Classical Mechanics: From Newtonian to Lagrangian Formulation
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Lecture - 27
Rigid body dynamics – 1

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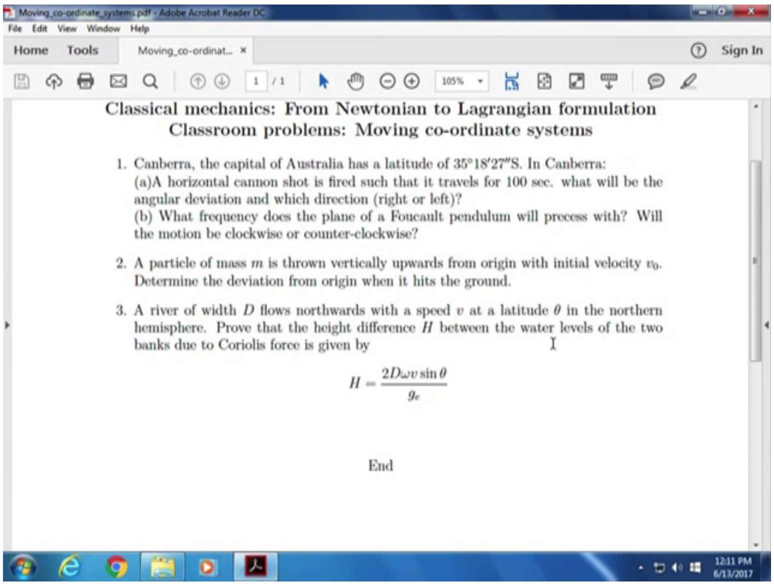

So, we look into the problem set and the last problem of this particular chapter is pending this problem says the river of width d flows northwards with a speed V at latitude theta in the northern hemisphere prove that the height difference h between the water levels of 2 bank due to coriolis force is given by this particular expression now recall that now notice that there is a ge here. So, instead of; o, this ge will come and this ge is nothing, but the corrected g value due to the centrifugal acceleration.

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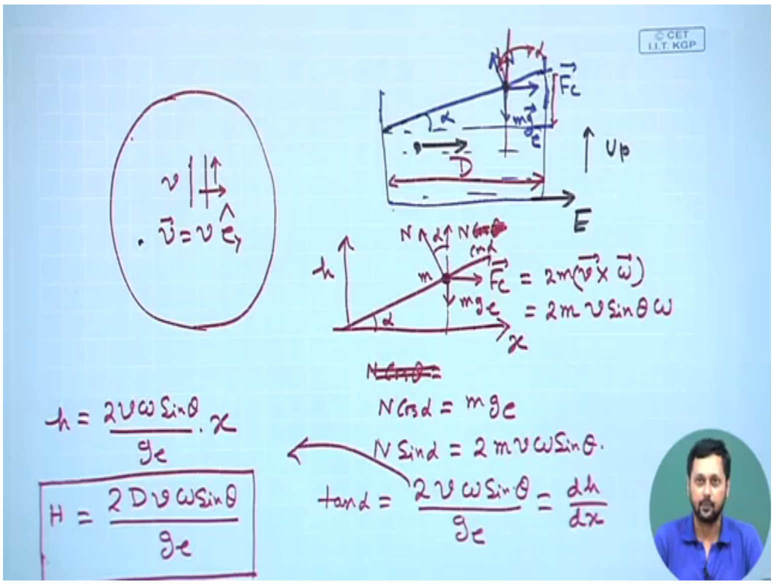

Now, le uts look into this situation this is our earth surface a river somewhere in this in this earth surface is flowing due north.

So, the flow of the river is in this direction with of some velocity V. Now as the water moves due to coriolis force, there is a acceleration of force on every water particle which is moving towards the eastern direction. Now due to this, if I now draw the cross section of the river how does it look; let us say this is my riverbed. Now if this is the water level we have if there is no coriolis force, due to this would have been the water level. Now due to coriolis force, the particles are experiencing a force which is towards east. So, this is my right. So, this will be my eastern direction this will be the up direction and it is moving due north. So, we are just drawing the cross section now because of that the water accumulation in the east bank of this river will be more compared to the west bank if this happens the river surface will be will have some finite tilt although small, but there will be a tilt here, right.

Let us call this angle alpha right now any particle on this surface say it will experience 3 forces there will be one force which will be due to Fc, the coriolis force that will be towards east there is a normal reaction which will be perpendicular to this surface given by N and there is a mg which is working downwards now this mg and Fc, they are perpendicular to each other now this g has to be replaced by ge as it is there; as it is given in the problem now this ge and Fc; strictly speaking, they will not be exactly perpendicular, but this effect is so small that we can neglect that now if I come back to this picture and try to show you that. So, this is not sorry. So, if I draw a perpendicular with respect to this axis this angle here once again will be alpha. So, if I draw it once again this part only, this is alpha and this angle is alpha also alpha.

So, this is N this is m ge and this is Fc right; now Fc is given by for this particle of mass m Fc is given by 2 m v cross omega now omega has 2 components one is along east and sorry one is along north and one is along one is the in the in the upward direction

Now, the component which is along north will not come into the picture because if you recall the velocity v essentially is towards the northern direction and northern direction is always given by y. So, v can be written as v ey cap. So, the only component which will survive is the z component which will be v sin theta times omega.

So, this is the magnitude of the coriolis force for normal reaction there will be a component which is given by N cos theta which will be countering this mg. So, we get N cos theta or cos sorry; not theta sorry, sorry, this will be alpha cos alpha. So, we have N cos alpha which will be m ge and N sin alpha will be exactly opposing this coriolis force and that is why this particle the force balance of this particle will be established. So, N sin alpha will be 2 m v omega sin theta right. So, from this what we can do is we can write an expression for tan alpha which is sin alpha by cos alpha it will be given by 2 v omega sin theta by g e, right.

Now, this tan alpha now if I; so, what we need to do is we need to we need to find the height of this v we have to find this height now look at this surface, they look at the surface the equation of the surface the slope at any point of the surface is given by this. Now if we consider this to be my; let us say x axis and this to be my h axis. So, this tan alpha; so, if this is my h axis we can also called it third axis this tan alpha is the slope at any point which will be given by dh dx, right. So, we can we get an expression just by calculating the force components we can calculate tan alpha which is the instantaneous slope of this surface which will remain constant of course,, but that is not strictly important which is given by dh dx. Now this relation if we just integrate this relation we get h is equal to it is a linear relations in theta by ge multiplied by x.

So, for the maximum height we have to put x equal to d which is the width of the river. So, h will simply be 2 d v omega sin theta by g e. So, this is the final expression which was which we were looking for. So, what we did here is we just calculated the; it is a very standard techniques in mechanics, I think you are all familiar with it we calculated the force components all force components of a mass point which is at the on this surface of the river and from there we have calculated the slope and just by integrating this we get this expression. So, with this problem we officially close the discussion of this moving coordinate systems the next topic which is also a very important topic. So, we will start discussing system of particles, but our main aim is discussion of rigid dynamics.

Now, before going into the direct details of rigid dynamics let us define a system of particle and let us try to look into some properties of the system of particle. Now, what is a system of particle?

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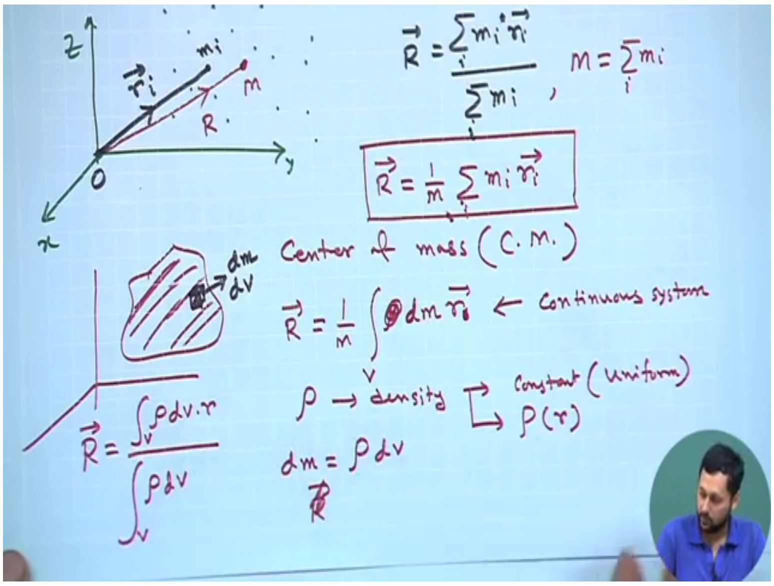

Let us say, we have a arbitrary origin; this origin is arbitrary. So, we just call it some x y z and we have point masses in the system. So, what is important that we know for any point mass the mass of which is m i; this is the ith point mass we should know the position vector ri with respect to this arbitrary origin O right. So, ith point mass has a mass of mi and the radius vector of ri in this is the case, then we can immediately define the radius vector or.

We can define the centre of mass which is given by r which is by definition r is equal to mi ri sum divided by sum over i mi. Now sum over i; the sum in the denominator I will just use this pen this pen is not good. Now total mass of the system m is given by sum over i mi. So, we can modify the definition of r it as one over m sum over i mi ri. So, this is the definition of centre of mass. So, in this system of particle we can define a centre of mass with a coordinate capital R and we can assume as if the total mass of the system is centred at this centre of mass; so, with a mass capital M.

So, this entire system of particle for its dynamical properties to study the dynamical properties of this system can be represented by a single point mass of mass capital m which is the total mass of the system and a position vector r which is given by this particular definition. So, this point is called centre of mass or simply CM right. So, c CM is defined by this particular relation. So, what we are going to do is now if we look into this equation closely we can write the replay for continuous system we can you know substitute this summation with an integration and we can write this as one over r integration over the entire volume rho or rather d m r i.

Where dm is the element of mass or dm r rather not ri there is no I anymore. So, this is the integration I mean the same relation written in a integration format and this is valid for a continuous system; what is the essential difference between a discretized system and a continuous system for a discretized system, the mass points each of the individual mass points can be identified and for a continuous system a mass point cannot be identified, but what can be identified is a density term rho; rho is the density of this system rho could be a constant.

So, it could be a constant or rho could be a function of r. So, you can have a velocity da; sorry, density which is changing with its width position or you can have a system with where the density is uniform or uniform or constant or uniform actually uniform is a more accurate term for this. So, you can have an uniform density system or a density which is a function of the position now if you introduce this density.

Then a mass element dm can be written as rho times a volume element v is the. So, what is it instead of the system of particle we have a system of continuous masses where a small point mass has a mass of dm a density of rho which is uniform across this system and the volume of dv. So, we can replace this dm in this equation and we can redefine r as not; I will just write it here maybe. So, we can redefine r to be equal to integration rho dv times r and again m is nothing, but rho dv integration rho dv and the integration is over the entire volume of the system . So, this is an alternative definition of centre of mass which is equally effective.

So, we are going to look into some of the examples.

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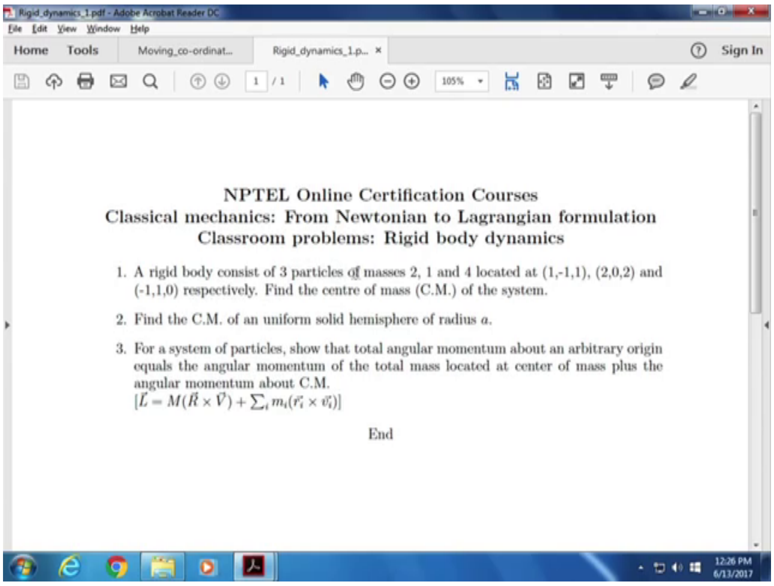

So, the first example will be rigid dynamics. So, we will going here. So, the first example is a rigid body consists of 3 particles of mass 2 one and 4 located at these 3 positions find the centre of mass of the system. So, what is given we have some coordinate.

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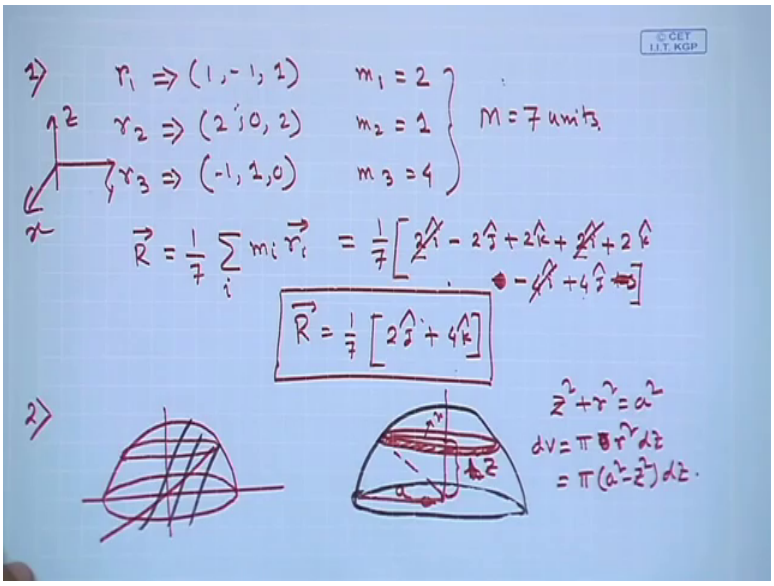

So, we know that r 1 is equal to 1 minus 1; 1 r 2 is equal to 2 0 2 r 3 is equal to minus 1 1 0 and we have mass m 1 which is given by 2 units m 2 given by one units and m 3 given by 4 units. So, total mass is 4 plus 2 plus 1 which is capital m is 7 units right which is a summation of these 3 .

So, r which is the centre of mass will be the summation 1 by 7; 7 being the total mass summation of i mi ri, right. So, if we execute this we will k . So, I have m one times r 1 which is. So, we have to write in terms of ijk. So, it will be 2 i cap minus 2 j cap plus 2 k cap for the second mass, I just have to mass equal to 1. So, I have to add 2 i cap plus 0 j cap, I am not writing it plus 2 k cap for the third mass it will be. So, this is a minus sign. So, this is minus 4 i capped plus 4 j capped and k cap equal to 0. So, I am not writing it. So, it will be 1 over 7. Now look into i cap, we have 2 plus 2; 4 minus 4; this will cancel out for j cap it will be minus 2 plus 4.

So; that means, 2 j cap and for k capped it will be 4 k capped. So, r is this; the location. So, if we know this if we if I plot this on my x, y and z system this points I can also find the centre of mass of this system now this is for problem one for problem 2 this is also related to centre of mass find the centre of mass of an uniform solid hemisphere of radius a uniform that is a very important word hemisphere means half of a sphere right. So, the radius is given as a what we can do is we can take a slice to solve this problem; what we can do is we can take a slice at some height h; sorry it is not a very good drawing I will just draw it once again . So, let us say this is your hemisphere.

So, what I can do is I can take a slice at a certain height this is my origin O and I can take at a height h or some arbitrary height z from the origin and for this particular slice we can assume a thickness of dz. Now the relation between this z and a being the radius going by the triangular triangle rule its z square plus r square is equal to a square right. So, if set square plus r square is equal to a if z is sorry if dz is the thickness then the volume dv is pi what is r square r square being the this radius is r. So, that is the see total radius of this hemisphere at the bottom is a and that is up here is r.

So, r square plus z square is equal to s square now dv is equal to pi sorry pi r square dz which is pi s square minus z square dz, right.

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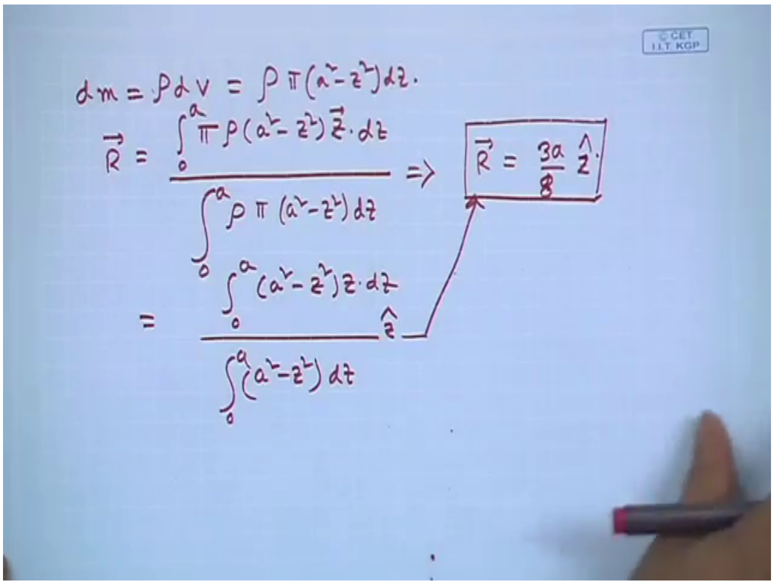

Now, mass will be given by the mass element will be it given by dm which is rho dv which is rho pi a square minus z square dz right. So, by definition of centre of mass r will be given by ratio of 2 integrals the first integral is 0 to a pi rho now please recall this definition. So, there is an r here. So, now, what is the vector we are using here we are using the vector instead of R, we can put z in this particular case because we have taken this as a variable z as the variable.

So, what we can do is we can simply put pi a square minus z square times z dz that is for the integration up and for the lower integration we can simply integrate dv rho dv which is pi a square minus z square dz. Now if you perform this integrals you will finally, get r is located at a position 3 a by 3 a by 8 z cap. So, I am just leaving it to you to perform this 2 simple integration because in the very next step pi and rho will cancel because rho is given to be uniform. So, is a constant? So, the integration will reduce to 0 to a; a being the range of maximum range of z. So, it will be a square minus z square times z dz divided by integration 0 to a; a square minus z square dz.

So, if you do this finally, you will get r equal to. So, z; please remember this z vector, we can simply decompose this and we can write a z cap outside. So, with this; this will be the final form. So, according to this result your centre of mass will be located on this line at a position which is given by 3 a times divided by 8; so, which is very close to the origin. So, the centre of mass is somewhere here, right .

So, now we have defined as we have defined centre of mass let us look into the some very crucial aspects of motion which is which we can get from this which the insight which we can gain from the centre of mass picture the first of all if we try to define. So, use of new edge.

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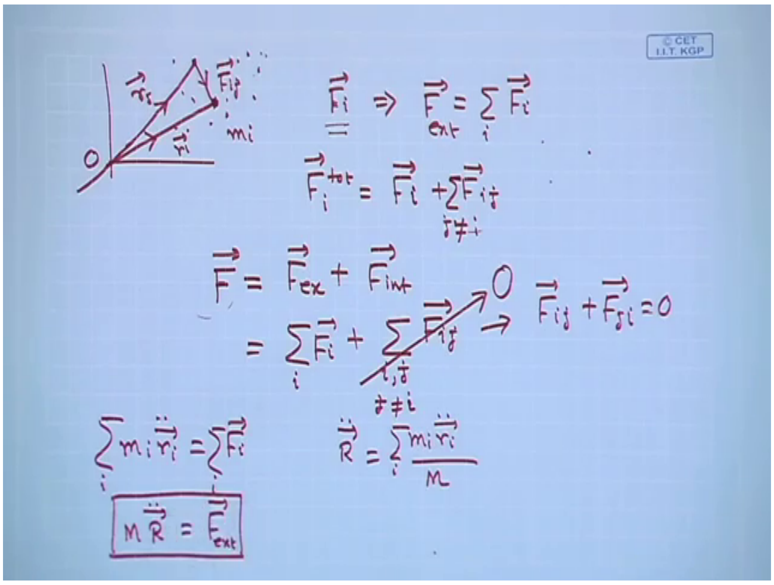

So, we will see if on a system of particle which is once again given as particles with some arbitrary origin O r i is the position vector of the ith particle of mass mi now if there is a force Fi acting on the ith particle then the total force F which is or F external.

So, please reme remember this Fi is an external force which is being acted on ith particle. So, total external force is a vector sum of all this forces acting on all other particles right. So, force on this particle that particle that particle that particle everything will be included in this vector. So, now if this is done, then we can oh and please recall that this is only the external force there is always an internal force. So, that total force F total on ith particle is a combination of this external I will I will just write it yeah. So, it is fine.

So, F i plus Fi j sum over j which is not equal to I what does it mean the second term. see this particle is experiencing an external force which is given by this value , but also let us say this is the jth particle which has a position vector of r j, they are interacting with each other along this particular line right and this force. So, the force of interaction is given by F ij right and if it is not only between this ith particle ith particle and this particle this jth particle could be any one of the system of particle except on itself. So, the same particle cannot insert an external I mean internal force on itself. So, j is not equal to i.

But any other particle; so, for example, this one or this one or this one or this one we will have an interaction with this particle, right. So, the total force is a summation of the external force plus all the internal forces due to all other particle except the self, right. So, if I execute if I try to find out the total force on this system which is given by F that will have 2 parts one is F external. So, I am talking about system of particle. Now please understand that there is no indices here plus F internal now this F external as we have seen can be replaced by sum over i F i and total internal force will be sum over i j. So, it will be a if this is the force only on ith particle for all other particle it has to be summed over i and with this condition that j is not equal to i it will be F ij right .

Now, what we need to understand that F i j; this particular sum will vanish why because this will come this will give rise to terms like F i j plus F j i right. So, F i j and F j i according to Newton’s third law; they are equivalent opposite in nature. So, the combination of this will vanish and this term if we execute the summation if we open up the summation and write term by term we can always form into such pair. So, as a whole this whole thing will not contribute right. So, only this external total external force that is that is that is I mean acting on this system of particle we will survive and at the end for the entire system of particles if we try to write the force equation for individual particle what is the force situation it is mi ri double dot is equal to F i. So, if I put summation on both sides now please recall the definition of moment of inertia what we can do is we can no sorry no sorry centre of mass moment of inertia.

We are not come to that yet r is equal to mi ri sum of over I divided by one over m so; that means, r double dot is equal to this. So, if we replace that into this equation. So, total. So, this can be substituted by m r double dot is equal to Fi sum over i Fi now the total force F is again sum over i Fi. So, this is simply f. So, we see as I as I said already for the motion of the system of particle under a net externally applied force F or we can write it as F external itself. So, external force F is a motion as if the entire mass is concentrated at some centre of mass with mass capital M and the position vector r which is given by this relation. So, that is how we can apply Newton’s law; Newton’s second law for a system of particle, right. So, we will continue from here in the next class for now.

Thank you.