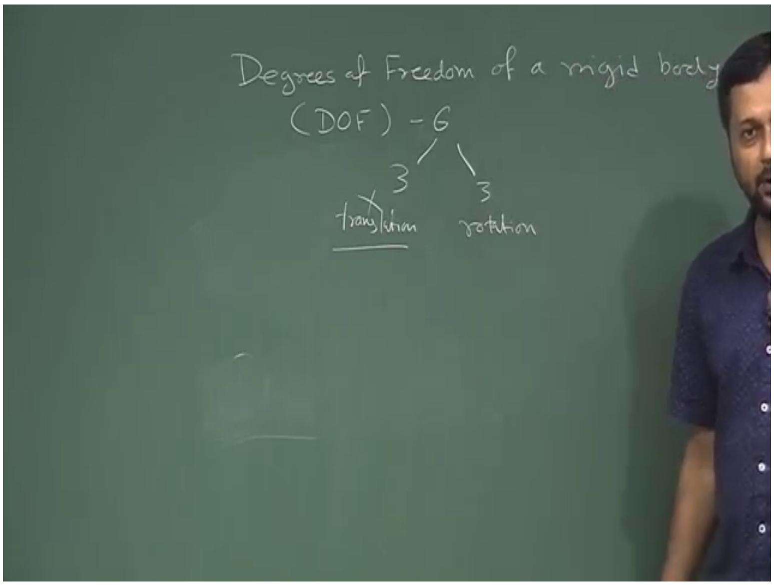

So, 3 for translation and 3 goes for rotation and moment you fix one point in rigid body this 3 translational degree of freedom is totally gone.

So, we will be discussing cases where one point of the rigid body is always fixed; that means, we have purely rotational motion and we have to deal with only 3 degrees of freedom and later on we will see that these three degrees of freedom are very good so; that means, we need three coordinates in general and later on we will see that Euler's angles can be can serve as this three coordinates and that will you know that will be very very advantageous in certain aspects, but right now we will just discuss the general dynamics rotational dynamics of rigid body.

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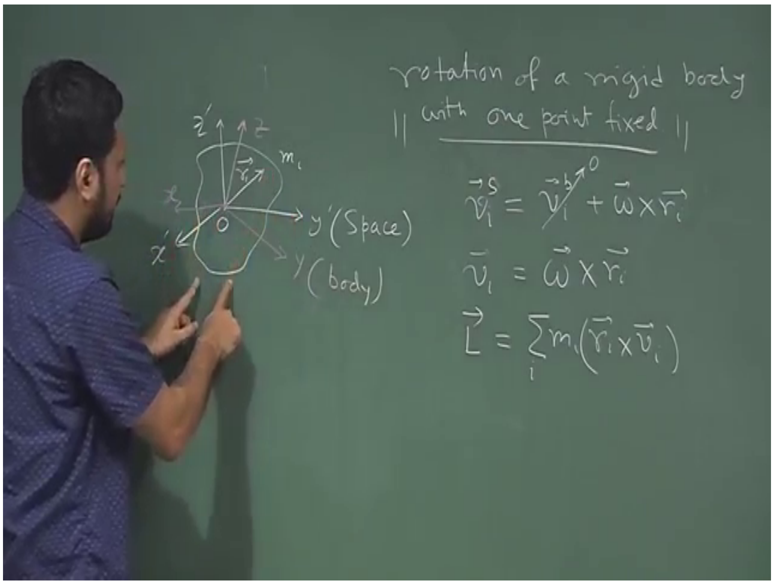

So, out of that 6, three we will never consider we will consider only this three right. So, we will just change the topic from degree of freedom to rotation of rigid body with one point fixed. So, this is the rib description of what we are going to do. Now let us assume that we have one set of axis which is x prime y prime and z prime and we have some kind of a rigid body and let us let us assume that the fixed point. So, one point fixed and the fixed point is always the origin it is not. So, this is something that we will be following throughout this chapter.

Now, and you have guessed it right because we are using x prime y prime z prime these are fixed primes. So, this axis system is fixed. So, we must have because we want to describe the rotation, we must have a set of axis which will be moving along with the rigid body right. So, let us assume I have a different color very good. So, let us assume we have z x and y axis and they are moving along with the rigid body right.

So, 25.14 any vector any position vector of let us say ith point mass which is given by ri and have a mass mi, this is rotating along with the. So, this position vector if it is taken in this moving coordinate system; that means, it is rotating along with the rigid body and as the entire axis system of x y and z they are moving with the rigid body, during the motion ri does not change right and now onwards we will call this the space set of axis and we will call this one as the body set of axis.

So, this is also a nomenclature will be following throughout this chapter. So, we have the space set of axis which are fixed in space given by x prime y prime z prime and we have a body set of axis which are moving along with the body, which is given by x, y and z and all the coordinates all the point all the coordinates on the velocities they are with respect to this body set of axis. Now because of that we can write the velocity v i right we will or rather we will give this suffix s for space.

So, if I want to calculate the yeah the velocity with respect to this of ith point with respect to this space set of axis which is the inertial axis. So, it will have 2 components one is vi body plus omega cross ri. So, these are all vectors this equation looks very familiar I guess because this is the exact same equation we derived for moving coordinate system and this body set of axis is the moving coordinate system of our interest.

Now, as this as we are discussing rigid body, the for with rotation this relative position of this of this point masses are not changing with time because it is a rigid body the relative position is fixed. So, that is why vib is unanimously 0. So, v which is the velocity as measured from a static frame or a space set of axis is only omega cross ri right and ri is measured as body with respect to body set of axis.

Now, what we will do now we will simply drop this suffixes once we understand the scenario, we will simply drop this suffixes and we will try to write the total angular momentum with respect to space set of axis. Now total angular momentum with respect to space set of axis we have already seen its sum over i ri cross vi, please remember ri is the same does not matter if we ri. So, if we measure ri with respect to a body set of axis or space set of axis its remains the same because it is a position vector and the origin of this 2 are common right.

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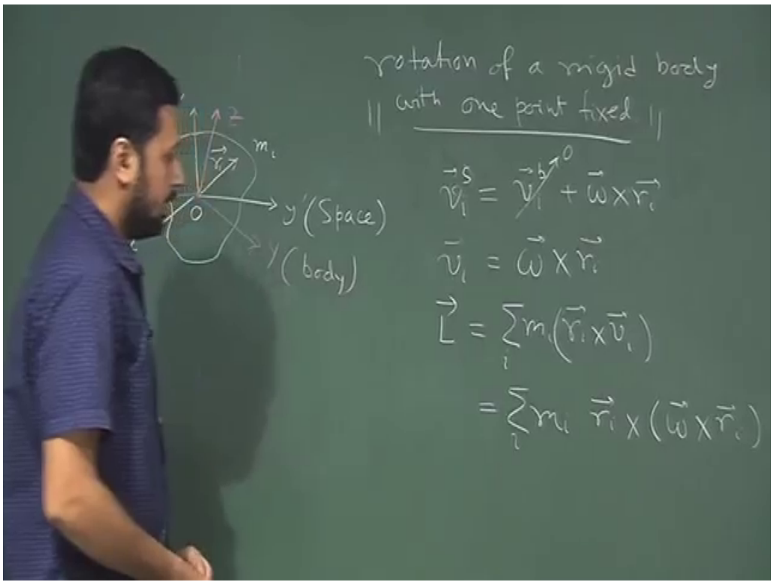

So, we will see that mi can be written as ri cross omega cross ri right. Now this is a triple I mean it is a triple product of vector and we can open this up in order to get some more insight. So, we will do we will start here from the next class and we will see what can be done.

Thank you.