Classical Mechanics: From Newtonian to Lagrangian Formulation
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Lecture – 22
Central forces – 15

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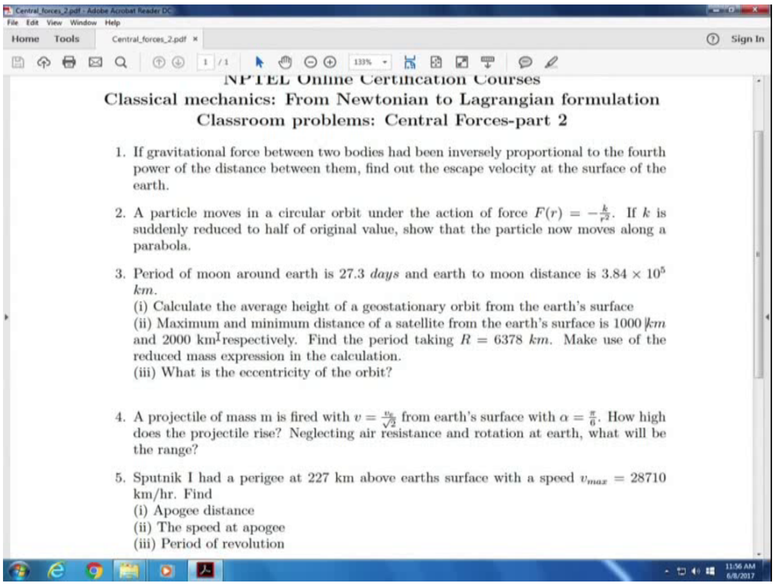
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Classical mechanics: From Newtonian to Lagrangian formulation
Classroom problems: Central Forces-part 2

1. If gravitational force between two bodies had been inversely proportional to the fourth power of the distance between them, find out the escape velocity at the surface of the earth.
2. A particle moves in a circular orbit under the action of force $F(r) = -\frac{k}{r^2}$. If k is suddenly reduced to half of original value, show that the particle now moves along a parabola.
3. Period of moon around earth is 27.3 *days* and earth to moon distance is 3.84×10^5 *km*.
   (i) Calculate the average height of a geostationary orbit from the earth's surface
   (ii) Maximum and minimum distance of a satellite from the earth's surface is 1000 *km* and 2000 km respectively. Find the period taking $R = 6378$ *km*. Make use of the reduced mass expression in the calculation.
   (iii) What is the eccentricity of the orbit?
4. A projectile of mass m is fired with $v = \frac{v_e}{\sqrt{2}}$ from earth's surface with $\alpha = \frac{\pi}{6}$. How high does the projectile rise? Neglecting air resistance and rotation at earth, what will be the range?
5. Sputnik I had a perigee at 227 km above earths surface with a speed $v_{max} = 28710$ km/hr. Find
   (i) Apogee distance
   (ii) The speed at apogee
   (iii) Period of revolution

So, we continue with the problem set the next problem. So, we for problem 3 we just did the first part and we found out that for a geostationary orbit the orbital average height of a geostationary orbit from earth surface is approximately or I should say average height we found out is 6.6 ro.

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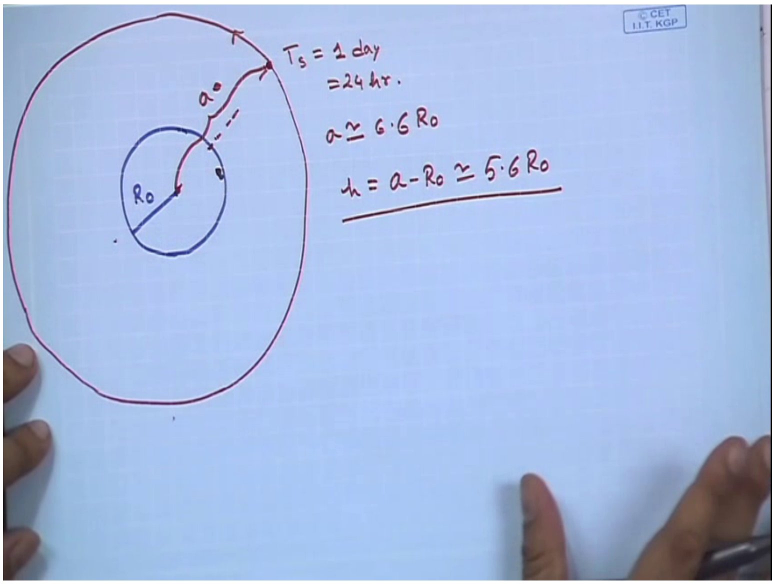

So, the situation is following. So, we have. So, this is our earth with the radius R 0 and a geostationary orbit is an a geostationary satellite is a satellite which stays let us say on this particular point on earth and it rotates round.

So, it rotates along with the earth. So, T s which is the time period of revolution is 1 day that is 24 hours and we have seen that the average radius which is given by this length a which is from the measured from the force center is approximately. So, a we have found out that approximately 6.6 Ro, right. Now if this is 6.6 Ro and this is Ro, then the actual answer to problem question 3 problem 1; what is the average height of a geostationary orbit from earth surface. So, the answer will be h which will be equal to a minus R 0 is approximately 5.6 Ro, right.

So, this is the answer now move to problem 2; problem 2 is for a satellite the maximum and minimum distance of a satellite from earth surface is given find out the period taking. So, I should get yeah. So, I have taken R equal to 3 6 7 8 which is more accurate number, but. So, far we have taken 3 6400 which is also it is not very bad approximation oh; oh, sorry, sorry. So, this second part should not be here it is my mistake this part should not be here. So, please ignore this.

So, we are interested only after this and then we have to find out what is the eccentricity of this orbit. So, question number 2 is if the maximum and minimum distance of a satellite is given and we have to find out the time period. So, let us do that. So, if we come back to this and let us start with this picture once again now for a satellite; so, this was a different picture lets draw fresh diagram start over.

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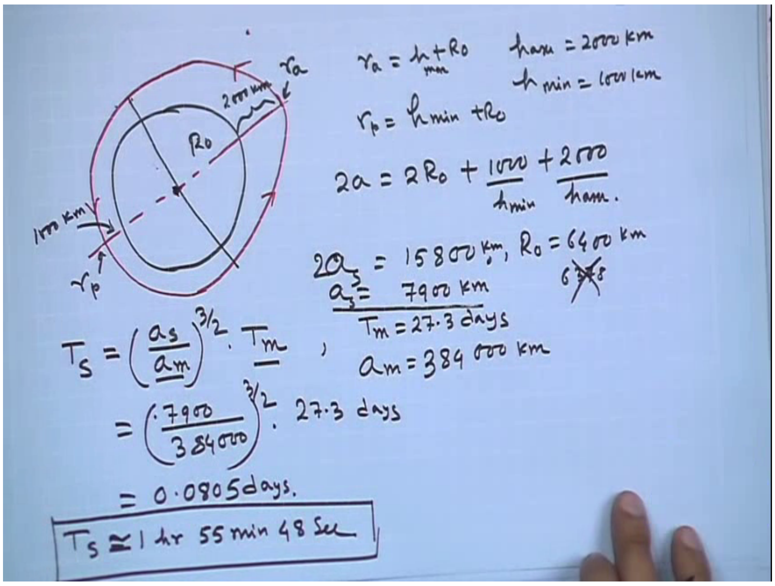

So, once again this is my earth surface there is a satellite and we know that in general the satellite path is an ellipse. So, let us say this is the apex line for that ellipse. So, it goes something like this sorry; my drawings are not very nice I know that try to improve on that, but not working that well. So, this is the point of evolution. So, this is my force center now this is my apogee distance this is my perigee distance right these 2 distances are given. So, 1000 and 2000 kilo meters respectively from. So, we have a radius r 0 and this one is given as 2000 kilometer and this one is given as 1000 kilometers, right.

So, r a is equal to h plus h max plus r 0 r p is equal to h min plus r 0 h max is equal to 2000 kilometer h min is equal to 1000 kilometers. So, once we put all this 2 a which is a distance from this to this will be 2 r 0 plus 1000 plus 2000. So, essential 2 r 0 plus h min plus h max. So, which gives you and a; which is if you go by this exact calculation. So, your a will be 15800, we actually lets use r 0 equal to 6400 only do not use the value of 6378 when you can do that, but let us use this because we are through and through we are using 6400 kilometers, right.

So, once we get these and. So, what we need to find out is a time period of the satellite now once again from Kepler's third law we know that the time period is time period square is equal to the average radius cube. So, average radius in this case is the length of the semi major axis we have to take this is a standard norm there are 2 lengths one is the length of semi major or major axis one is the length of minor axis, but typically in Kepler's law it has been found out that the length of major axis is what we need to take into account semi major axis.

So, T s is equal to as by am whole to the power 3 half times T m right. So, we put as oh; oh. So, what is am and T m am is the average distance from earth to moon and T m is the time; time period of the moon why it is important because moon is also a satellite it is a natural satellite, but it is a satellite of earth which is orbiting earth in a much higher orbit somewhere there I mean at a long distance from earth surface, but according to Kepler's law, it does not matter if it is close by or if it is very far from very far from the earth surface force center if it is a satellite or if it is something which is revolving around a particular force center it has to follow Kepler's law.

So, moon is something that is that can be taken as a reference. So, we know that from this problem that T m is equal to 27.3 days and am is equal to 384000 kilometers. So, we have to put it put these values back here and if we do that we get. So, 7 nine oh sorry this is this is 2 a equal to this. So, a will be equal to 7900 kilometers. So, a is equal to this or as is equal to this right. So, 7900 by 384000 whole to the power 3 by 2 into T m is 27.3 days.

So, from this we get T s equal to 0.0805 days. Now if we want to express this in terms of the numbers or some hour, minute, second, then we can get T s will be equal to; it will be something approximately 1 hour 55 minutes and 48 seconds. So, this is an approximate number, but very pretty accurate. So, this is the time period of evaluation of this particular satellite.

So, this time period of evolution is determined by a direct use of Kepler's laws of planetary motion that is the third law good; now the second or yeah last part of this problem we have to find out the eccentricity of the orbit now for the eccentricity expression please remember that r a.

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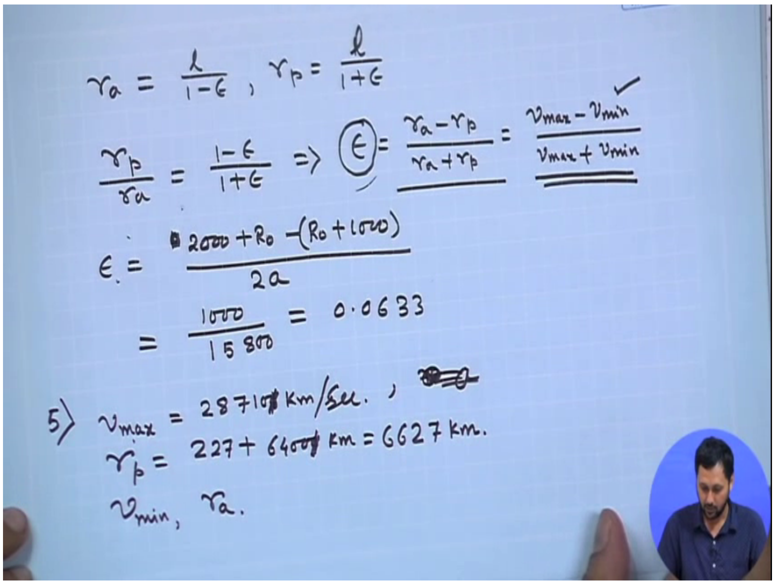

That is the apogee distance is given by L into one minus epsilon r p is given in as one by one sorry, L by one plus epsilon.

So, from this we see that r p by r a is equal to one minus epsilon by one plus epsilon. So, if we do some simplification here is it right, yeah, yeah. So, if we do some do simplification we will see epsilon is given by ra minus rp by ra plus rp and also if you recall ra is the distance which is farthest from the force center and rp is the distance which is closest from the force center, right. So, if this is the case we if we if you recall we have a velocity V min in the farthest point and a velocity or speed V max in the closest point and if we can also write eccentricity is equal to V max minus V min by V max plus V min we did that in the class.

So, right now will be using this equation; so, epsilon will be equal to. So, ra is; what ra is 2000 plus r 0 minus r 0 minus 1000 or rather minus r 0 plus 1000 divided by ra plus rp is nothing, but 2 a which we have already calculated here r 0 r 0 cancels. So, it is 1000 divided by 15800 which is approximately equal to 0.0633.

So, you see that eccentricity is small we. So, r orbit is not very much like an ellipse I would say it is pretty much close to a circular orbit anyway. So, this is done now we move to the next problem or actually in the in the same line we can finish problem number 5 at first and then we can move to problem number 4, right. Now in problem number 5; what is given in 5 its given that the sputnik which is Russian satellite sputnik one which is a first of the Russian satellites in outer space it has a perigee of 227 kilometers above earth's surface with a speed of V max is equal to 28701 kilometers per hour and we have to find the apogee distance the speed at apogee and period of evolution right.

Now, how to approach this problem? So, what is essentially given is if you if we go back to this particular picture let us say. So, let us say this is the orbit, we are discussing the perigee distance is given which is the point of closest approach to earth which is at 227 kilometers and V max is given. Now at first, this might look very you know the problem where proper information is not given because at first you might think that this is not sufficient information to calculate this. So, what we need to calculate is the apogee distance which is ra speed at apogee and period of evolution.

So, what is given is V max is equal to 27100; sorry, 1710 kilometers per second V sorry; and a sorry and r p which is the perigee distance is 227 plus 6400 kilometers which is 7266 kilometers, right. So, these are the only 2 information; we have now how can we use this information to calculate the speed at apogee the apogee distance period of evolution things like that how is it possible is it is possible because if you now come back here and look into it we have this particular expression.

So, if we can somehow get the value of eccentricity epsilon from there we can. So, we already know V max. So, if we know epsilon we can immediately calculate V min and once and once we know V min calculation of r min; so, ra is not very difficult. So, what we need to calculate is V min and ra period of evolution is pretty straightforward afterwards; now how can we do that please recall that. Now let us look at this situation the distance we know this is the perigee distance for this particular case the perigee distance we just calculated is 6627 kilometers and V max is given, right.

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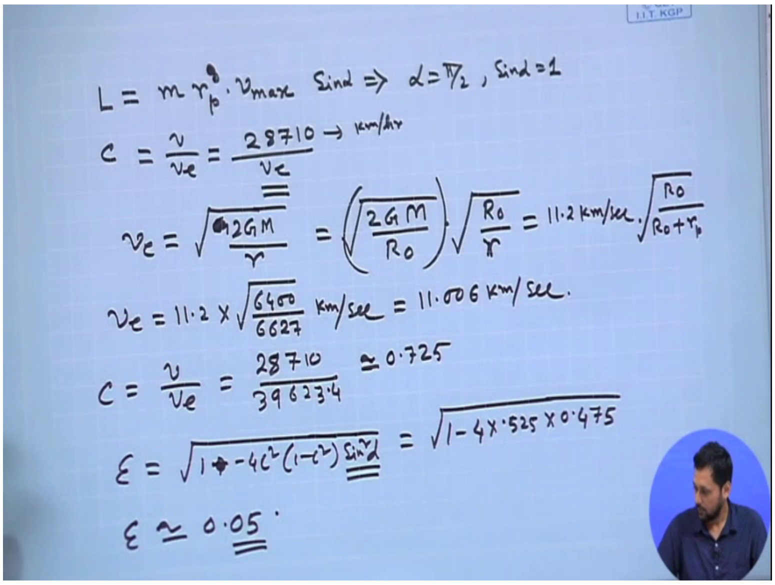

So, we can calculate the L angular momentum from this information because at perigee the angle is ninety degree what is the expression of L at this perigee is m r; m r which is rp into V max, right. So, this is the information we have in hand. Now with if we put values of course, we do not know m which is not terribly important V and of course, there is a sin alpha which will be alpha equal to pi by 2. So, sin alpha will be equal to 1, right.

Now, C is given by V by [vocalized-noise] e which is 28710 divided by the escape velocity. Now this is something please remember, I told you previously that escape velocity; what we know is the escape velocity from earth surface. So, escape velocity from earth surface is something we all know for this particular case the satellite when it is at its perigee position it is 227 kilometers above the earth's surface; so, definitely the escape velocity which is which the value of escape velocity will be changing.

So, first of all they for this problem we need to calculate the value of escape velocity from this height for this particular height now what is the expression for escape velocity the escape velocity expression is root over G; sorry, there is a 2 here 2 G M by r for a distance r. Now what we can do is once again we can put the values of gravitational constant mass of earth, but which is not terribly needed it is not important to know all these values what we can do is we can simply write this in this particular form into root over Ro by r right and this we already know it is 11.2 kilometers per second. Now we just have to multiply this with r 0 divided by r 0 plus r. So, V e is equal to 11.2 into root over r 0 is 6400 divided by it is 6627 kilometer per second.

Now, if we do that we will get a number which is. So, I do not have this calculated, but I can we can do that it is not so difficult. So, it is 6400 divided by 6627 whole square root multiplied by 11.2. So, it will be 11.006 kilometer per second or we can approximately take it take it as 11.

Now, so, we see which is given by V by V e will be 28710 divided by this is kilometer per second. Now if we have to change it to kilometer per hour which is the speed given which is the unit of the speed given for the satellite. So, please remember this V is 27 kilometers per hours; we just have to multiply this 11 point something star 3600. So, it is 39623 approximately 0.4 right.

So, if we do this or from this calculation, it will be 28710. So, it will be approximately 0.7245; we will just call it 0 point proximately 0.725; all these approximations will introduce some error into the calculation, but it is because we are not you know any way we are not doing it very accurately; now epsilon equal to 1 plus or 1 minus 4 C square 1 minus C square sin square alpha, right.

So, again sin alpha will be equal to 1. So, in this particular case it will be one minus 4 C square C is equal to this. So, C square will be 0.525 into 0.525 into 1 minus C square is just this minus 1 into 0.475. So, if we execute this point star 0.525 into 4 and plus 1 and if we take root of this. So, eccentricity epsilon is approximately equal to 0.05 yeah approximately 0.05.

So, you can always think it is you probably you will you can do this calculation by yourself and check the number, but what we what is important that in this way just by I mean knowing the speed at perigee or speed at one of the epicedial points and the I mean the distance of that particular epicedial point, it is sufficient information to get to the eccentricity of the orbit that is all we need and also please remember that in the in this type of calculation we have to recalculate the escape velocity in this manner.

Now, once we do that and now we are in a position to put the values in this particular expression.

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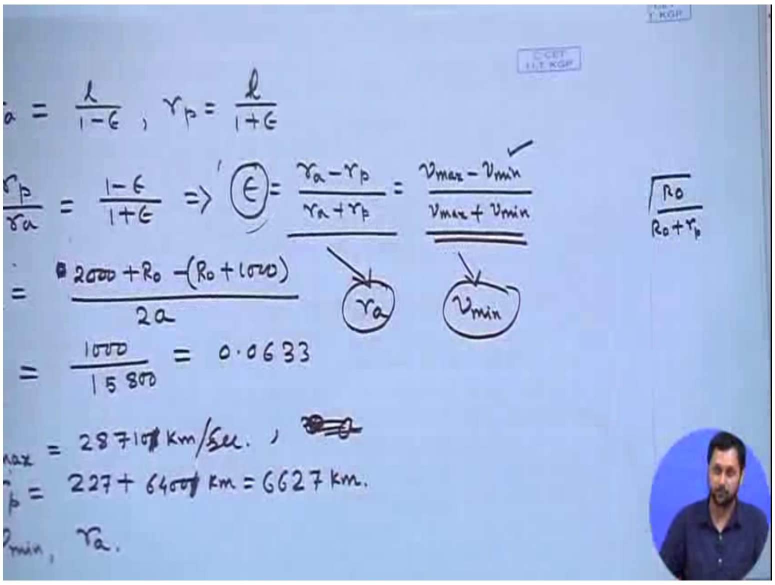

So, once for the apogee in this case, we if we use this particular expression then we can calculate the apogee distance and if we can use this particular expression, then we can calculate the minimum value right or minimum value of the velocity. So, I am not doing it you can finish it yourself that is not a problem, but once you get eccentricity and you have these 2 expressions in hand its really really easy, fine.

And for the third part period of evolution once you have the for this last part once you have the apogee distance and the perigee distance and if you add them up together then you will get 2 twice 2 times a from there you can calculate the length of semi minor axis or semi major axis and immediately that is useful for calculating the time period because we know that we can use this information that moon has a time period of 27.3 days and the earth to moon distance is this, right.

So, I am leaving it on to you to finish this problem the numerical calculation, but it will be easy.

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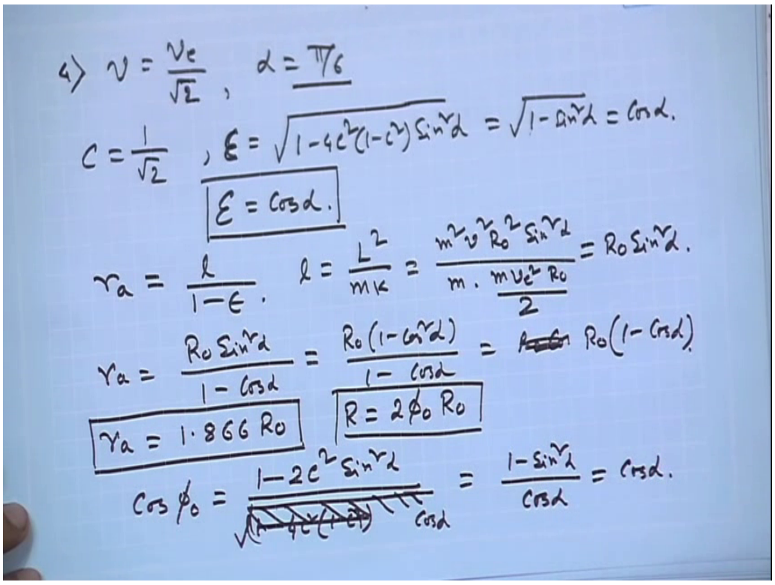

Now, let us come to problem number 4 this is the last problem of this problem set what is given is V is equal to V e by root 2 and alpha is equal to pi by 6, right. So, see it is and we will see that this will lead to a you know ballistic missile case. So, it is it will it we will treat this as a projectile.

So, we see C is equal to one by root 2 and epsilon if we put the value as one 4 C square one minus C square sin square alpha as C is equal to one by root 2 this whole thing will reduce to root over one minus sin square alpha which is nothing, but cos alpha right of course, it will be plus minus, but epsilon cannot be negative. So, we have to say epsilon is equal to cos alpha. So, this is the first part, right.

Now, what we need to calculate is how high does the projectile rise and what will be its range; now there are 2 parts. So, immediately we have already calculated that epsilon the value of epsilon now we have to calculate how high does the projectile rise this is given by the apogee distance which is L by L by 1 minus epsilon. Now L is L square by mk and just like we did during one of our previous classes. So, we have to substitute for L and mk L and k once we do that L will be m square V square r 0 square sin square alpha m and k can be replaced by m V e square r 0 by 2 we put r 0 because here the projectile is fired from the surface of the earth.

So, once we do this cancellations we are left with r 0 sin square alpha. So, ra which is this is r 0 sin square alpha divided by one minus cos alpha and you can please see that r 0 sin square alpha can be written as r 0 1 minus cos square alpha 1 minus cos alpha which will be r 0 sorry r 0 1 minus cos alpha. right.

So, the maximum height which is achievable is r 0 1 minus cos alpha. So, if we put the value of alpha equal to pi by 6 in this expression then it will be something like 1.866 r 0. So, this is the maximum height at which this ballistic missile will rise and for the range we have to find out 5 0 because please remember range is given by range r is given by 2 phi 0 r 0. So, we have to calculate phi 0.

Now, there was this expression for phi 0 which is cos phi 0 is equal to 1 minus 2 C square sin square alpha into root over 1 minus 4 C square one minus C square which is the expression for epsilon. So, we are we should not write this because this is nothing, but root over epsilon. So, what we can do is we can simply write this as. So, the numerator stays like this, but denominator we can just substitute it by cos alpha and again C is equal to 1 over root 2. So, C square is half. So, it will be one minus sin square alpha by cos alpha. Now 1 minus sin square alpha is cos square alpha. So, this is essentially cos alpha.

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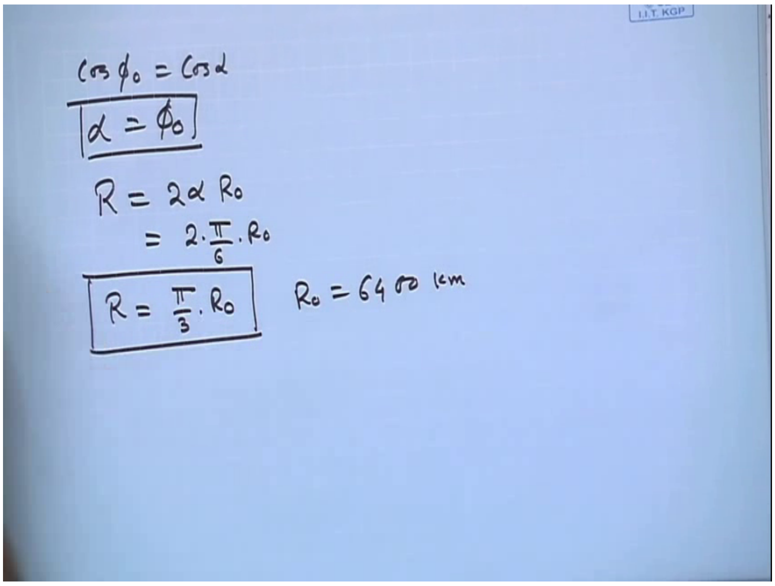

So, you see that cos phi 0 is equal to cos alpha. So, alpha is equal to phi 0 right. So, the range which is 2, we put alpha equal to 5 0 or 5 0 equal to alpha 2 alpha r 0 which is 2 into pi by 6 into r 0. So, range r is equal to pi by 3 times r 0, right. So, this is the expression for range of the projectile; we can put values r 0 equal to 6400 kilometers pi we know and we can get a number for it.

So, we concluded this discussion of central orbit with this problem. So, this was the problem set we discussed in today's class of course, we had one more problem said beforehand which we have already discussed. So, one thing many things we could not cover during this discussion for example, we could not cover scattering we could not cover orbit transfer in details and also I did not we I do not have time to go through the details of stability of the orbit, but anyway these are some advanced topics which probably will find out in your later courses on classical mechanics.

Of course, things related to special and general theory of relativity can also be included in this discussion of central forces which we could not do during this class, but whatever we have learned, I think this is enough for I mean this is good enough for you guys at present and hopefully we will move on I mean hopefully, we will continue our you know good learning experience in the next chapter it will be motion in rotating coordinate systems.

Thank you.