Classical Mechanics: From Newtonian to Lagrangian Formulation
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Lecture – 20
Central forces – 13

So, we continue our discussion on central orbit. Now in central orbit, we in the last 2 classes, we have discussed equivalent one dimensional problem and we took up the special case of inverse square force field, but and then we moved on to the orbital sciences like we derived equations for elliptical orbit circular orbit or derived condition for it elliptical orbit and circular orbit we will continue with that, but prior to that we will spend a few minutes on the equivalent one dimensional problem once again because I forgot to mention I forgot to you know focus on certain features of this problem.

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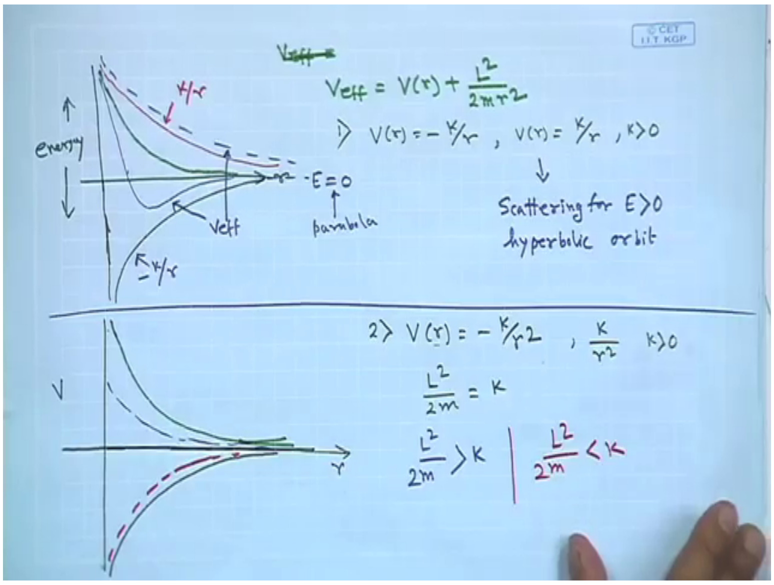

Now, if you recall this was my let us say this is my r axis and this is my energy axis. Now what we could do or what we did actually let us say this is and for V effective this is given of the soil right little lower V effective is V of r plus L square by 2 m r square right. So, this is the functional form for this particular effective potential. Now L square by 2 m r square I mean L square by 2 m as we have discussed, it is a positive definite quantity. So, we always we will always get potential which will look like this. So, this dependence is one over r square dependence.

Now, we have let us say we have a V r we discuss the case case one V r is equal to minus K by r. So, this is an attractive inverse square force law; that means, the potential force is 1 over K by r square. So, the potential is minus K by r and this; the potential looks something like these and overall the effective potential is something like this. So, we have discussed this case in details now what happens that if instead of having V of r s minus K by r if we have V of r is equal to K by r or K is greater than 0. In this case, it will; also I mean instead of being in the negative side K of r or V of r will always remain in the positive side, but it will fall slowly like this. So, this is K by r and this one is minus K by r right.

So, in this case the effective potential which is represented by this dotted line will look something like this. So, V effective will be this in case when we have a inverse negative attractive potential and V effective will be this when will have an in sorry yeah when we will have a repulsive potential then this will be the shape of V effective and you can immediately see from the shape of this potential that bound states are not possible for this particular potential because there is no such dip there is no such minimum in this potential.

So, only possible motion under this dotted line or if the potential effective potential is something like a dotted line is a scattered orbit which could be a and it cannot even be a parabola because for parabola the condition is energy total energy has to be at E equal to 0 and this is the condition for parabola. So, if we have a repulsive inverse square force field which will give this potential we can only have scattering for energies greater than 0 and with hyperbolic orbit. So, here we have scattering for particle energy E greater than 0 and hyperbolic orbit.

So, this is one case; let us discuss the case; if we have a potential which is given by V r is equal to minus K by r square. So, if the force is given by inverse cubed, I mean it if force is attractive with inverse cube of r, then this will be the form of the potential now as always your centrifugal term will be this one will be this one and your attractor this potential V of r the plot will looks like this.

Now, in the hypothetical case when L square by 2 m is equal to K if this particular condition is satisfied then 2 sides at each of; each point will cancel out exactly and it will look like as if the net or what I mean is the effective potential will be a flat line if L square by m twice m is greater than K then the effective potential will be something like this and if L square by m twice m is less than K. So, this lower part is dominant then my effective potential will be something like this, right.

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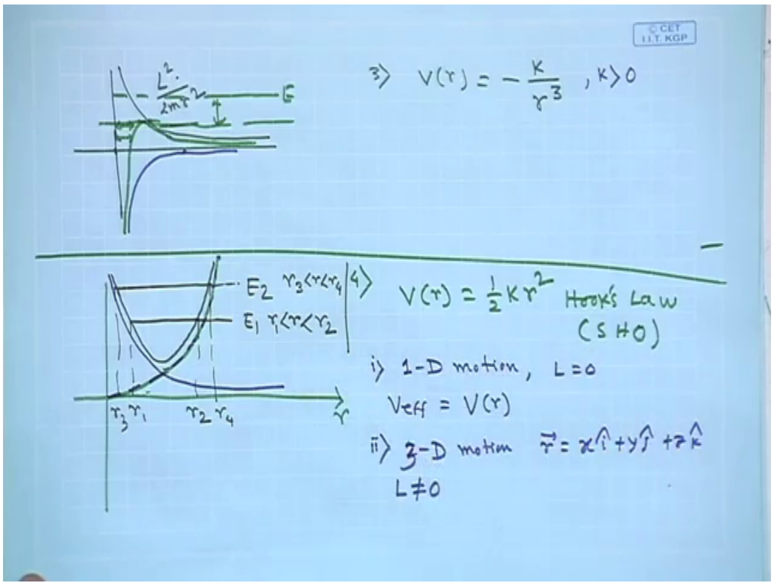

So, this is another case the third case is when we have V of r to be equal to minus K by r cubed, right. So, what happens before that what happens if we have K by r square with K greater than 0 instead of having a repulsive a attractive potential, if we have a repulsive potential. So, the potential will always be on the upper here upper part of this diagram. So, this is my r and this is my V diagram and once again there will be only possible motion will be a scattering motion with some orbit, but this time, it will not be an hyperbolic orbit because I mean it might or might not be an hyperbolic orbit we do not know our priority once is because this is not an inverse square potential, right.

Now, for K by r cubed; now we have slightly different situation this is my L square by twice m r squared term which falls with falls off with 1 over r square and for if K is greater than 0, then we have a term which falls with 1 over r cubed in the negative direction. Now if this is the case, then my effective potential will look something like this one. So, you see in this picture this one falls faster. So, at lower side of r this will be dominated and for higher side of r this will be dominated I mean the upper part will be dominating. So, the effective potential will look something like this.

So, in this case, we can have a bound motion this is possible, but the bound motion will be one of the lower bound will be the origin where there is a discontinuity in the force. So, we can have a bound motion in this region or we can have a scattering in this region and for energies slightly higher than this we will have nothing, I mean it the particle which comes with an energy higher than this threshold this particular threshold it will it cannot feel the presence of this potential.

So, if the energy is somewhere here this particle which comes with this particular energy E which is greater than this threshold will not experience the presence of this potential at all for energies which is below this limit; they can have either a scattered motion; if they are coming from this side or they can have a bound motion in this region, right and in this case for bound motion the particle will pass through the force center during its path.

So, this is case number 3; also if we have a positive sign here if we have a pure repulsive potential once again the net potential will be in this side; in the upper part and there is always a scattering. Now the fourth one which we are going to discuss is slightly different we will take half K r square, this is the familiar hooks law which produces simple harmonic oscillation, right.

Now, here dimension here in this particular case the we first draw this half K r square we know that it is a parabola something like this right now this effective potential depending on which dimension the motion takes place that shape of a with effective potential will change if we have one d motion then L is equal to 0. So, V effective will be equal to V of r this. So, if we have r square equal to x square for say. So, we will have only one possible potential; only one contribution in the effective potential because L is equal to 0 and my effective potential and my actual potential will coincide.

But if we have; let us say 2 d or 2 d motion or any other dimension of motion higher in 2 d or 3 d motion where r is in general represented by xi cap plus yj cap. So, odd and set K cap 2 d or actually let us call it 3 d motion in this case we will. So, when the motion is not confined on a straight line we will have an L which is not equal to 0. Once again we have this term back and my effective potential will look something like this same for any higher orders of r if we have instead of hooks law if we have K r to the power 5 or something we can always have a potential I mean in principle I mean which looks something like this. So, we have a bound state where origin is also not accessible because at origin this potential well at or at r equal to 0 goes to infinity also the higher values of r is also not accessible because this side also it goes to infinity the only possible motion in this potential is the bound portion between this 2 levels for this energy or between this 2 levels for this energy; so, either between r 1 and r 2, r 3 and r 4.

So, if E is energy is equal to E 1, then the motion is bound between r one and r 2 and if E is equal to E 2, then the motion is bound between r 3 and r 4 right. So, this is how the effective potentials for different problems or different sets of potential I mean different sets of force law different types of force law looks like. So, I wanted to give you a very brief description of this and now if we go back to our; the last problem of our problem set one.

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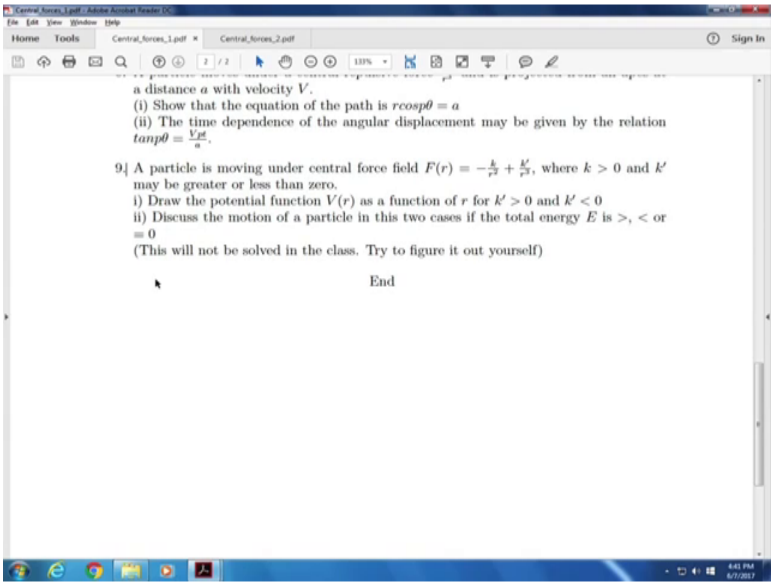

I am not going to solve it for you because it is something that you need to figure it out yourself; it is not a problem exactly, this particular problem; it is not a problem exactly, but it is kind of a quantitative description of what is going to happen if we have this type of potential where K is positive definite and K prime might or might or might not be greater than 0; what will be the different shapes first of all we have to for part 1, we have to determine the potential function I should write effective potential V effective. So, we have to draw the different types of potential function and we have to you know put energies which is higher equal to or less than equal to 0 and then we have to see whether we will have bound motion and bound motion possible in this case.

So, it is a example that you should or it is an exercise that you should try yourself. So, I am just closing it for now let us move on with our discussion on orbital motion. So, I have divided this discussion on central force into 2 parts; one where one is up to the description of effective potential and the second part is when we are discussing the orbits because orbit is something that you do not find in typical textbooks it is not been discussed, I would suggest the textbook of gradually almighty which a which is history in the reference book list that book has a very nice description and also, you will find it some of it will be definitely available on internet also there are a certain a group of text books which are classical mechanics or mechanics for engineering students which in which the topic will be covered, but for now for you for physics students I think gradually almighty is a very very good book for this.

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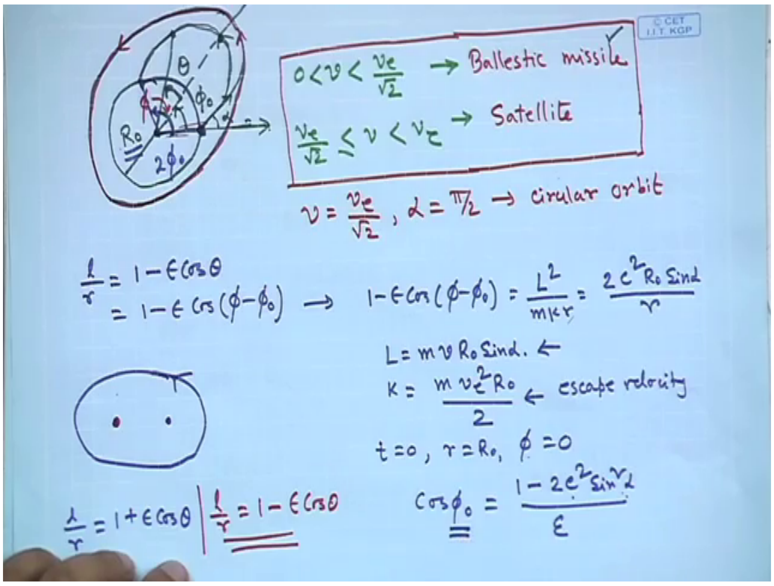

So, let us continue our discussion; now in the last class, what we did was we figured out what are the conditions for circular and elliptic orbit. So, let us say; this is our earth with radius r 0. So, at from this position something is fired; I mean an object is fired at an angle alpha with the horizontal direction depending on its velocity whether the velocity V is greater than I mean V false in the range of ve by root 2 or in the range of ve by root 2 to ve, it will have for and bound or circulating orbit.

So, we have seen that if this is the scenario then we have a bound orbit. So, the orbit will look something like this and if it is if this is the case, then we will have a circulating orbit right. So, this is an orbit which is this is an elliptical orbit which is typical for a satellite and the red marked one which is typical for a satellite. So, this will give us a satellite and this will the orbit which is closed here which is not a circulating orbit which will come back to the earth surface once again is the orbit of a ballestic missile.

So, we have discussed this already and we have found out the; so, this ranges is an outcome of solution of solution of some equation also we have seen that if V is equal to V E by root 2 and alpha the launch angle is equal to pi by 2, then we will have a circular orbit. So, this is a special case that we have seen already.

Now, let us try to focus on this particular case satellite we have seen. So, in for satellite we will come back to this letter for write right now let us focus on the ballestic missile and try to see if we can get an expression for its range now what happens is we can modify this picture slightly. So, this; please remember, this remains the force center. So, we can draw a line which goes through the farthest point or the perigee distance of a ballestic missile from this line we can draw a angle call it phi 0 and any position any point on this which is measured a once again with reference to this particular axis. So, basically we have let us call this theta this angle between this 2 line this line and this line theta and let us call this angle to be equal to phi.

So, basically we have 2 reference angles one which is measured with respect to the vertical direction at the point of its launch that is called phi and another is the angle theta which is measured along that slide. So, the line which joins the force center and the perigee position is definitely the apoapsis line and according to this line the equation of the orbit will be as we have already seen L by r is equal to one minus epsilon cos theta y 1 1 minus please recall that if this is your ellipse and it its going in this direction if this focus is your force center, then the equation was L by r is equal to one plus epsilon cos theta , but if this is your force center then your equation was L by r is equal to 1 minus epsilon cos theta.

So, in this case our force center if you try to just rotate this picture and fit in here you will get the you will you will see that this equation is the valid equation. So, this is my equation and we see that theta is equal to nothing, but phi minus phi 0, right. So, we can write this as cos phi minus phi 0 why phi and phi 0 because these are the angles which are easy to measure that that is something we know a priori theta is something that we have to construct we first know we need to draw this line and then we need to construct theta right. So, we just a pure mathematical calculation.

Now, what happens is we take this equation or we can just yeah; we take this equation and we can start putting the values of L and see what happens. So, one minus epsilon cos phi minus phi 0 is equal to L by r please remember L is equal to L square by m K r. Now we substitute the values for L and K L is equal to m V R 0 sine alpha and K is equal to m ve square R 0 by 2 this comes from the straight forward calculation of angular momentum and this comes from the escape velocity definition.

So, this is how we define this escape velocity once we once we put these values in here in this expression and simplify; we see that the expression becomes 2 c square R 0 sine alpha divided by r. So, I am not showing this calculation, but you can do it is pretty trivial pretty straightforward now if we use the initial condition at that at t equal to 0 at time when the projectile shot was fired r is equal to R 0 and phi is equal to 0, then that gives us and if we put it into this equation; this gives us cos phi 0 is equal to 1 minus 2 c square sine square alpha by epsilon.

Once again I am showing you; I am not showing you the details of calculation, but it is pretty simple you get this expression then you put these 2 boundary condition you put output r equal to R 0 and theta is equal to 0 and you immediately see this is the expression; now if this is the expression, then if this is the expression.

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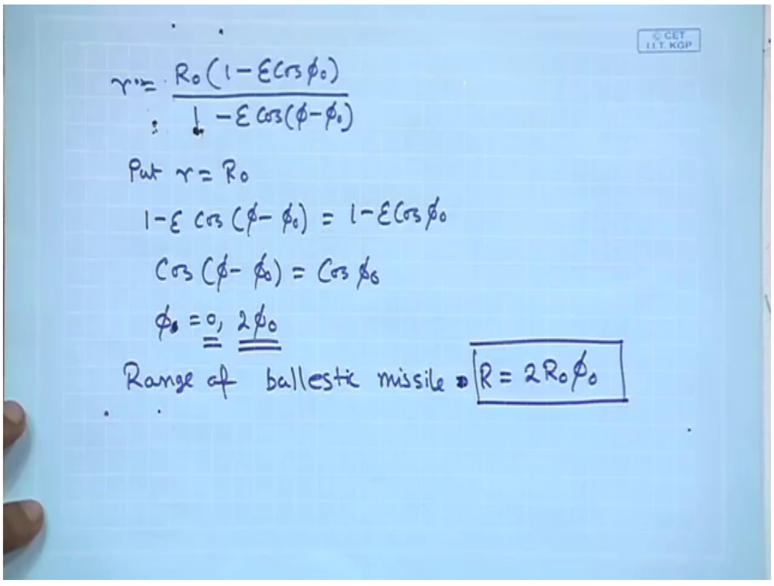

Then we can write r to be equal to R 0 1 minus epsilon cos phi 0 divided by 1 minus epsilon cos 5 minus 5 0; why this is because if you know; if you look into this expression. So, now, use this in. So, this was my starting point of starting equation.

Now, you have to substitute this into this equation and you will get this relation at the end right. So, now, if we now what we what is our aim we want to see how far this projectile actually travels. So, we want to want to make have an estimate of this length right. So, in order to have an estimate of this length we have to look for the extremum of angle. So, if I know this angular displacement, we just have to multiply if we know this angular displacement here we just have to multiply it with this R 0 and this will give us the total length travelled along that surface right.

So, we get this particular expression and we put r equal to R 0 here. So, if I put r equal to R 0, here I see that one minus epsilon cos phi minus phi 0 is equal to one minus epsilon cos phi 0 so; that means, cos phi minus phi 0 is equal to cos phi 0 and this could be true if phi 0 is equal to 0 or 2 phi 0 or sorry phi is sorry, sorry, phi is equal to 0 to phi 0, right.

Now, phi is equal to 0 if you look carefully phi is equal to 0 is this particular position well from. So, that is that that is the point where the projectile was launched. So, this angle has to be equal to 250. So, we need to have 2 solutions for this equation or 2 points where the projectile touches earth this was is the end this is the initial point. So, this is the point of our interest now if we if we accept this as our answer then the range of ballestic missile equal to which is which we can denote by r is equal to 2 R 0 phi 0, right

So, this is our relation and phi 0 is we can get the value of phi 0 by examining this particular expression cos phi 0 is equal to 1 minus c square sine square alpha by epsilon; also we have seen that this can be written in terms of this initial parameter. So, once we know the initial launch launching condition we know that we know alpha and we know c and in principle; that means, we also know epsilon we can immediately calculate phi 0 and from phi 0, we can just calculate the range as 2 r times phi 0.

So, this is the that is it for today. So, tomorrow’s class, we will take up problems and we will solve many problems related to this orbit transfer sorry not orbit transferred that that. So, that is one thing we have not covered. So, we will talk about orbital motions escape velocity and other topics.

Thank you.