Classical Mechanics: From Newtonian to Lagrangian Formulation
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Lecture - 28
Rigid body dynamics – 2

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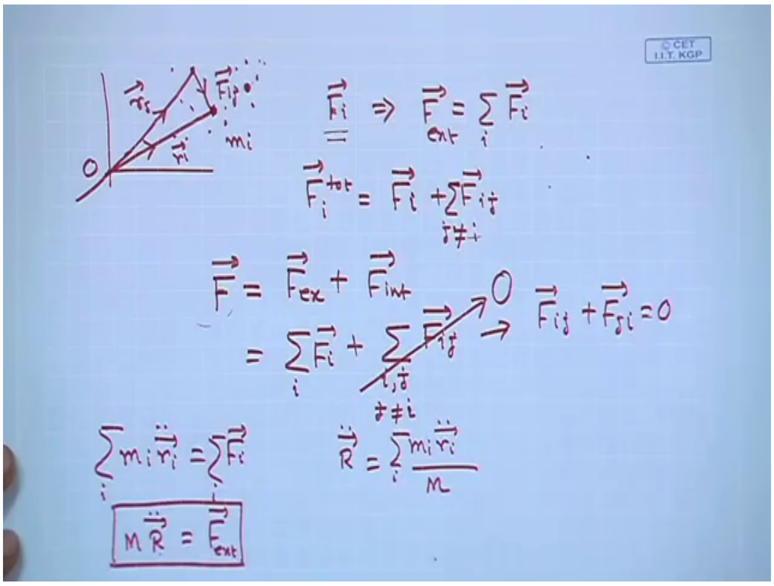

So, we have seen the equation FR double dot is equal to F external is valid for a system of particles. So, we can write the conservation of linear momentum, if total external force is equal to 0.

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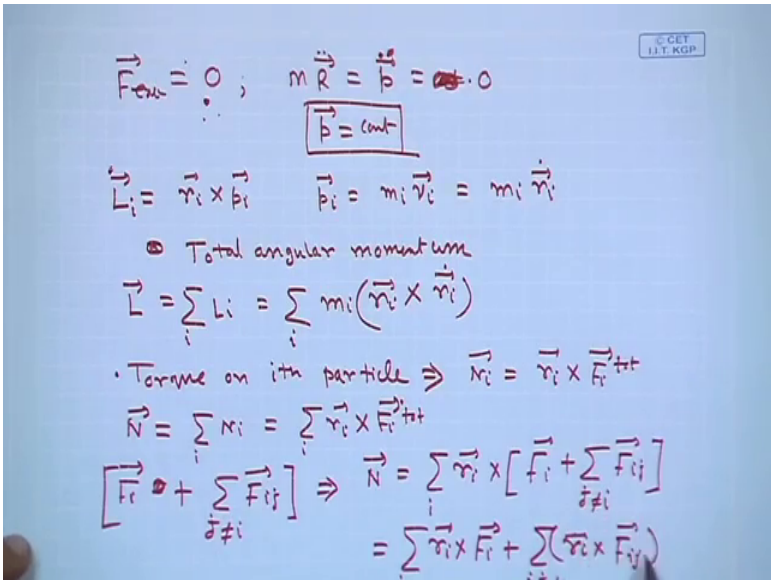

So, if F external is the total external force of a system of particle is equal to 0 then we see MR double dot which is nothing, but p double dot is equal to constant, p double dot is equal to 0 or sorry p dot is equal to 0 which gives you p equal to constant. So, for a system of particle conservation of linear momentum helps.

Now, let us try to define angular momentum L for a system of particle. Now angular momentum L how will you define it. So, let us go back to this picture once again say angular momentum is can be defined only with respect to some origin, in this case this is the arbitrary origin we have chosen we could have chosen it here also or here also anywhere in this in this page anywhere or inside the in the middle of somewhere inside the system of particle.

But let us say this is the origin. So, angular momentum L will be defined for a particle for the ith particle will be defined by ri cross pi, pi being the linear momentum of the ith particle pi is nothing, but mi vi or mi ri dot right. So, if I. So, this is the ith particle now if I try to define a total angular momentum for a system of particle which will be simply Li sum of over i mi ri cross ri dot right. So, this is the definition.

Now, let us try to define a quantity let us say the force Fi on ith particle. So, the force Fi is acting on ith particle if it is acting you know when the force is acting unless and until the force is in the same line as the or unless and until the force is central in nature; that means, it acts a only along this radial vector itself, there is a external torque which is result.

I mean which is there due to the result of this external force Fi right sorry. Now if we try to calculate this torque on the ith particle, which is given by Ni simply is ri cross Fi that is the torque of this particular force on ith particle about the original. So, like angular momentum torque has to have also has to be defined around the fixed origin unless the unless and until we do that it has no meaning. Actually that is true for any quantity when the position vector the velocity all has to be defined with respect to some origin. So, the total torque N of the on the system of particle will be sum over i Ni and it is given by sum over i ri cross Fi write. So, once again if we try to define the total if I try to write a relation including total torque and total angular momentum of system of particles.

We have to remember that the force Fi on ith particle also has an additional component what which is sum over j which is not equal to i fij, right. So, the torque; so, this is the total force actually not this, this is the external force which is acting on ith particle and this is the total force. So, actually. So, I am bit wrong here that. So, we have to consider total force. So, F total is has 2 components Fi plus this.

So, if we now, try to estimate total torque which will be given by i ri cross Fi plus sum over j which is not equal to i F ij right now if we compute this term now this sum will go in. So, we can break it into 2 sum one is ri cross Fi, Fi being the external force on ith particle plus sum over i and j not equal to i ri cross F ij right .

Now, this F ij as I said we can always break it into the form F ij minus F ji. So, this can be done; now if I examine this sum this particular sum.

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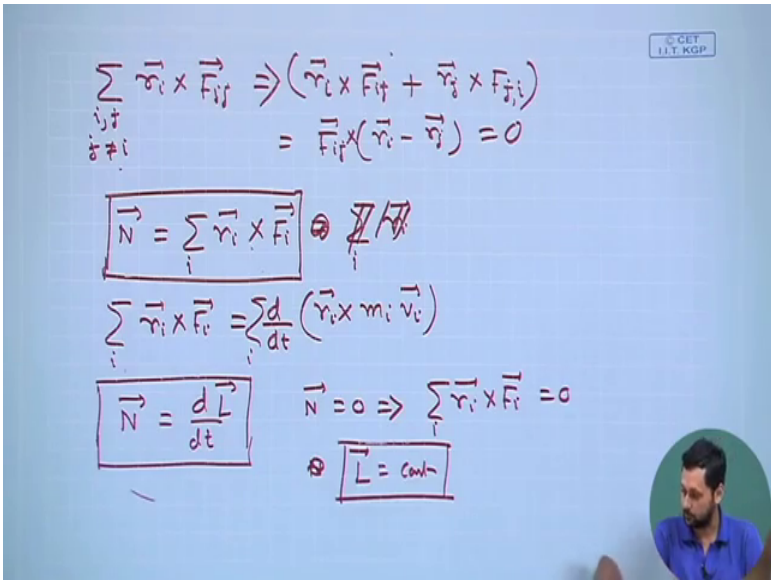

I will write it again sum over i and j j not equal to i ri cross F ij. So, essentially I am trying to calculate the torque due to internal forces now this will have pairs like ri cross F ij plus rj cross F ji because this in this indices i and j they are interchangeable and there are identical I mean they are exchangeable indices. So, we will have pairs like them. So, this particular term will give me pairs like this. So, this entire term can be broken into terms pairing pair terms like this. Now if I examine this particular term F ij is equal to minus F ji.

So, we can write F ij cross ri minus rj, and ri minus rj is nothing, but if I go back to the this original picture of system of particle ri minus rj is along this radial vector along the vector which joins the particle i and j. Now this F ij is a force which once again acts along the same line. So, r. So, if I now if I look at the cross product F ij was also a force which is acting along this line and ri minus ij is either in this direction or in this direction depending on how we define I mean which which one is i which one is j, does not matter if it is if ri minus rj is along this direction or in that direction, thing is this cross product irrespective of it is a I mean it is along or opposite to this particular direction this cross product will unanimously vanish.

So; that means, the total torque N of a system of particle once again can be reduced to. So, which was these 2 term, this term will be uniformly vanishing. So, it will be nothing, but sum over i ri cross Fi. Now which if you remember this ri cross; now ri cross Fi if I remember the first relation we wrote this sum over i Fi can be replay represented by ri oh sorry, sorry, sorry, sorry, sorry. So, this is the this is one relation right now there is a relation which is which connects this angular momentum at the this total external torque with the angular momentum, and that relation for ith particle is ri cross Fi which is a torque of ith particle is d dt of ri cross mi vi which is the angular momentum of ith particle.

Now, we know we have already proven it during our discussion of central orbit, that this relation has to be valid for each of the individual particles of this system of particle. Now if I run the summation over i in both sides, we immediately see the left hand side gives you the total external torque n and the right hand side this gives you if the time derivative will come out and this will give you the total angular momentum of the system of particle. So, the relation which we have derived for one individual particle I mean which we knew that it is true for one individual particle holds true for the entire system of particle as well and this is primarily because the torque due to the internal forces all the intact of due to internal forces actually vanishes right.

So, this is one important relation with. So, we have to we see that the Newton's second law which is which relates the change rate of change of momentum with external force that is valid for a system of particle, as if the entire you know mass is concentrated at a particular point which is called the Centre of mass of this system and we have seen that the total external torque is also total external torque due to the external forces for a system of particle is equal to rate of change of angular momentum right. Now also what we can do is we can write a conservation principle for angular momentum, if we get N is equal to 0 that essentially means total torque due to all the external forces equal to 0, then that tells you that L is equal to constant.

So, this is likely the angular momentum conservation we have a linear momentum conservation we have an angular momentum conservation for the system of particles as well right. Now the next important topic is the motion with respect to centre of mass. So, far we have considered a system where the centre of mass is. So, we have an origin which is somewhere here and let us say this is that is why my system of particles belong and this is my origin and I have calculated the. So, whatever calculation I have done I have done it with respect to an arbitrary origin. Now let us assume a situation where I am for each of this particle of this centre of sorry system of for each of this particle in this system.

I have 2 coordinates available one is taken with respect to an arbitrary origin which we have done already, and the second coordinate is with respect to the centre of mass; now see what happens.

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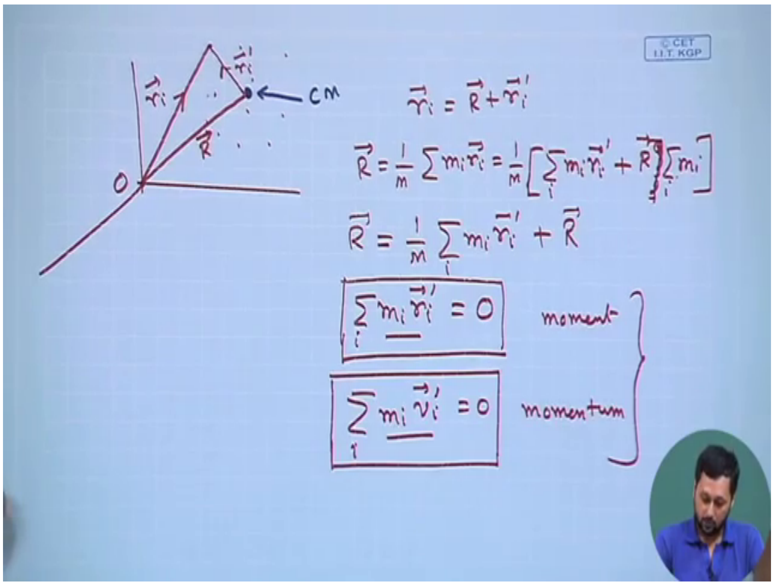

So, this is my system of particles. So, this blue dot is my cm right. So, for any particle here let us say my with respect to my arbitrary origin, please understand that centre of mass is also defined with respect to this origin. So, the position vector is ri and with respect to centre of mass the position vector is let us say ri prime and capital R is the centre of masses coordinate. So, we can write ri is equal to R plus ri dashed. So, if I go by the definition of centre of mass which is sum over i mi ri, now put this value of ri in here you what we will get is sum over i mi ri prime plus R, right.

Sorry R sum over i mi now the second term this will simply give you the total mass of the particle. So, if you open the bracket the first term will be 1 over m sum over i mi ri prime plus R. So, you see this from this equation what we can infer is sum over i mi ri prime is equal to 0 also this and if I take the time derivative of it, then the time derivative of position vector with respect to centre of mass will give you the velocity. So, I can also write a relation that mi vi prime is equal to 0. So, this quantity is called the moment and this quantity is called the momentum. So, both moment and momentum for a system of particle with respect to with respect to the centre of mass vanishes.

There is a very important relation will be using it during our discussion of rigid dynamics a lot now. So, this one is not very intuitive, but this one is very intuitive think of a situation, let us say you have a toy you have a toy explosive which actually explodes in the middle I mean. So, you I think we all have this Diwali [FL] we have fun with it sometime. So, we have this explosive small explosive we put fire to it and throw it away on the midway it explodes. Now let us say it is moving with of particular velocity right now moment it explodes the total force which due to due to which is explodes it is an internal force, right.

Now, because of this internal force all the parts of it they start moving in different directions. Now consider the centre of my consider this motion from the centre of mass frame; from centre of mass frame the position vector of all the particles are changing constantly also the velocity is changing. Now according to relation I have just derived any particle which is of that the any piece of that explosive which is moving away from the centre of mass with some velocity some position vector ri and some velocity ri it ri dot or which is given by vi the total moment or total momentum calculated with respect to the centre of mass must vanish and this is in accordance with the conservation of linear momentum. So, for that particular piece of explosives the total you know the total momentum which is carried away in all the directions will vanish.

If you recall during our discussion of systems with variable mass we took up a problem which was like a raindrop is evaporating a raindrop in free you know in vacuum is evaporating. So, let us say and the one of the basic assumption in solving this problem was the net momentum carried away in all the directions vanishes and that is a very valid assumption given that the momentum is carried away in all directions uniformly. So, that is why the momentum change with respect to centre of mass frame is 0. So, with we have this with have a formal proof for it.

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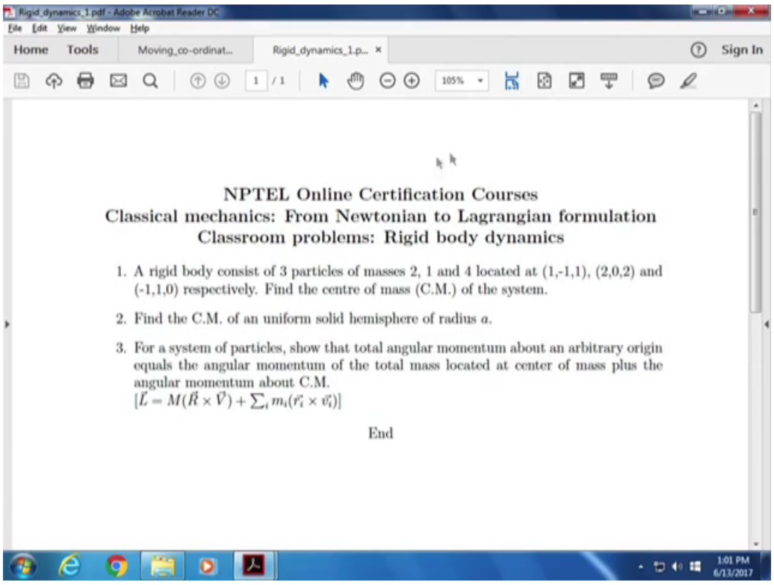

Now let us move to the problem set once again we have a problem which is also related to this small derivation; for a system of particles show that the total angular momentum about an arbitrary origin equals the angular momentum of the total mass located at centre of mass plus the angular momentum about centre of mass. So, essentially what we need to show is the total angular momentum L which is L has 2 components, one is the angular momentum which is about centre of mass and the second is the angular momentum which is measured with respect to centre of mass all right. So, so sorry I am I have made a mistake here it will be ri prime. So, it will be ri prime vi prime I will correct it in the final assignment right. So, in order to prove this let us look into this ri.

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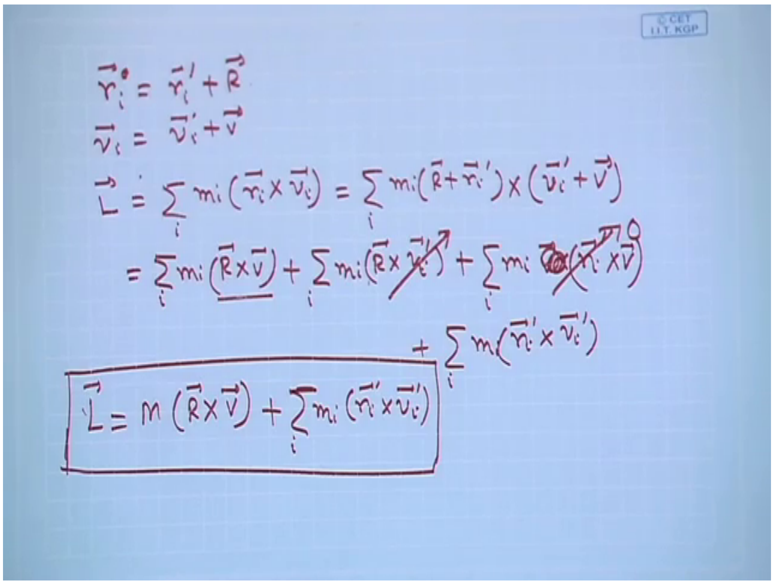

So, ri is equal to ri prime plus capital R V.

So, vi will be equal to just the time derivative of this. So, vi prime plus capital v. So, L will be equal to sum over i mi ri cross vi. So, this is the definition of L we know now we substitute ri with this relation. So, once we do that we get mi r plus ri prime, and also is substitute the same for v we get vi prime plus capital V; capital V is the velocity of the centre of mass. Now if we open the brackets and do it term by term the first term will be R cross v second term will be m i R cross vi prime plus sum over i mi V cross ri, sorry, sorry, ri prime cross v that is from this term.

There will be an additional term mi ri prime cross vi prime. Now the relation which we just proved that the moments and momentum around centre of mass vanishes by that particular relation we can set these 2 terms equal to 0 why because we can take mi inside, mi inside this and we can write it as ri cross sum over i mi vi prime which is equal to 0 similarly this one can be written as sum over i mi ri prime cross v capital V which is once again equal to 0. So, what we are left with is. So, these 2 quantities are i independent of i. So, we can take execute this summation. So, this is m R cross V plus this additional term which is sum over i mi ri prime cross pi prime.

So, hence we have proved what we have to prove and we will see later that this quantity itself will give us some interesting you know theorems of moments of inertia, but that comes later. Now with this we formalize the discussion of systems of particle; let us move into rigid body now what is the rigid body? A rigid body is essentially system of particle, but with a very you know specific property for example, let us say this water bottle is it a rigid body it is not, but because it is full of water I mean a there is some certain amount of water in it and we cannot call it a rigid body, but if I somehow manage to drain this water or I somehow drain this water, I have an empty bottle that is a rigid body what are the difference? The definition of rigid body is a system of particle where the relative separation between the points are fixed.

So, now look at it again, when there are water molecules inside we know that its liquid and in liquid the only water molecules the relative separation or relative position of the of liquid molecules changes; that means, when we have water inside this system it does not qualify for a rigid body. Moment we drain the water out its a dry system, it is a solid bottle it is a plastic material where the met the separation between this 2 particle they are fixed again plastic is also not a very good example of a rigid body, we are say polymer material polymer has certain properties which might not qualify for rigid rigidity, but a metal object or an wood piece of wood for example, yeah and so, even this piece of brick these are these are very good examples of rigid body.

Now, what is uh what are the terms we need to know about rigid body there are 2 very important parameters in a rigid body which are which we are going to discuss now.

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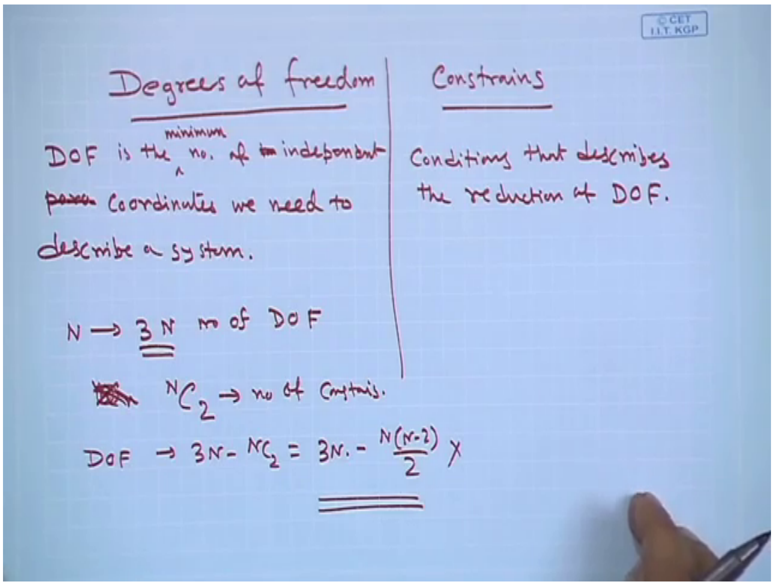

One is degrees of freedom and the second one is constrains right. Now; what is degrees of freedom? First of all we have to understand we lets go by the formal definition; degrees of freedom we also call it DOF is the number of or minimum number of independent parameters or rather, I will call it coordinates we need to describe a system.

So, it is the minimum degrees of freedom is the minimum number of independent coordinates we need to describe a system, and constrain is constrain there is a in terms of if I want to correlate these 2 the best definition of constrain is the conditions that describes the reduction of degrees of freedom. Now let us try to understand this one by one, first of all degrees of freedom let us say if I have a free particle in space free particle means what is the definition of particle first of all. Particle means it is a point mass it does not have any physical dimension, there is a only a mass which is very very tiny for that particle how many degrees of freedom do you need or how many independent parameters do we need to specify this system completely the answer is 3.

If we can write xyz coordinates, we can write r theta phi coordinate, but if we know three coordinate parameters three independent coordinate parameters for that particular particle, we know this position of it completely, we it may define the position of it completely in square. We have 2 particles independently moving you know I mean let us say they have no relation. So, we need six parameters now in a system of particle if there are n number of particle capital n number of particle. So, in principle we need three times capital N number of parameters to describe the system completely given that there is no re no relation between each 2 particle. But what happens in a rigid body for rigid body because there are so many constrain condition, there is an additional constrain condition. So, if I try to formalize it we need for a n particle system.

We need 3N number of DOF degrees of freedom or rather the degree of freedom of a system of N particle is 3N, but for a rigid body what are the number of constrains we have? We have a number of constrain as n C 2, oh sorry, capital N C 2 number of constraints right constrain is the condition that describes the reduction of degree of freedom for example, if we fix in a system of particle, if I now if I go back to the original example we have 2 particles we need six independent parameters. If I now fix the length of this if I now the fix the separation between these 2 particles number of degrees of freedom needed to describe the system reduces.

So, this is the constrain condition, when I fix the distance between these 2 or fix one particle; that means, we immediately fix 3 parameters of it. So, even for a 2 particle system we just need three independent parameters because the first particle is not moving. So, that is the constrain condition. Now for a system of n particles for a rigid body the reduction of constrain number of sorry number of reduction of number of degrees of freedom due to number of constrain is not straightforward, we cannot just calculate it by this relation.

So, let us say we need three n number of degrees of freedom and N C 2 number of constrain. So, we can simply say that the degree of freedom of a rigid body is 3N minus N C 2 which is 3N minus N into N minus 2 by 2, but it is wrong this is not right because there are there are logical reasons for it if you go through I mean the master book of classical mechanics that is Goldstein book, you will have an idea why this relation is wrong we are not going into this. So, today’s class we are stopping here in the next class what we will do is, we will first define the degrees of freedom of a rigid body and we will continue from there.

Thank you.