Classical Mechanics: From Newtonian to Lagrangian Formulation
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Lecture - 24
Moving Co-ordinate Systems – 2

Yesterday’s class was about moving coordinate system and we continue on the same line.

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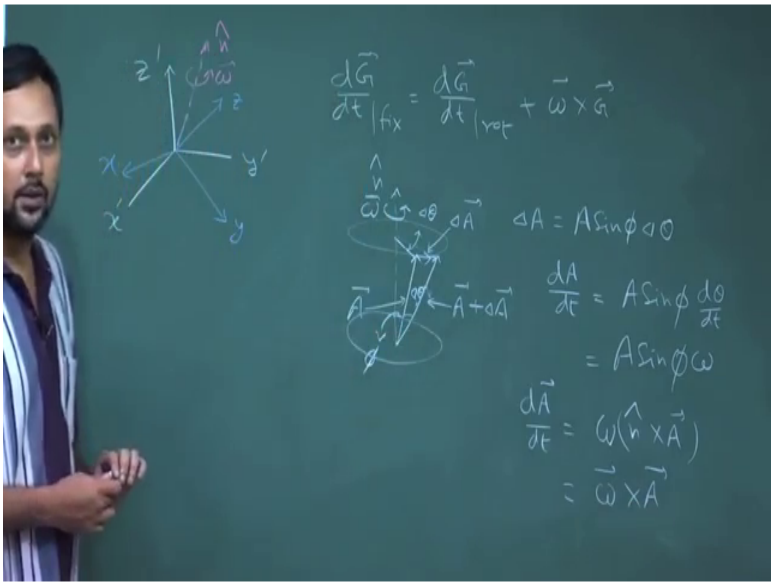

So, what we did in the last class is we considered a situation where there are 2 coordinate spins one is stationary which we leveled as x prime y prime and z prime and there is another coordinate system which is moving we call that say called this x, y and z and the movement is taking place around end axis which is given by n capped and the velocity of movement instantaneous velocity rotational velocity omega.

So, in this situation we have shown that time derivative with respect to this fixed frame and this moving frame of any arbitrary vector G is related by dG dt fix is equal to dG dt rot plus omega cross G. Now we prove that assuming that by one particular property of a rotating vector if this is fixed direction around which a particular vector is rotating. So, this is one position of the vector. So, so essentially the vector whichever is rotating around this particular axis has to stay on this cone on this surface; I will destroyed little bigger for you guys, right.

So, let us say this is the direction n capped and this is one position of the vector and this is another position of the vector. So, let us call this vector A here and A plus delta A here as the direction is changing. So, this vector A is not fixed anymore. So, the deviation from this here to here is given by delta A right. Now what we did was we took the angular deviation theta and we have shown that this delta A can be written as the magnitude of delta A can be written as a delta theta and the vector direction is on this way. So, that is if we take the cross product of omega and A sorry; omega is in this direction omega is along the direction of n capped. So, if we take the cross product of omega and a in the limiting case the vector dA will have the same direction.

We can also have an alternative proof for it if you simply join this and let us call this angle delta theta instead of this angle it will be sorry, this will be delta theta. So, it is the same thing essentially because this and this angles they will be there has to be equal it is just the 2 d projection of this particular case. So, we can and if this angle is phi then this length we can very easily show that this nth is A sin phi and. So, dA will be A sin phi times delta theta.

So, it will be dA will be equal to A sin phi delta theta right. So, dA dt will be A sin phi d theta dt which is nothing, but omega because omega is the rate of change of theta or yeah rate of change of this angle this particular angle or this particular angle whichever you want to look at it and once again if you want to calculate the want to look explore the direction of this changing vector then you can write this as. So, this can be written as A sin phi omega because d theta d d theta dt is nothing, but omega and in vector term dA dt is nothing, but A sin phi sorry not A sin phi, but a omega n capped cross A; this will give you A sin phi n capped has a magnitude 1.

So, n capped cross A will give you A sin phi and this n capped times omega is nothing, but omega. So, this will be omega cross A. So, either we can directly take this angle and d theta and compute delta equal to a and delta A and show that dA dt is equal to omega cross A or what we can do is we can take the 2 d projection of this particular problem in this plane or a plane down here as you wish that is totally up to you and we can again proof dA dt is equal to omega cross A.

Now, the important result is dA dt is now using this important result we can finally, conclude that this is the case for any vector G which is moving or which is represented either in the fixed coordinate system or in the rotating coordinate system now as this relation is true for any vector G what we could do is we replaced G with r.

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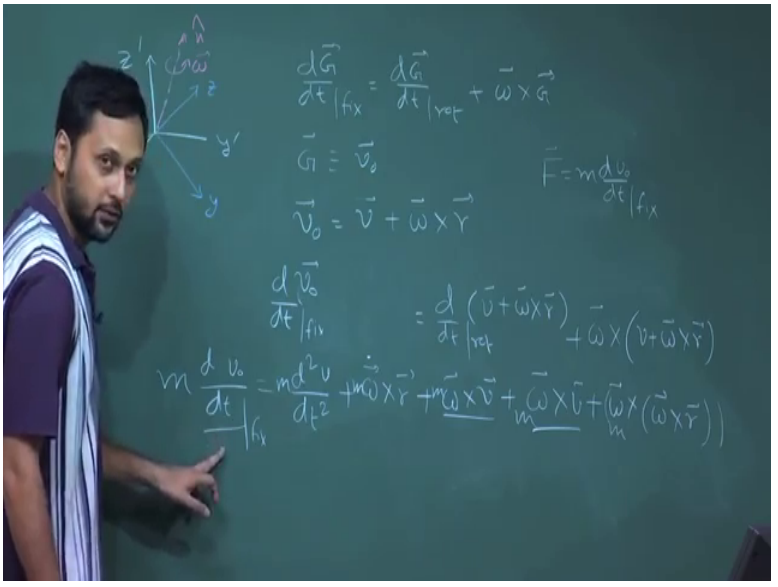

And we showed that its V 0 is equal to V plus omega cross r right.

Now, what is omega cross what is V 0? V 0 is the velocity of a moving body which is measured with respect to this fixed frame and V is the velocity which is measured with respect to this moving coordinate system rotating coordinate system. So, we can now instead of taking G equal to equivalent to r we just take it as V 0 and using this form of V 0 we can write ddt of V plus omega cross r equal to ddt dot V plus omega cross r plus omega cross V plus cross r . So, this we can write and simplifying see actually this side we do not have to do it sorry we can just keep this as V 0.

So, now this will be nothing, but d 2 V 0 dt 2 right and on the right hand side we have. So, this V without a suffix; so, this is the velocity as which is measured in the rotating coordinate system. So, if we take this time derivative once again it will be simply d 2 V dt 2 and next term we will have a omega dot cross r plus omega cross r dot plus omega cross V plus omega plus omega cross r, right.

Now, omega cross r dot; please remember this r dot is once again the velocity which is measured in the rotating coordinate system. So, r dot is the time derivative which is taken in rotating coordinate system that is the velocity V in rotating coordinate system. So, we have this term twice. Now this term if I if now if I multiply this whole thing sorry this is not d 2 V dt 2 is simply dV dt sorry. So, if I multiply this whole thing with mass of this particle which is invariant because mass is the quantity which is invariant under any coordinate transformation. So, well in the non relativistic world in the relativistic world we have a different story altogether we are not going into that.

So, if the mass is constant. So, if we multiply this both side of this equation with mass what we get is a mass term everywhere here and here right. Now look at this term m dV dt and this is measured with respect to the fixed coordinate system. Now Newton's law is valid in a inertial frame which is in this case the inertial frame is the fixed coordinate system the other coordinate system is rotating. So, it is not an inertial frame now according to Newton's law f is equal to m dV dt.

Now, in this case in the current notation we are using it will be dV 0 dt fix. So, this term on the left hand side gives you the net force which is acting on the on the particle of mass m. So, that is the true force because this particular term its I mean once again this is the acceleration term multiplied by mass in the rotating coordinate system which has the same dimension of this one, but this is the actual force because this is measured in a inertial frame now if we rearrange this equation from here if we rearrange this equation replacing this by f and keeping this into one side and rest in the other side.

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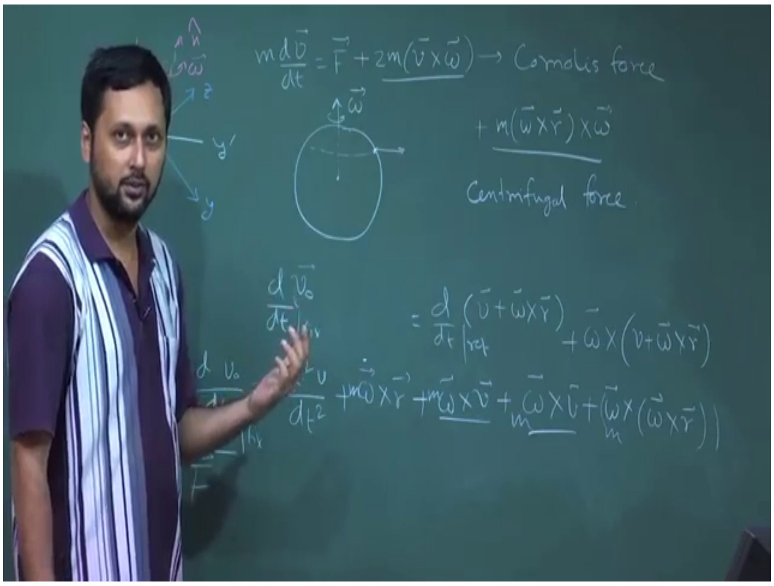

So, we can write this as m dV dt is equal to f plus now it will be twice m into omega cross V. Now when we change sides the cross product will also change side. So, it will be twice m V cross omega right; this will also change side. So, it will be m r cross omega dot right plus the last term will be m omega cross omega, right.

So, this is the actual force and all these three additional terms are the fictitious or pseudo forces which is a result of this rotating coordinate system now please understand this; this force is measured in the fixed frame in the inertial frame and the pseudo forces, they do not exist in the inertial frame because right. Now we have written in this equation in a particular form where we measured the velocity itself in the moving coordinate system all this additional fictitious force terms they appear.

Now, consider earth as an example of moving coordinate system as we discussed in yesterdays class also what happens we have this is our earth let us say let us consider it to be a sphere this is the direction of omega omega being the velocity of rotation for earth and. So, it is rotating in this particular manner in a this particular direction now omega is constant. So, at least for earth this term has no meaning because this is mr cross omega dot if omega is a constant then this term will vanish, but these 2 terms will survive. So, especially when we are writing this equation for earth we can get rid of these 2 term this particular term, but these 2 terms will still survive.

Now, this term is called Coriolis force I think you are familiar with this and this coriolis force causes the monsoon this coriolis force causes cyclone to rotate in a particular direction this coriolis force causes river to bend in one seven the one of the one side of the river bend riverbank to go up and other side to go down and also for this coriolis force horizontal cannot short deviates from its line particle falling object deviates from its path all this happens for this coriolis force. So, will discuss that briefly and this one is the familiar centrifugal force which acts on any object on this planet.

Now, because of this particular force; so, I will go into it in rather in details what happens because of this force this force is actually if you if we draw this you know the components of omega and r carefully let us say on any object at this particular position this force will have a direction in this way also the magnitude will be very small because it is of the order of omega square omega itself is a small number will see very soon this will have a magnitude which is small, but please understand this although small it is non I mean non trivial there is a effect of it.

So, the acceleration due to gravity which is G which is the gravitational pull of earth is along this particular line because of this one presence of this force the acceleration will not be along this particular line, but in a line which is slightly deviated from it please understand the magnitude of this force is very small. So, if we draw it at scale it will be almost nonexistent, but because of centrifugal force our true you know direction of true gravitation acceleration is not exactly towards the center of earth, but slightly deviated from it.

So, we will take it up later on, but right now let us focus on coriolis force on earth. So, we will first have to define a coordinate system on our planet and then we will take it up in a more systematic manner right.

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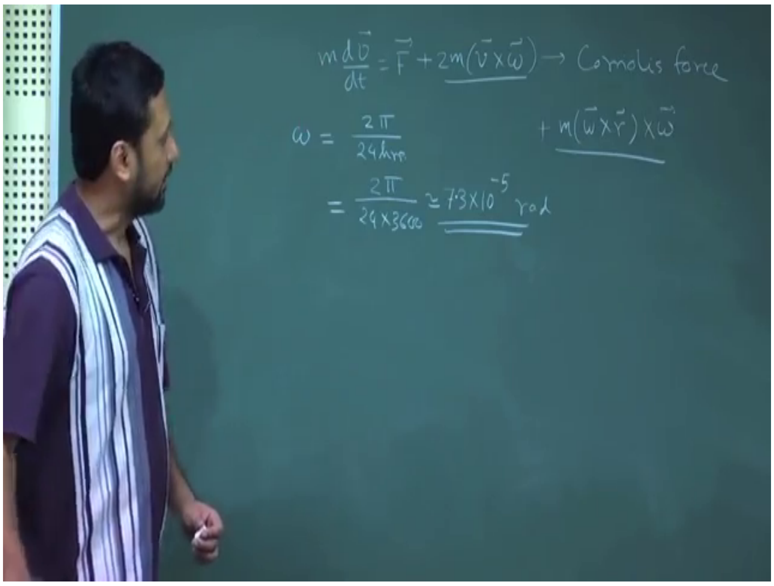

First of all what is the magnitude of omega; omega is the earth speed of rotation and this has a magnitude of 2 pi by. So, the complete rotation takes place in 1 day. So, the magnitude is 2 pi by 1 day that is 24 hours which will be have the numbers somewhere here, right. So, it will be 2 pi by 24 into 3600 and if you calculate this it will be 7.3 into 10 to the power minus 5 radiance right; 7.3 greater than minus 5 radiance. So, please keep this number in mind will be using it a lot, right.

Now let us first start defining a coordinate system and then will be discussing coordinate force this.

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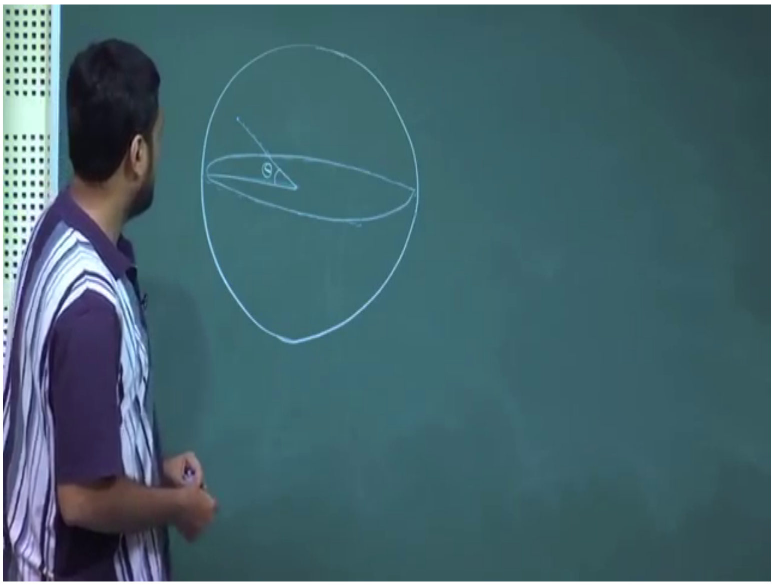

Let us go to this particular point on earth which is at some latitude this is better some latitude theta. So, this is the equatorial plane of earth and we are at this particular point p which is at a latitude theta cannot surface now what happens is ok.

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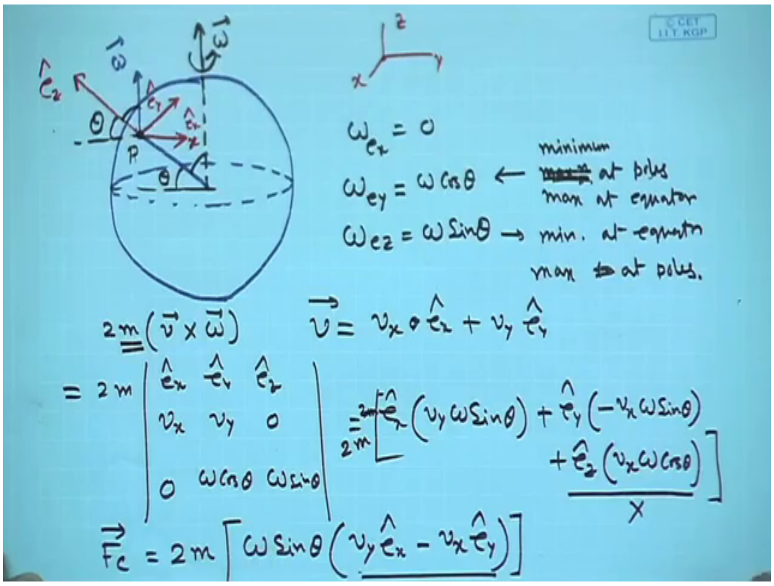

I think I will just switch to pen and paper. So, this is our planet, again my drawing is not never very good excuse me for that. So, this is the equator plane and this is a point of our interest. Now where is omega; this is the center. So, the axis of rotation is this. So, this is again the direction of omega which is rotation in this particular direction.

Now, let us try to define our coordinate systems which is along the same line which joins this point to the 4 z. So, if I extend this line above the ground please understand this blue part is inside the ground and this red part is above the ground. So, this is my e z direction let us call this coordinate system x, y and z now it is a right handed coordinate system, Now how do we draw a right handed coordinate system typically, this is my x, this is my y, this is my z. Now try to place this point here and try to align y towards the north pole if you do that this is your y you will see that your x will be somewhere in this direction, it does not look very the angles does not look very right, but this angles are all ninety degree. So, this is my x or rather I will call it e x here and this is my ey. So, this is my ey direction this is my ex direction.

Now, try to place this omega vector at this origin right. So, this is once again your omega vector and now where is it yeah now this angle is your latitude which is given by theta. So, this angle once again is theta. So, this angle will be 90 minus theta right. So, if you recompose this omega vector or the angular velocity vector into this new coordinate system the components of that will be omega along e x equal to 0 omega along ey is equal to c it will be ninety minus theta. So, this angle will be theta. So, it will be omega cos theta and omega ez equal to omega sin theta.

Now, if you look into I mean if you are studying books by for example, many other textbooks, but you will see that this component has been resolved though components of omega has been resolved with respect to the col attitude which is this angle. So, instead of having omega cos theta and omega sin theta they have omega sin theta and omega cos theta here. So, which is also true please of you have to make sure you are you you understand which angle it is been resort around whether it is the latitude or the co latitude I prefer latitude because that gives you a straightforward physical picture.

So, we have omega cos theta and omega sin theta and if you studied this components carefully see theta is the latitude. So, if you go to the poles theta will be equal to ninety degree or either plus ninety degree or minus ninety degree in both cases omega ez will be maximum. So, we have maximum at poles and if you come to oh sorry, sorry, sorry, sorry, wrong it will be minimum at poll minimum at polls and maximum just give me a second yeah that is true minimum at poll and maximum at equator; equator is the this plane in the center 0 zero degree where theta is equal to zero degree right and this one this is minimum at equator and max at poles, right.

Now, the coriolis force is given by 2 m root over V sorry 2 ms times V cross omega no root over sorry now if we consider an object which is moving in the in the horizontal plane of this on this coordinate system. So, this point p is not a fixed point we can move it anywhere on the earth surface in the northern hemisphere southern hemisphere. So, we have we essentially we have the same type of component result we can resolve the same components for omega. Now also please note that omega ez, this is the component which depends on sin theta. So, as theta b is positive in the northern hemisphere and negative in the southern hemisphere omega ez changes its sign in different hemisphere whereas, omega ey; the y component remains invariant does not matter whether you go to northern hemisphere or southern hemisphere omega is at remains the same.

So, now, if we consider the velocity V to be resolved into 2 components vx ex cap plus vy ey cap then we can compute this to be equal to 2 m ex cap ey cap ez cap vx V y 0 and this will be 0 omega cos theta omega sin theta which will be equal to forget about 2 m it is always there. So, it is 2 m here. So, ex cap will be vy omega sin theta minus 0 plus ey cap there is a twice m here outside. So, just keep that in mind ey will be this minus 0 minus vx omega sin theta plus ez cap will be equal to ok.

So, this it will be vx omega cos theta. So, we see that there will be components not only in the xy plane, but also in the z direction. So, that z direction component in z direction eventually means there will be a vertical you know shift of this particle of mass m which moves with this particular velocity, but for now we just neglect this we just ignore this term and we try to see what happens what deviation the particle suffers in the xy plane. So, so the coriolis force Fc if we write it once again we can write this as omega sin theta vy ex cap minus vx y cap.

Now, look at this look at this particular vector over here and so, this is a vector here. So, we have this.

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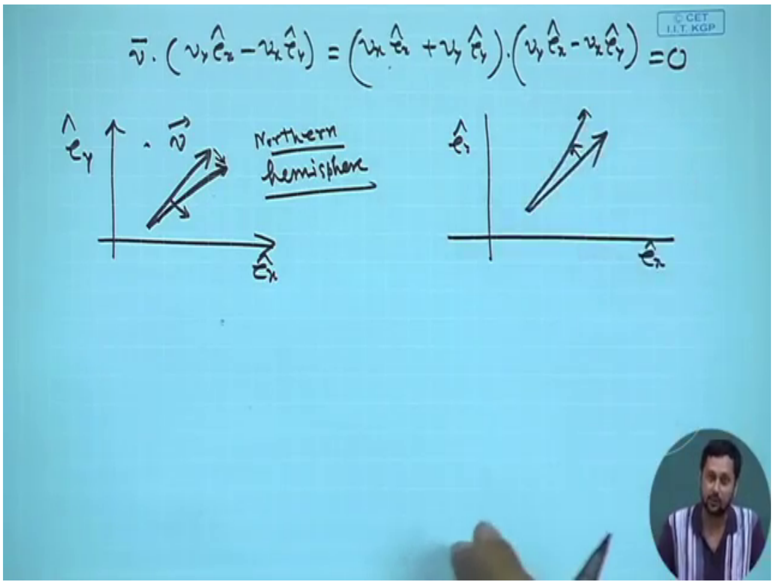

Now, this vector if you take the dot product of velocity with this vectors ex cap minus vx ey cap which will be V has to be replaced by vx ex cap plus vy ey cap right, dot vy ex cap minus vx ey cap you immediately see that this will give you x 0, right. So, you see if in the xy plane I am just drawing a trajectory of the particle in x and y plane if this is the initial direction of velocity V we get a direction of the force f we get a particular direction of the force f which is by the by this particular dot product we immediately understand thus that the deviation is always in a perpendicular direction.

So, if we fire a horizontal cannon shot with an initial velocity V in the xy plane it got deviated by certain amount in the xy plane now try to now try to see in terms of this particular picture let us assume that the canon shot is fired along ey. So, initially it has only y component if only V y component survives then in the coriolis force you see there is always only a vx component. So, if we fired the cannon shot along the northern direction because y represents direction towards north. So, if we fired the cannon shot towards the northern y direction. So, we will it will get a deviation in the x direction.

Similarly, if we fire a cannon shot only in the V x direction which is the east direction please understand that this is north and this is east this particular direction is east right. So, if we fire it along the e x direction. So, vy is equal to 0 then we see the deviation is along ey direction minus ey direction minus ey is south so; that means, if we fire it along this direction the deviation is down south here. So, that is in the northern hemisphere now if we come in the south southern hemisphere what happens is when we change theta 2 minus theta this particular term changes direction.

So, in the southern hemisphere what happens is if we fire a horizontal. So, in this is this is the case of northern hemisphere. So, we get a deviation towards the towards right hand direction in the northern hemisphere now what happens in southern hemisphere in southern hemisphere because of this particular term it will change its sign. So, if we start with the vx. So, we will get a minus vy, but there is a negative sign coming from this one also. So, effectively we will get a deviation in the y direction.

So, in southern hemisphere if you start along the east direction you get a deviation in the north section you if you start along the y direction or sorry north direction then you get a deviation in the southern direction. So, if you are firing in a general direction this in the southern hemisphere then your deviation will be in the left hand side. So, essentially you can think of it this way your velocity vectors if you are standing on the northern hemisphere your velocity vector and your omega; omega is towards I mean general direction of omega is upwards.

So, your V cross omega will be a vector which will be going to the right of your velocity vector whereas, if you are standing somewhere in the southern hemisphere omega is actually going inside the ground here. So, your V cross omega will be a vector which will take you which will which will direct towards the left of your original velocity direction. So, we will continue in from here in the next class right, now let us stop.

Thank you.