Classical Mechanics: From Newtonian to Lagrangian Formulation

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Lecture – 21
Central forces – 14

Hello friends and welcome back to the class. Now if you will; let us quickly recall the concepts we have learned about orbit transfer and a theory of orbits first of all the eccentricity epsilon.

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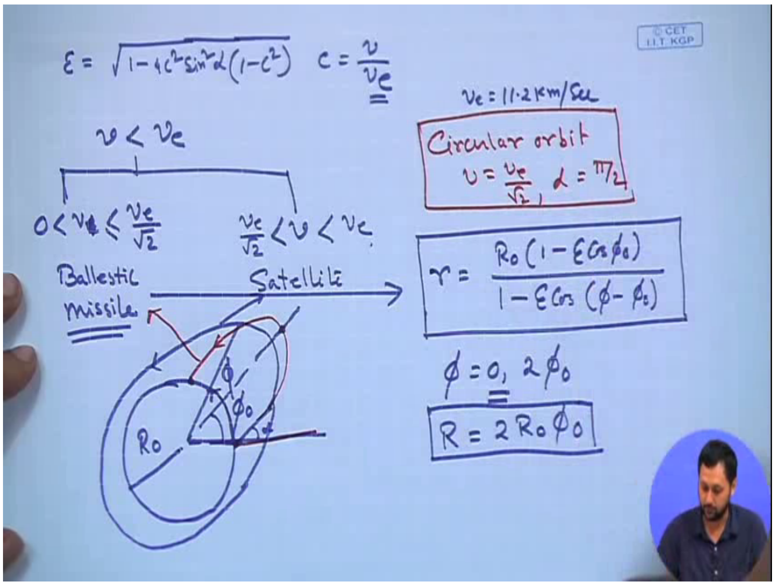

We can write it as 1 minus 4 C square sin square alpha into 1 minus C square where C is equal to v by ve; ve being the escape velocity. Now again this escape velocity right. Now we are discussing a projection from earth surface. So, that is why we have to calculate this, we have to use the ve equal to 11.2 kilometers per second, right. So, this is the escape velocity from the surface of the earth.

But we will be facing situations where the projection is not taking place on the surface of the earth, but the it is taking place at this at some certain height from the earth. So, this ve has to be recalculated that we have to keep in mind and also we have found out the condition for elliptical orbit, we have seen that if the velocity v is less than ve, then we have an elliptical orbit. Now inside that also there are 2 regimes if the velocity is velocity stays between ve sorry v less than ve by root 2 then we have actually it should be ve less than equal to root 2, right.

Then we have a closed orbit which is called a Ballistic missile whereas, for velocities ranging between ve by root 2 and ve we have a satellite which is orbiting around earth right. Now we also found certain condition for this and we have seen that condition for getting a satellite. So, we did that by keeping in mind this particular picture. So, this is our earth with radius R 0 and what we have seen is if this is my launch position, the velocity of the launch is v with an angle alpha with the horizon horizontal direction, then we have seen that it can go like this in a revolving orbit or it can come back like this in a closed orbit.

So, this one is Ballestic missile and this one is satellite right this we have seen. Now also we have found out for Ballestic missile; there are certain parameters which can be calculated we have seen that for Ballestic missile the relation is r for any point on the orbit we have defined this angle as phi 0 that is the angle of the perigee or apogee sorry and any other angle is measured this angle as phi. So, this is measured the angle angular deviation is measured with respect to the position of launch we measured the angle with respect to the respect to a horizontal line which is drawn at the position of launch.

So, using this, we have seen R can be given as R 0 1 minus epsilon cos phi 0 divided by 1 minus epsilon cos phi minus phi 0. So, this is the equation we have got. So, and from this equation, we have seen that it the R equal to 0 for phi equal to either 0 or 2 phi 0 phi equal to 0 is the initial condition. So, the phi equal to 2 phi 0 is the position where the Ballestic missile come comes back to that surface once again and from there we have calculated the range R is equal to 2 R 0 oh sorry; 2 times R 0 phi 0. So, R; this R is the range and this R is the radius of the earth.

So, these R this we have seen and these are all for this particular velocity range when the velocity is below ve. So, now, what happens is when we have velocity equal to or greater than C or greater than ve.

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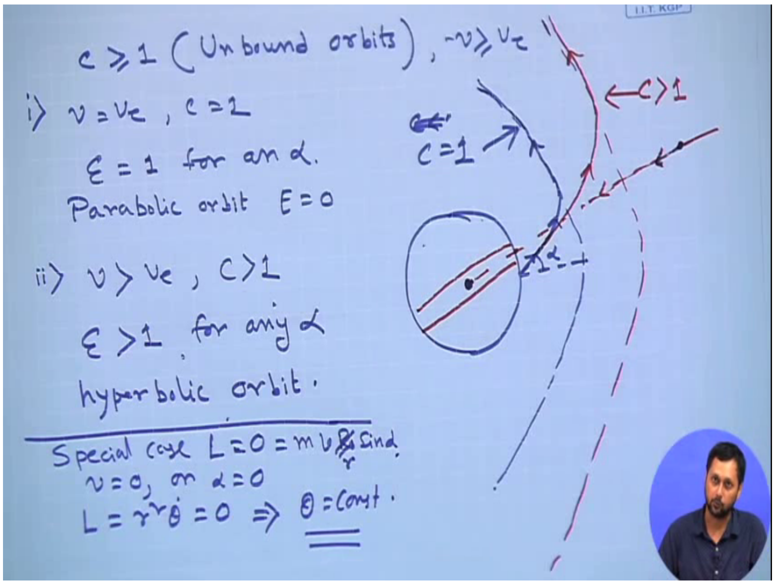

So; that means, we have a situation where C is greater than equal to 1 and of course, we understand that this will lead to unbound orbit. So far, we have discussed only bound orbits and also in this particular case when ve equal to root 2 and alpha equal to pi, we have circular orbit that we have also discussed the condition for circular orbit is achievable when v equal to ve by root 2 and alpha equal to pi by 2, right.

So, this is what we have seen now for unbound orbit which is given by C greater than equal to 1 or v greater than equal to ve. So, there are 2 cases first case is when v is equal to ve, we get C equal to 1 and from the expression of eccentricity if we put C equal to 1 in this we immediately see that the second term in this; this eccentricity expression if we put C equal to 1 this term will vanish and we get epsilon equal to 1 for any alpha. So, then we get a parabolic orbit and also we have discussed that in this case the total energy of the orbit will be equal to 0.

What happens if we have v greater than ve; that means, C greater than one we have epsilon greater than one for any alpha and we have a hyperbolic orbit? So, these 2 cases can be described by this picture once again this is our earth. So, this centre of the earth is the force centre. So, let us assume that this is the launching position it makes an angle alpha with the horizontal. So, for C equal to 1 or v equal to ve, we will have a parabolic path. So, how does this parabola look like please remember we first have to. So, we have to draw a parabola width. So, we can do it like this we can draw a parabola where with one of this force centre a at one of the focus.

So, if we draw a parabola something of like this right. So, essentially this orbit will be something like this yeah. So, after initial launch with an angle alpha it will go and eventually merge with this parabolic line similarly a hyperbola will be I will draw it with red. So, if we have a velocity which is greater than ve with the same launching angle let us say then the projectile the or the object with after launch it will go like this and essentially merge into to this hyperbolic orbit and move away from the force.

So, this is the case where C is greater than 1 and this is the case when C is sorry C is equal to 1 this and this fine. So, now, also we have a special case where when L is equal to 0. Now L which is given by m v R 0 sin alpha we can replace this r with some arbitrary radius arbitrary distance r. So, this can happen if v is equal to 0 or alpha is equal to 0. So, this or both; so, this is the situation which will give us if L is equal to 0 now if you recall L is equal to r square theta dot which will be equal to 0; so, which will give you theta equal to constant.

So, this will give you an orbit on which theta does not change and if we. So, one particular case could be let us say if this is our earth and we drop an object from outer space with 0 initial velocity so; that means, this particular condition v equal to 0 satisfies so; that means, L will be equal to 0. Now if we just release it from outer space. So, it will be attracted towards the force center. So, essentially it will travel along the line with travel along the straight line which connects this point and the force center with a straight line where theta will be equal to constant.

So, this is a case this is a special case with which is a motion with L equal to 0 and from our school; I mean plus 2 standard physics we know that if we allow this object the I mean if we dig a tunnel in our along the surface of the earth or along the length of the earth and if we allow an object to move back and forth, the motion will be a simple harmonic motion that we know. So, let us not go into the details of that.

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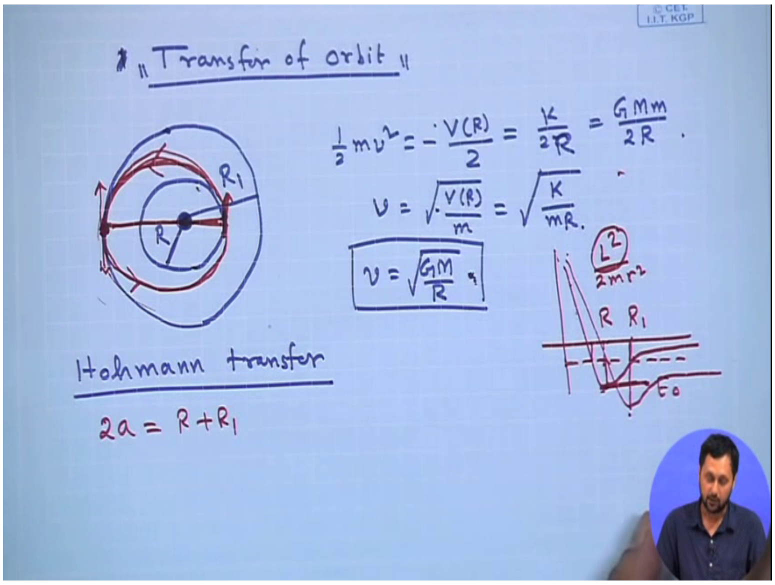

So, the last topic which I want to cover is transfer of orbit here I will not be employing any mathematics because that is not terribly important here first of all we are not going into the deep I mean we are not covering it in depth this particular topic.

And. Secondly, the calculations which can be performed are also sometimes a oversimplification. So, let the best thing is if we just keep everything simple. So, the situation is as follows we have let us say this is our planet our earth and one particular orbit is one particular satellite is moving in a near about circular orbit with a fixed radius R. Now what is the velocity on a circular orbit? If you recall if the this radius is R, then we can get we can write this relation that half m v square which is equal to V minus V of R by 2, I mean this is not very important to understand the nature of orbit transfer, but also I wanted to just discuss this briefly.

This happens because you are in a circular orbit the total energy is given by minus v R by 2 and this will give you a velocity V which is equal to root V R, there is a minus sign by m and if we put values it will be simply root over. So, we can if we take v R to be equal to minus K by r. So, it will be K by 2 R. So, it will be K by mr in this case all what we can oh sorry capital R or if we put K equal to G m m by K is equal to G mm, then velocity v will be equal to root Gm by R. So, this is the velocity.

Now, as we see that the velocity is inversely proportional to the radius. So, what if we want to go to a higher or lower orbit let us say, if we want to move to a higher orbit with certain radius R 1; how do we do that there are of course, many ways of doing it, but there is a very special type of orbit transfer or very common type of or there is a very common way of doing it there are of course, other ways which is called Hohmann transfer. So, what happens in Hohmann transform is, at some point of this orbit a rocket is fired now for a circular orbit at any point on the orbit if you fire a rocket the additional velocity will work perpendicular to that particular direction.

Now, firing a rocket means you are putting additional energy to the to this system now if you recall from the equivalent energy picture this is the equivalent energy picture or a effective mass potential of inverse square force field circular orbit is here higher to that. So, if this is this energy E 0; this is the corresponding radius R of the circular orbit right now if we want to if we put some additional energy into the system then what happens we can move it slightly higher. So, it will be an elliptical orbit with 2 radiuses R 0 and R 1.

Now, it in Hohmann transfer what is done typically is the velocity of the rocket or velocity of this fires a additional thrust is as just as such that the ellipse is drawn something like in this manner. So, this is an ellipse. So, this is not a very clear picture. So, this is the force center as we know. So, this is my perigee distance and this is my apogee distance. So, we see that the relation is 2 a which is here to here is equal to R plus R 1 this is R; this is R 1, right. So, this is how this energy of this particular. So, earth and from here we can always write the total energy of this elliptical orbit and then from there we can calculate what is the velocity needed what is the firing speed needed in order to move to this particular orbit.

Now, what happens is from here we can from here once. So, once the rocket move x this elliptical path comes to this particular position there is a rocket which will fire in the opposite direction so; that means, if this is the direction of the satellite the rocket will fire in this direction. So, that the velocity component in this direction this is needed because once again we have to come to this we have to go down to this particular energy level, but this time with another R.

So, what happens is essentially when we are changing the velocity we are adjusting the L square by 2 m r square component. So, remember this L is a function a L is something which is in our hand; this we can control. So, by carefully adjusting the velocities we can compute another L which will give rise to another effective potential with a slightly higher radius which will be given by R 1. So, please remember that this effective potential what we draw is a strong function of L l square or L square by 2 m r square, right.

So, if we want to you know if we change the velocity of the satellite in question or that is in our hand essentially. So, what we are doing is we are modifying this effective potential. So, essentially in a Hohmann transfer what we need to do is we need to go from this particular potential to this particular potential. So, that we start with a circular orbit with radius r which is a minimum of this potential a effective potential and then at the end we go to this particular potential with a radius I mean minimal at R equal to R 1. So, in brief this is Hohmann transfer also we can do some single a simple calculation for this single bond maneuver that is the transfer between a circular orbit to elliptical orbit, but we are not going into the details of that.

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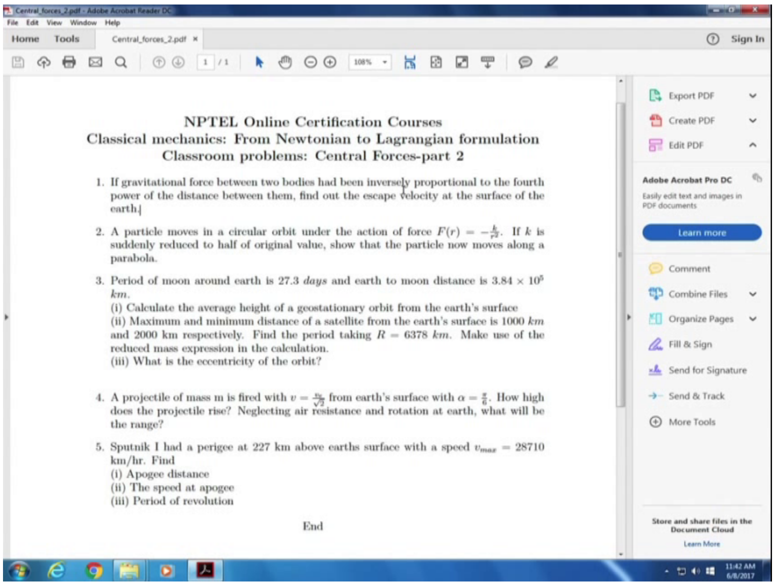
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Classical mechanics: From Newtonian to Lagrangian formulation
Classroom problems: Central Forces-part 2

1. If gravitational force between two bodies had been inversely proportional to the fourth power of the distance between them, find out the escape velocity at the surface of the earth.
2. A particle moves in a circular orbit under the action of force $F(r) = -\frac{k}{r^2}$. If k is suddenly reduced to half of original value, show that the particle now moves along a parabola.
3. Period of moon around earth is 27.3 *days* and earth to moon distance is 3.84×10^5 km.
(i) Calculate the average height of a geostationary orbit from the earth's surface
(ii) Maximum and minimum distance of a satellite from the earth's surface is 1000 km and 2000 km respectively. Find the period taking $R = 6378$ km. Make use of the reduced mass expression in the calculation.
(iii) What is the eccentricity of the orbit?
4. A projectile of mass m is fired with $v = \frac{v_e}{\sqrt{2}}$ from earth's surface with $\alpha = \frac{\pi}{6}$. How high does the projectile rise? Neglecting air resistance and rotation at earth, what will be the range?
5. Sputnik I had a perigee at 227 km above earths surface with a speed $v_{max} = 28710$ km/hr. Find
(i) Apogee distance
(ii) The speed at apogee
(iii) Period of revolution

End

So, let us move on and try to solve some problems. So, we start with this new problem set where we have five problems. Now out of that five we will see how much we can do and in the class I wish to cover all of it. So, the first problem is that if the gravitational force between 2 bodies has been inversely proportional to the fourth power of the distance find out the escape velocity at the surface of the earth. So, is a hypothetical situation.

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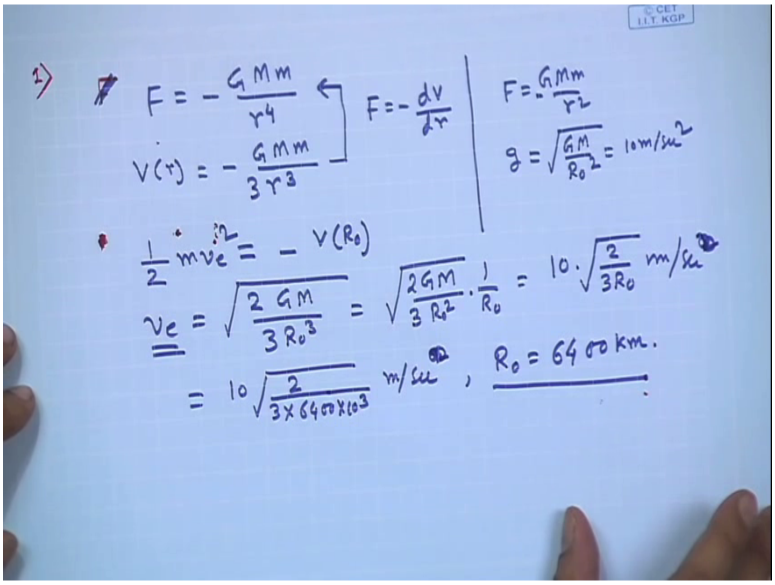

Now, if we start calculating it first of all what is the nature of force we have. So, F. So, I just take this pen. So, F is equal to minus GMm by r to the power 4. So, typically it is r square as we all know and we are discussing in hypothetical case. So, it is Gmm by r to the power 4.

But this is R to the power 4; there is always we can find out the potential function which will be minus G m m by R cubed and there is a factor of 3 coming because this 2 are related by f is equal to minus dv dr right. So, with this we can write the expression for escape velocity as half mve square equal to minus v of R 0 please remember we are calculating we are doing this calculation for earth surface right. Now comparing this 2 we get an expression for velocity and we use this expression for V R and the expression for velocity is 2 Gm by 3 R 0 cubed, right.

Now, we can try to put the values of gravitational constant mass of earth so on and so forth or what we can do is we can we can simply modify this by rewriting 2 G m by 3 R 0 square into 1 by R 0. Now if you recall now I am just dv, I mean I am coming back to the actual scenario where F is equal to G M m by R square with a minus sign here the gravitational constant V g is given by root over G M by R 0 square right and we know that the value is 10 meters per second square right. So, G M by R 0 assuming that the radius of earth remains constant even under this particular force which is a again it is an hypothetical situation.

So, G M by R 0 square root the whole thing is 10 right now if this is 10 then we have 10 multiplied by 2 by 3 R 0 meter per second square. So, this will be our answer now if you compute this we have to put the values. So, it will be 10 divided by 2 into 3 into 6400 kilometers. So, we have 10 to the power 3. This will be in meters per second we have taken oh sorry; it will be second because its escape velocity I am sorry we have taken R 0 as 6400 kilometer which is the standard value for average radius of earth which is once again not very accurate its actually 63797 something, but we can for this calculation we can take 6400.

So, if you put this numbers you will get a value for escape velocity; I leave it to you, I want you to finish this yourself right; now let us move to problem number 2 problem number 2 says a particle moves in a circular orbit under the action of this force F r equal to minus K by R square which is the standard inverse square 4 attractive force which is given if K is suddenly reduced to half of its original value show that the particle now moves along a parabola. So, it is a tricky problem think of it this way what happens the; what we need to show is that the particle which is moving originally in a circular orbit.

Now, during its motion at some point suddenly the force constant K which is it will be replaced by K by 2. So, it reduces by half now we have to show that this path will change into a parabola now I can tell you that we do not need to do I mean we do not need to show that I mean we do not need to calculate the equation of the path instead.

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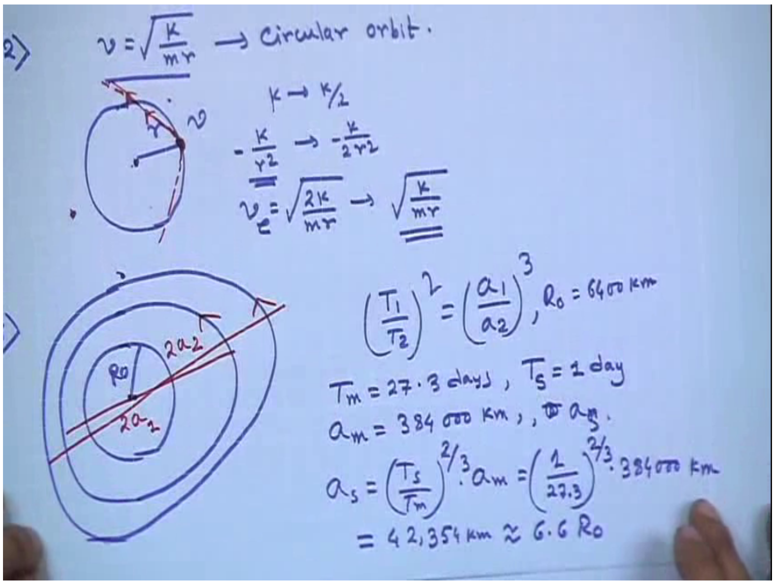

What we can do is we recall that for circular orbit v is equal to root over K by mr. So, we are doing problem number 2 here v that is for a circular orbit.

Now, what happens in a circular orbit or rather now this is the situation here. So, we have a circular orbit with radius r and any point we have this velocity or speed rather we should call it speed because which is technically more correct right now if K goes to K by 2 now at this particular situation when the force is given by. So, force when K goes to K by 2 the force was K by r square minus now the force is K by 2 r square right now if we look at the express a escape velocity expression for these 2 cases initially the escape velocity ve which is given by root 2 K by mr for a for a distance r from the force center.

Now, the escape velocity will be given because K has been reduced to half it will be given by K by mr because we are substituting K equal to K by 2, right. So, now, compare this. So, this is the escape velocity and this is the original velocity. So, what happens is the particle which moves in a circular orbit with this particular force law with force constant with this particular force law as the force constant suddenly reduces to half the instantaneous velocity which was at any point of the circular orbit which was given by K by mr; now becomes the escape velocity of that particle.

So, what happens is now for escape velocity when a particle moves with the escape velocity in a inverse square force field we know from our discussion earlier that the path will be a parabola, right. So, that is why let us say at this point where the force constant has been reduced from K to K half, it will immediately start traveling in a parabolic path. So, your path will look something like this. So, it will go instead of going in the original path it will start traveling in this path. So, this is how you prove it. So, it is not a very straightforward proof as we can see its kind of an intuitive answer we had to provide, but it is a very tricky; I mean it is a very and it is a beautiful question because its initially when you look at it you; you might have thought that you have to solve equations for circular orbit and you have to put to show that the equation becomes changes into something like an elliptical orbit that nothing is needed.

All you need to show that the a particle which was moving with certain velocity when this force constant reduces to half it becomes the escape velocity of the particle and that is it and when it moves with an escape velocity you immediately know that its moving in a parabolic orbit hence proved. So, problem number 3 is period of moon around earth is

27 and 3 days earth to moon distance is 3.84 into 10 to the power 5 kilometers. So, there are 3 parts to this problem, we will do for this class we have maybe only 2 minutes left. So, we will focus only on the first problem.

What is the calculate the average height of the geostationary orbit from earth surface. Now this is an application of Kepler's law remember in Kepler's law for if this is your force center and we have one elliptical orbit and other elliptical orbit, I mean it does not necessarily be on the epicedial points need not be on the same plane, I just draw it like this. So, this is one elliptical orbit then other elliptical orbit, then if these are 2 a for first orbit and let us say; this is 2 a of second orbit then according to Kepler's laws T 1 by T 2 square is equal to a yeah a 1 by a 2 cubed.

Now, what is given here is the orbital length the length of orbit and or rather a of the orbit for moon and the time period is given. So, tm is given as 27.3 days a m which is the average distance of earth to moon which is given as 384000 kilometer the geostationary orbit we know that for geostationary orbit it is an orbit which stays on earth on a single position of earth and evolves along with the earth. So, for geostationary orbit T satellite will be one day.

So, if you are not familiar with the term geostationary I would suggest that you kindly look it up and you must have heard of Indian; insert a satellite insert satellite is a geostationary satellite which during all the day I mean every minute every second after its launch its being placed on a single I mean it is the relative position with respect to earth of that particular orbit is I mean a particular satellite is not changing. So, that is the definition of a geostationary orbit. So, geostationary orbit has a time revolution or time period of one day. So, insert is placed directly on India. So, there are each country has their own geostationary orbit which will play which will be placed directly on that particular country and it will evolve along with that as the earth moves earth rotates it will also rotate along with that. So, we have Ts equal to 1. So, we have to calculate a as ok.

So, when we put it in this equation we get as equal to Ts by tm whole to the power 2 third into am which will be 1 by 27.3 whole to the power 2 third into 384000 kilometers which will be 40; which will be 40.354 kilometer approximately 6.6 R 0 assuming that R 0 is equal to R 0 is the radius it is 6400 kilometers. So, we see that if this is R 0, then 6 R

0 which is somewhere here along above this height a geostationary orbit must rotate. So, each of these orbits has a very specific name to it we are not going into the details of it, there is a lower circular orbit there is an upper circular orbit so, but geostationary orbit is something that we are interested in and we have seen that it is 6.6 R 0. So, for now we have to stop here we will continue once again.

Thank you.