Classical Mechanics: From Newtonian to Lagrangian Formulation
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Lecture – 23
Moving Co-ordinate Systems – 1

The topic of discussion in today's class is moving coordinate system.

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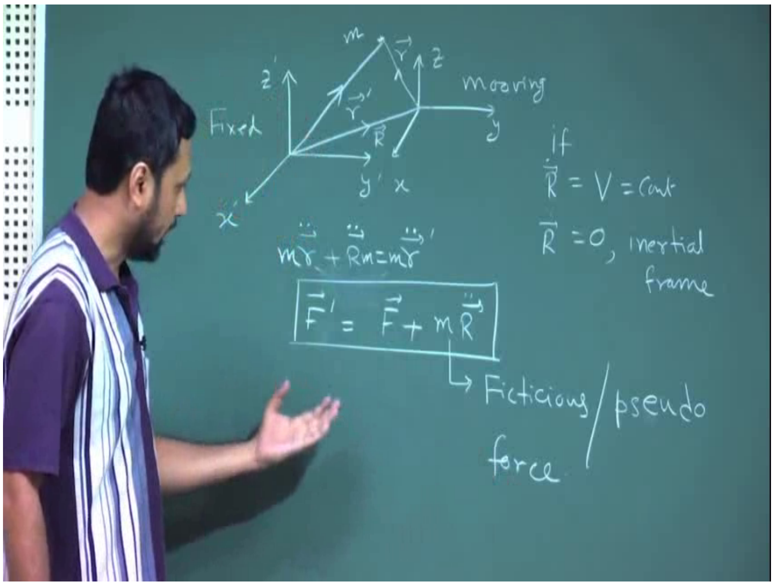

Now, during the review of elementary mechanic, mechanics principle Newtonian mechanics principle, you probably recall this picture we have a frame which is fixed and which is given by x prime, y prime, z prime. And there is another frame which is given by this set of axis x, y, z. Now a particle which is moving in space the position vector measured with respect to this frame is r prime and with respect to this frame is r.

Now if the location of this origin of this particular frame is capital R with respect to this frame. Then we can write r plus r is equal to r dashed. This is just for vector addition. Now if this is the case then r double prime, r double dot is equal to capital double dot equal to capital sorry small r prime double dot. Now if we multiply this with the mass of this object m in both sides. So, this is the acceleration which is measured with respect to this particular frame. So, let say this frame is moving with respect to this frame this is fixed and this is moving. So, we see that m r double r m, r r prime double dot is the force which is measured with respect to this primed frame inserted on this particular object.

Now, if we similarly under on. So, this one gives this; this one will give a force which is measured the force which is measured from this particular frame, inside being work working on this particular object. And of course, there is this additional term mr double dot. So, we see that now forget about the arrows. Now this primed frame or this particular frame let us consider this to be fixed and this to be. And we have discussed that if this is moving with an uniform velocity; that means, if r dot is equal to v equal to some constant, then we immediately see r double dot is equal to 0 and these 2 forces measured forces are equal.

So; that means, if this moving frame is moving with an uniform velocity, then it is called a inertial frame, but if the velocity is not constant; that means, the acceleration there is a non 0 accelerating term; that means, this equation in this equation all the term survives and the force inserted on the object which is measured from I mean depending on whether we measure the force on this object, with respect to a fixed frame or respect to a moving frame the amount of force changed. And this additional term is called a fictitious force, fictitious or pseudo force it is not a real force it is coming up only because this at this frame is moving with some non uniform velocity. So, this fictitious or pseudo forces are what makes the life makes life really difficult, when we are sitting and when we are trying to measure something with respect to a moving coordinate system.

Now in this chapter, we will be discussing a special type of moving coordinate system or. So, which are actually rotating coordinate system? So, the situation is as follows; I will just keep it there.

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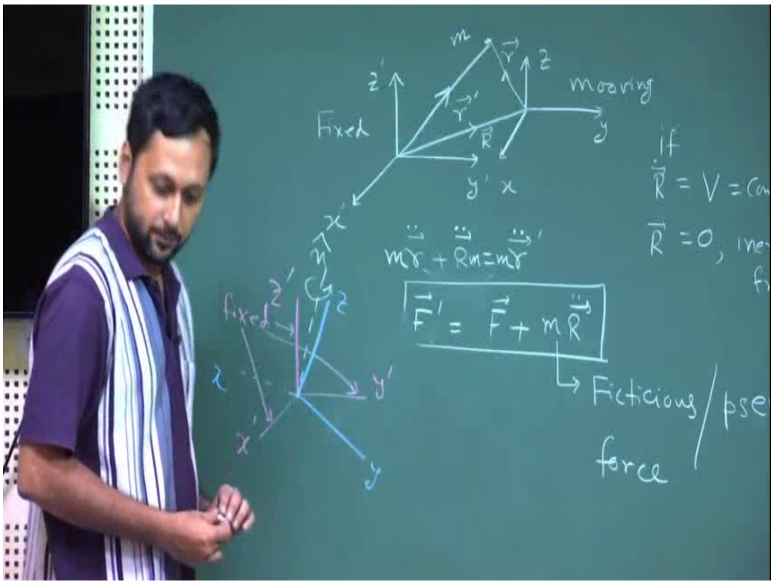

So, let us say this is our fixed frame of reference, and this is our moving frame of reference, which share the origin common. So, the origin is common. So, if this is my x prime y prime z prime, then this is my x, y and z. So, and see here what we have discussed is a general motion it could be a translation motion, it could be a rotation motion, but here the situation described is a pure rotational motion the origin is common.

And let us assume that there is some axis along which this one frame is executing a rotational motion with respect to the fixed frame. So, this partly frame; this one is fixed. So, this 3 sets of axis; they are fixed in space and the other 3 sets of axis; they are rotating in space and the rotation rotating with this the origin remains common point between these 2 frame. And there is an is axis of your continuous rotation, let us call it n cap and the rotation is taking place around this particular axis. So, this is the situation we are going to describe in today’s class. So, so this is an eque; this is the case which is case of rotation.

Now let us assume a situation where the rotation is with an uniform velocity. So, let us assume that the velocity is the rotational velocity which is described here is omega which is some kind of a constant.

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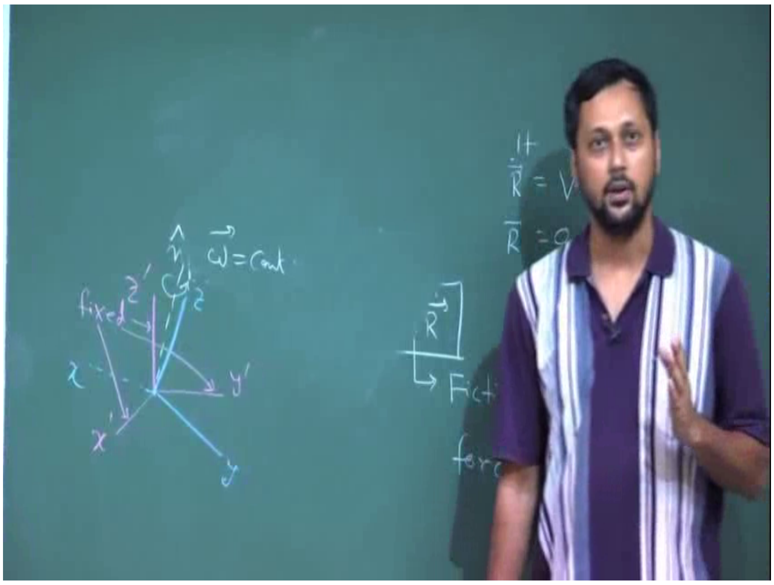

So, let us see; let us assume omega equal to constant into I mean omega vector is constant just a minute. So, this is my direction and the rotational velocity omega which is the velocity of rotation of this rotation is a constant. So, omega vector constant means not only the direction the direction of the vector is constant, but also it is magnitude is a constant of time is this equation I mean. So, my question is does this uniform rotation still in I mean, gives rise to a fictitious force the answer is yes. Because any rotation even if it is a uniform rotation it will definitely give rise to a fictitious force for the very simple reason.

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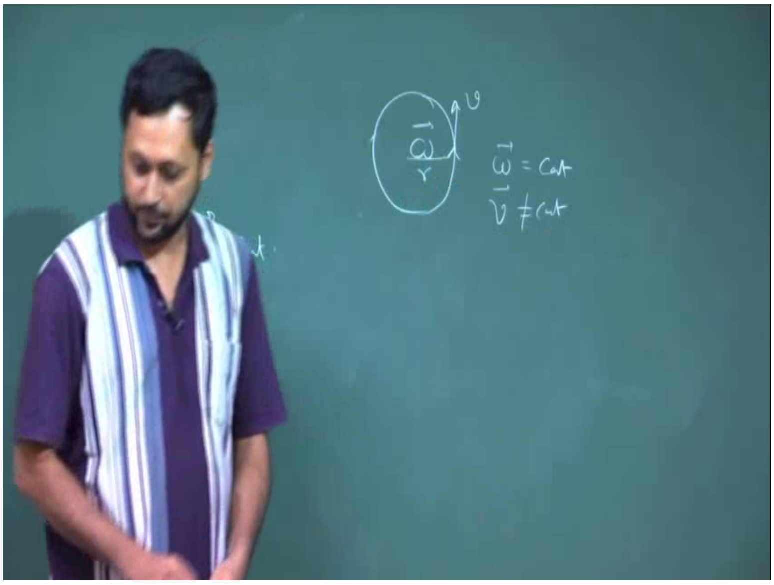

Let us assume a point moving on this circle with an uniform velocity omega. So, the direction of velocity; so, probably we all know this, if the circle; I mean if the point is moving on this particular frame in clockwise direction then the direction of velocity is axis of velo; axis of this angular velocity omega is out of the plane. And if it is going in a clockwise direction, then the direction is inside the plane. So, if I hope we are familiar with this notation. Now if omega is constant in this case think of it the velocity the corresponding linear velocity if the radius vector is r, then they are related I mean the speed v and this omega is related by v equal to r omega. Fine, but the direction of v is changing continuously on this circular loop right isn't it. So, although the magnitude of velocity remains fixed, the direction is constantly changing. So, we can say even if for omega equal to constant v is not a constant, right.

So, that is why, even if there is an uniform rotation around the fixed axis with a uniform angular speed, does not mean that it is you know inertial frame inertial system. A velocity a rotational velocity will always give rise to a rotational motion will always give rise to a linear velocity which is not a constant with time. So; that means, there is always the pseudo force acting on this.

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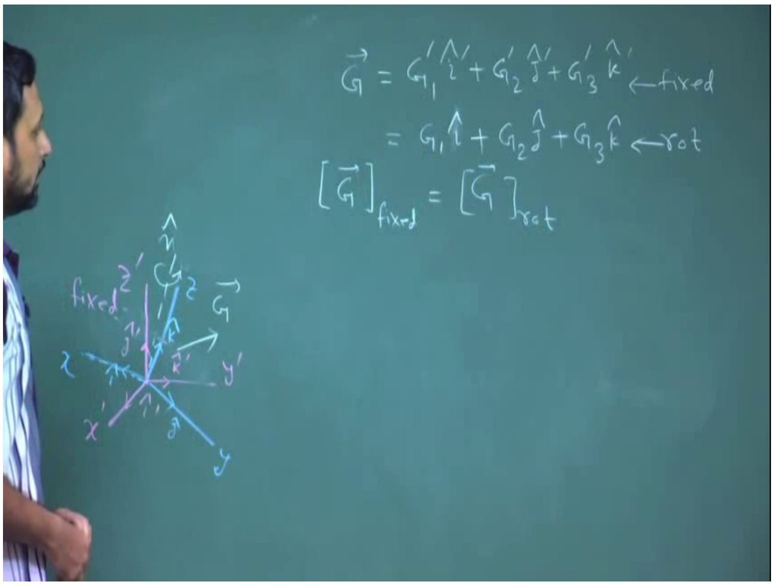

So, our job is to find out the super force. Now let us for a second forget about for now let us forget about this constant omega it is not important we will just do a derivation for a general omega let us assume that there is a vector which is ok.

So, let us assume that there is a vector, which is given by represented by G. Let us say this is the vector some vector. So, so it need not necessarily be starting from the origin it could be anywhere. So, this is a general vector. Now this vector will have components in both the frames. So, G will have we can resolve the components of G in the fixed frame that is the primed frame or we can resolve the components of G in the rotating frame.

Now, once we do that and please remember, whether although the frame is rotating does not matter if we write the vector G. The vector G remains invariant whether we represent it in this frame or the other frame. So, we can write G to be equal to G 1 prime i cap plus G 2, prime j cap plus G 3 prime k cap. So, sorry i cap prime j cap prime and k cap prime. So, let us assume that for the fixed frame, we have k. For example, k cap prime i cap prime and j cap prime and for the moving frame let us have i cap j cap and k cap. So, now, onwards whenever we will be referring to a moving coordinate system will be ah sorry whenever we will be discussing a fixed coordinate system. If there is a fixed and a moving coordinate system the fixed coordinate system will be denoted by prime and the moving coordinate system will be denoted by this non primed numbers; non primed indices.

So, we just resolve the components, this is in the fixed frame. And this is in the rotating frame. So, the equation what we are what I have written here is nothing, but equivalent of writing G fixed which is the vector G in fixed frame equal to G rot; the vector G in rotating coordinate system. Now if we want to take the time derivative of this vector, how do I; how are you going to do that?

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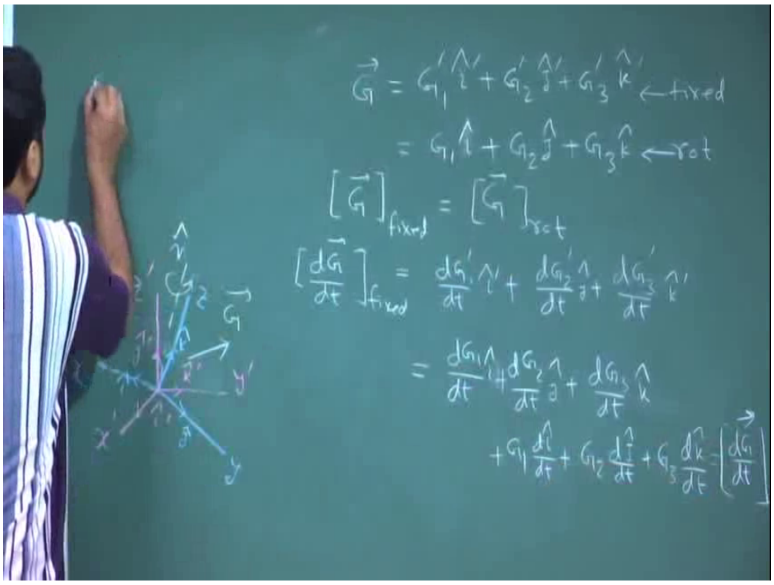

So, of course, we can just take time derivative of both sides of this. So, which will be given by dg dt for left hand side. We will have digitally fixed which will be given by time derivative of this one.

Now, please now note, i prime j prime k prime. These are unit vectors in the fixed frame. And because they are fixed in space the time derivative is always 0. So, if we take the time derivative of this one, it will be simply d G 1 prime dt, i cap prime dg 2 prime dt 3 cap prime plus d G 3 prime dt k cap prime right. And on this side, now this will be equal to dg dt dropped the because if we take the time derivative in rotating frame it will also with the same time derivative, right. So, we can write this as i cap plus dg 2 dt j cap plus dg 3 dt k cap plus 3 additional terms. Because i j and k they are also changing direction with time. So, we have G 1 di cap dt plus G 2 d j cap dt plus d 3 G 3 d k cap right and this is equal to dg dt rot.

So, one side we have dg dt fixed one side, we have dg dt rot which are which has to be equal because irrespective of which frame, we represent the vector the time the vector itself and also their time derivatives has to be equal. So, this 2 has to be equal, right.

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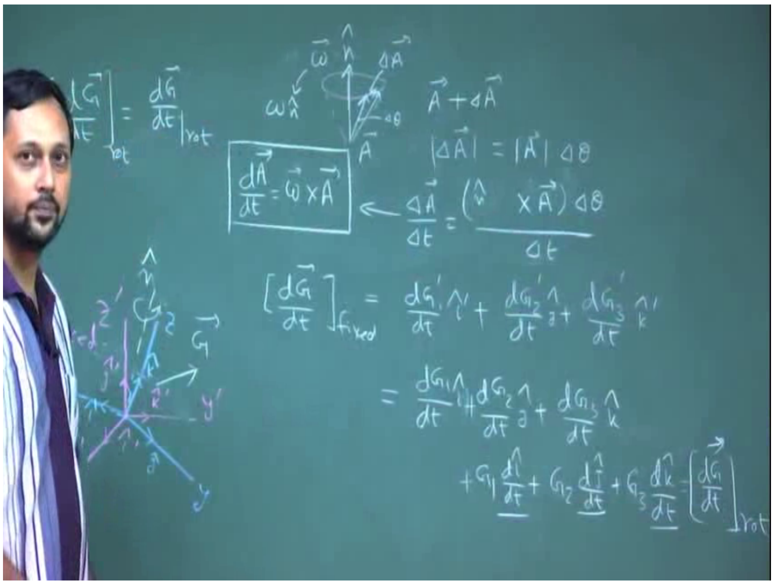

Now, if we examine this the first term is if we now write dg dt rot separately, we see that dg dt rot is equal to the first 3 terms is nothing, but dg dt with respect to rotating frame right. Look at the first 3 terms it is the instantaneous representation of the vector the time derivative of the instantaneous representation of the vector in the rotating frame and for the other 3 terms we have to see.

Now, let us consider this for moment. Let us say we have this some vector a, and this is an direction n there is a vector a let us say this angle is some fixed angle. Let us call it phi and this is some vector a. Now this vector is rotating around this particular fixed direction with maintaining this fixed angle. So, the situation is something like this. So, let us say I have a red pen in hand and I have a green pen in hand so let us say this red pen represents the fixed direction and the green pen represents the rotating vector around this particular direction. And this angle is fixed. So, it is rotating something like this. So, the rotation is something like this. So, the angle is fixed and the direction of this vector is change.

now if the direction of this vector is changing with time. So, we have a here and at a later time it is it moves to a position sorry, this picture is not very clear. I will try to draw better picture right. So, whatever happens this is n and this is a, the tip of the vector a will remain on this cone, right. So, in a later time it will go here. Now this vector is not exactly a vector, but it is a vector which is a plus delta A, delta A is this one right. Now what is delta A delta A if you minute right. Now if this angle is your delta theta. Then we can write delta A the magnitude of delta A will be times the magnitude of a, times delta theta right and what is the direction of this delta A. The direction of delta A in the limiting case; try to realize this is the direction which is perpendicular to both n cap and a right.

Because. So, in the limiting case when a and a plus delta A are almost coinciding on each other is some something similar, to the construction we did for when we were determining the time derivative of the derivative of theta cap and r cap in plane polar coordinate system. So, in the same notation we can on the in the same line of thinking we can write delta A vector is equal to A times delta theta, sorry this will be delta A will be equal to yeah. So, right it will be the magnitude will be a delta theta and the direction will be n cross a cap, right; now n cross a cap a cap being a unit vector along a direction. So, we can absorb this here and we can rewrite it as n cap delta theta cross A, right.

So, change in delta A the magnitude of this will be as I discussed already my magnitude will be A times delta theta and the direction will be perpendicular to both A and delta A. So, we can write delta of a vector is equal to n cap cross A times delta theta, but actually we can write delta theta outside also n cap cross A times delta theta. Now if we try to take the time derivative of da, I mean time derivative of a then we have to divide this thing by delta t on both sides. This whole thing by delta t and we have to take limit when delta t goes to 0. Now if we do that we immediately see your da dt is nothing, but n cross A times theta dot.

So, this theta dot is this angle the angular change of this vector and theta dot is nothing, but omega if omega is the instantaneous angular velocity of rotation. So, what we can do is we can and we can substitute this with omega. And please note that omega vector is nothing, but omega n cap right. So, because n cap defines the direction of omega this is the direction of this rotation vector. And omega which is equal to theta dot is the instantaneous speed of rotation right. So, we can write omega vector as omega cross n cap. So, with this we can simply write da dt equal to omega cross a. So, this is a very important relation.

So, the result says for pure rotational motion around the fixed axis the time derivatives can be represented by omega cross a right. Now we use this result for this 3 time derivative. So, di dt will be omega cross i dj dt will be omega cross j and dk dt will be omega cross k. Now if we employ that here. So, I am removing this part, we do not need that anymore whatever we have to prove we proved. So, you see this first term of dg dt rot is dg dt measured at the with respect to the instantaneous xyz axis and the second term will be G 1 remains G 1 into omega cross i cap plus G 2 into omega cross j cap plus G 3 into omega cross k cap.

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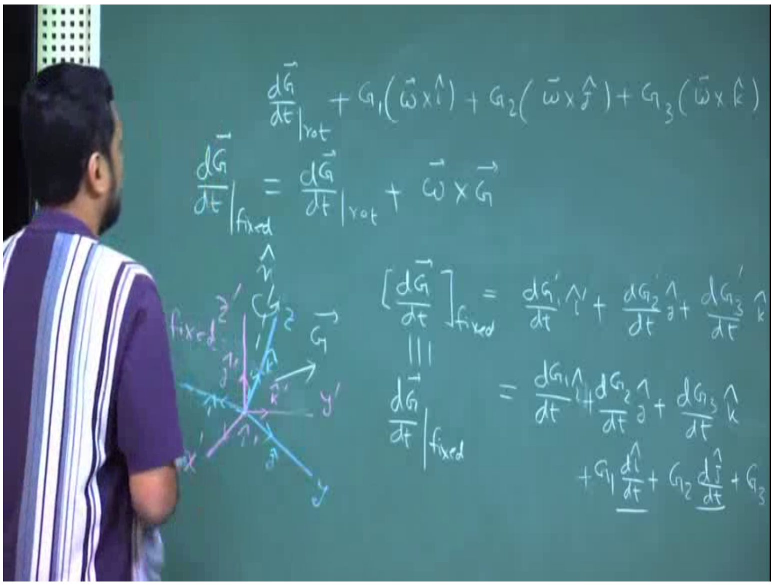

So, if we write it little more we rearrange it, we take omega cross out from each of this and write omega cross G 1 i cap plus G 2 j cap plus G 3 k cap. Then what we get is this dg dt rot plus omega cross right. And now for the left hand side dg dt the time derivative with respect to rotating frame is equal to time derivative with respect to fixed frame. And that is equal to equivalent of writing. So, this one instead of taking with respect to rotating frame we can take it with respect to fixed frame. So, instead of dg dt ro rot we can write dt dg fixed.

Please understand that this is equivalent of writing, dg dt fixed. The entire bracket when we use and entire bracket; that means, that the time derivative is executed with respect to the fixed frame, and when we write a subscript type; that means, the time derivatives measured with respect to the instantaneous position of the value instantaneous position of the fixed frame. Now for fixed frame the instantaneous position and the overall time derivative is the same because it is a fixed frame for rotating frame, it will be there is one part which is the instantaneous time derivative and there is a term which is omega plus G right.

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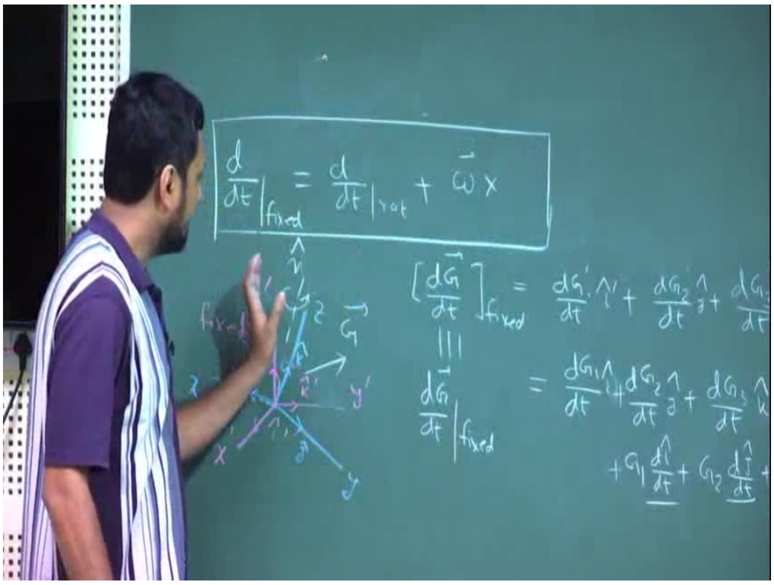

So, we finally, get the very important relation what we are looking for right which is please note that if it is derived for a general vector G we have not assumed any specific properties or any specific characters of G so; that means, it will be true for any arbitrary vector in the 3 dimensional space. Now if we understand this then we can essentially you know we can operate this particular I mean we can have this type of operation for any vector in the 3 dimensional space. So, we can replace this we can get rid of this G and we can create an operator which is ddt fixed equal to ddt rot equal to omega cross plus omega cross.

So, what it means is the time derivative which is measured the time derivative with respect to a fixed frame of reference is equivalent of time derivative measured with respect to a rotating frame of reference, plus omega cross omega being the instantaneous velocity of rotation cross that particular vector. Now if we operate this on the standard position vector r.

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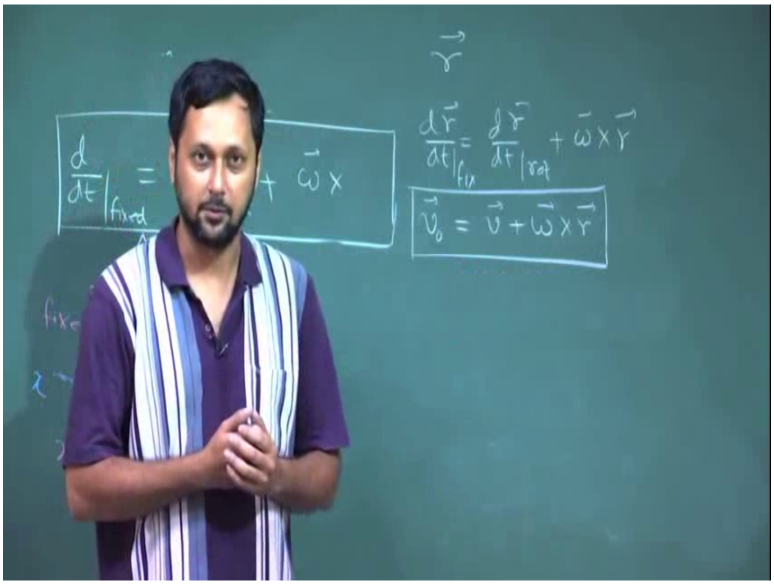

What do we get we get dr dt is equal to dr dt fixed, we are just writing f for it equal to dr dt rot? So, let us plus omega cross a or omega cross r.

So, what is this dr dt fixed means; that means, if we are measuring a velocity. So, dr dt is velocity. So, if the velocity is measured with respect to a fixed frame, let us call it v 0. And so that will be a combination of the velocity the same velocity please understand the vectors G or r or whatever we put in here this vector is frame independent. The vector remains vector is a quantity. So, vector does not matter if we measure it in with respect to this frame or that frame the vector remains represents the same quantity. So, the velocity we see which is measured with respect to a fixed or inertial frame if that is v 0. Then the same velocity measured with a non inertial frame which shares the same origin as with the inertial frame will be v, where v is the velocity measured in that non inertial frame plus omega cross r.

. So, now, you bring it in terms of. So, for example, let us take an example our earth is a rotating coordinate system we take it as fixed, but it is not strictly speaking it is not a fixed frame. So, when we are measuring some velocity on earth. So, in principle as our frame of reference is earth. And now if we measure the measure the velocity from outer space which we go to some spaceship which is fixed in outer space. So, that is the same velocity if we measure from outer space will be v 0 the same the velocity we are measuring from earth is v. So, we will see that this is related by v 0 equal to v plus omega cross r. So, in that case omega will be the angular velocity of earth rotation. So, next class we will start from this point and we will see how this ah how this will lead to any other fictitious forces.

Thank you.