Classical Mechanics: From Newtonian to Lagrangian Formulation

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Lecture – 02
Problems on simple kinematics motion under resistance

So, we have discussed the basics of Newtonian mechanics and how it works; what are the fundamental principles of conservation of energy in the last class. Now let us start with doing some simple problems here.

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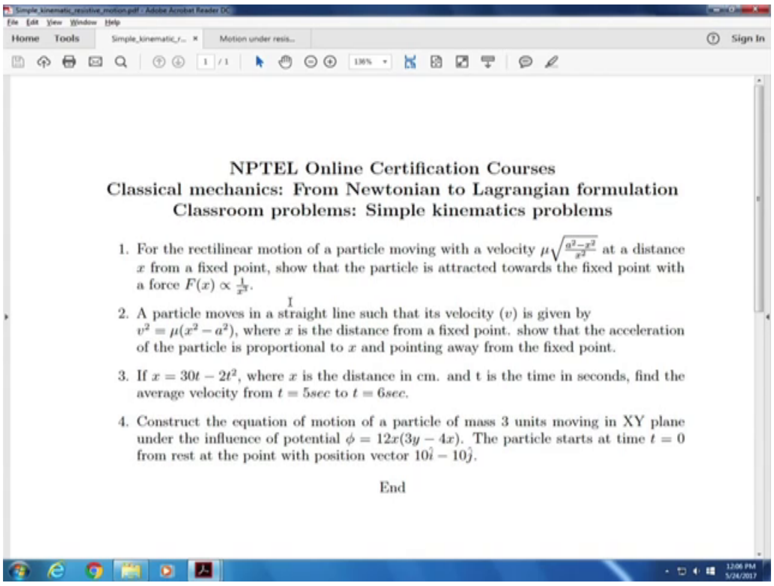

So, I have prepared 4 problems for today's class out of that probably, we will do 2 or 3; let start with the first problem. Problem says for the rectilinear motion of a particle along with the particle moving with a velocity mu times root over x square minus x square by x square at a distance x from a fixed point show that the particle is attracted towards the fixed point with the force which is proportional to 1 by x cube.

So, essentially what is given here is velocity as a function of x from a fixed point; let us assume that the fixed point is the force centre.

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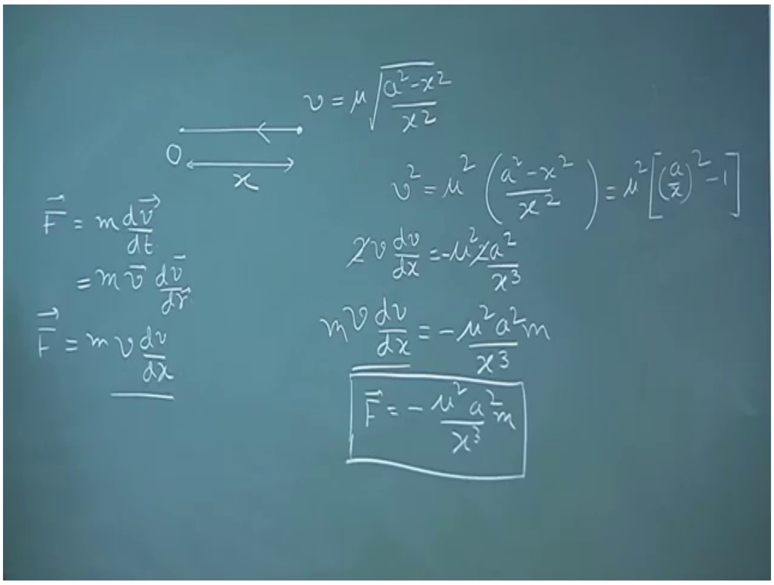

So, when we solve this problem; let us assume this is our fixed point, we said this as our origin o and here let us say this is the instantaneous position of this particle at a distance x from this fixed point. So, the velocity v at this point is given as some constant mu times root over x square; sorry; a square minus x square; a square minus x square by x square.

So, what we need to do is we need to find the law of force; it is very simple to solve this problem; what we are going to do is we simply compute v square which will be mu square a square minus x square by x square and then we take partial derivative of this side and that side with respect to x and we end up with 2 v dv dx is equal to mu square.

So, first for simplicity we can quickly rearrange this as mu square a by x whole squared minus 1. So, only the first term here will contribute; the second term essentially is a constant term. So, if we take the derivative, it will be a square and it will be derivative of 1 by x square which will be x cube with minus sign ho and then there will be a factor of 2 coming from here. Now this 2 will cancel out with this one; essentially leaving us with v d v d x is equal to minus mu square a square by x cube and this is if you now concentrate on the left hand side, if you remember Newton's laws says F is equal to m dv dt and what we did we rearranged this as mv dv dr which in one dimension essentially becomes F equal to m v dv dx. So, this is how.

Now, look at this term; if we multiply this term with m and this term with m; m being the mass of the particle. So, we can replace the left hand side with F which will be simply minus mu squared s square by m by x square. So, this is our final result and you see that there is a negative sign here. So, this negative sign implies that the force is towards the fixed point negative sign in this equation always means that it is an attractive force if it is; if there is no when negative sign; that means, the force is out from this force centre.

But because if there is a negative sign here we can straight forward tell that the force is directed towards the fixed point. So, we have got our desired result from this right; now let us move to the next problem in the next problem. So, in the next problem is also very very much identical to the previous problem; here the velocity is also already also given as a function of x and if you do this if you follow the exact same procedure then you will find that it is proportional the force the which will be vd vd x will be proportional to x and it will come with the positive sign.

If you just simply perform the derivative here and that positive sign means the force is pointing away from the fixed point this pointing away from the fixed point will come because the force will come with a positive sign unlike the previous problem which came with the negative sign. So, we are not doing it I am leaving it leaving it to you to perform this problem which is very much identical to the first problem let us move to the third problem third problem says the distance x is given as a function of time t which is x is equal to minus sorry; 30 t minus 2 t square.

So, we have to find the average velocity between time t equal to 5 seconds to time t equal to 6 seconds.

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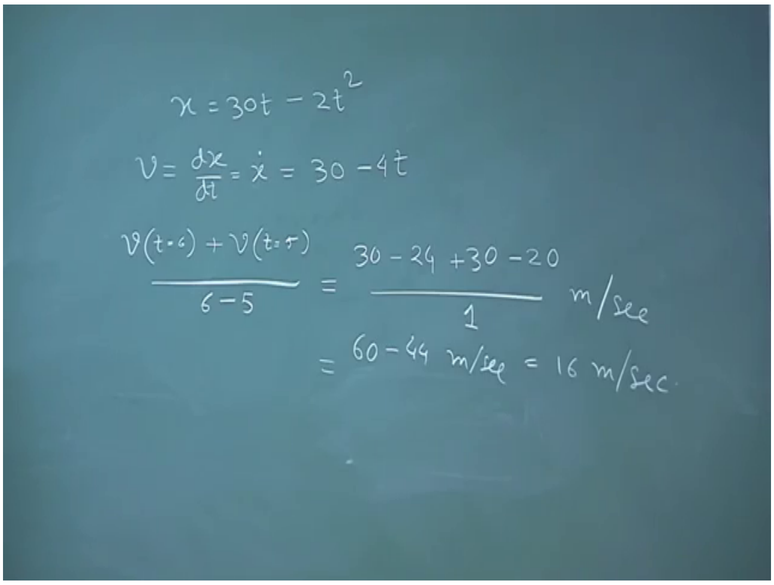

So, let us start with the expression of x which is given as thirty t minus 2 t square t being the time here. So, velocity v will be simply dx dt or we can also write it as x dot which will be 30 minus 4 t. Now what we need to do is we need to compute the average velocity between t equal to 5 seconds and t equal to 6 seconds.

So, the average velocity will simply be velocity at equal to 6 plus velocity at t equal to 5 divided by 6 minus 5 that is the time difference. So, in this case this will be 1 and the upper part will simply be 30 minus 24 plus 30 minus 20; sorry, it will be a 6 and 5; 20 divided by 1 and let us assume that unit is in meters per seconds. So, I am just putting in unit has meter per second, it could be kilometer or centimeter per second does not matter really; which I mean I am saying this because the unit of x is not specified.

So, I am just assuming that it is meter per second; it could be any other unit so essentially which gives 60 minus 44 meter per second which is 16 meter per second. So, it is also a very straight forward very simple problem and now let us move to the third problem of this problem set. So, the third problem in the problem set says or rather the fourth problem, we have skipped the second problem. So, now, we are already at the fourth problem construct the equation of motion of a particle of mass 3 units moving in xy plane under the influence of potential pi which is a function of x and y and the particle starts at time t equal to 0 from left at the point with position vector 10 i cap minus minus 10 j cap.

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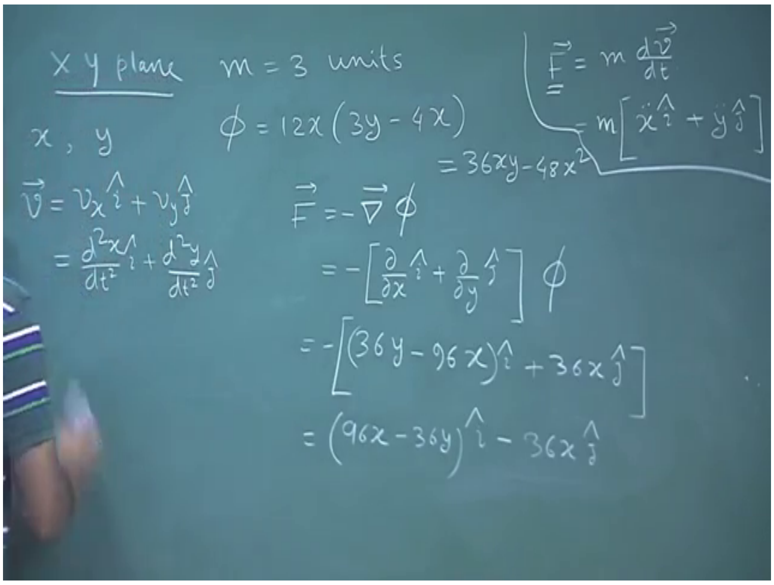

So, what is given here is a mass of the particle which is m is equal to 3 units again once again no particular unit is specified we are just keeping it as a 3 dimensionless number phi is given as 12 x 3 y minus 4 x. So, this is the potential in under the; under in under the influence of this potential the particle is moving in the xy plane.

Now, how do you compute how do we compute the equation of motion we know that the equation of motion essential is F equal to m dv dt in this case v as the particle moves in the xy plane; this is very important because as the particles moves in the x y plane, we know that the velocity will have or position vector will have 2 components which is x and y essentially.

So, we can decompose the velocity v as vx i cap or and v y j cap which will be nothing, but d 2 x d 2 y cap plus d 2 y j cap. So, if we plug this back into this equation what we essentially get is F is equal to m x double dot i cap plus y double dot j cap x double dot as you can recall it is the second time derivative of x and y double dot means second time derivative of 1. So, what we need to do is we need to find out the left hand side of this equation and then we can compare term by term we can come; we can take the left hand side; sorry, x and y component of the left hand side to be equal to mx double dot and my double dot.

In order to do that we need to compute F equal to minus grad phi and this delta operated here is del del x i cap plus del del y j cap also there is a k component, but because we are working in the xy plane; we are omitting the k component for now; ideally there should; I should write; there is another component which is del del z k cap, but right now we are working in the x y place. So, we are just not using this part.

So, once we operate this on phi what we get is minus. So, the first term of phi is essentially 36 xy when we take del del x of that we get 36 y i cap plus; please remember there is a minus sign here. So, we will just keep it like this and the second term is minus 48 x square. So, oh sorry; I just made a small mistake here. So, what we need to do here is we will have to write this 12 sorry; 36 xy minus 48 x square, right. So, actually there will be one more term coming in because this term will also contribute. So, it will be minus 96 x if we take del del x i cap plus; now in the second term it is a partial differential with respect to y.

So, only the first term will contribute not the second term. So, we get 36 x j hat; now we include the negative sign and write the complete expression which will be 96 x minus 36 y i cap minus 36 x. Now once we plug it back into this equation here.

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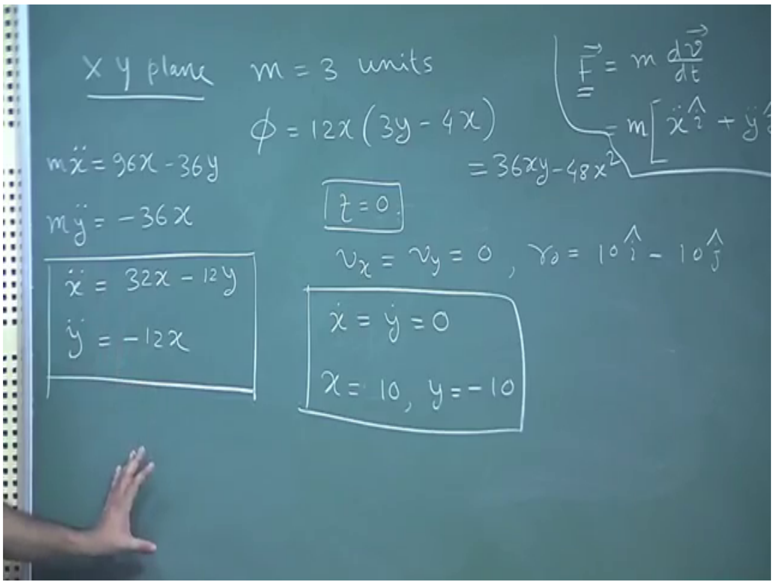

So, what do we get we get 2 equations of the form mx double dot will be equal to 96 x minus 36 y and m y double dot is equal to minus 36 x. Now recall that m is equal to 3. So, we put that and essentially we reduce this equations to x double dot is equal to 32 x minus 12 y y double dot is equal to minus 12 x. So, these are the sets of equations.

But also this is not complete as of yet because if you recall; there are few other boundary conditions given the particle starts look at the last line the particle starts at time t equal to 0 from rest at the point with position vector 10 i minus 10 j. So, that essentially means. So, in order to solve this second order differential equations what we have on board we need for it we need all together 4 boundary conditions because these are all 2 coupled second order differential equations and in order to have this 2 sets of boundary condition we have the following relations t equal to 0 start; please remember the statement of the problem.

Particle starts from rest; that means, vx equal to vy equal to 0 or x dot equal to y dot is equal to 0. So, that; that means, there is no motion and also the initial coordinate is given as r 0 is equal to 10 i cap minus 10 j k. So, at t equal to 0 x equal to 10 y equal to minus 10. So, we have 2 sets of coupled differential equations and all together 4 boundary conditions in terms of x dot equal to y dot equal to 0 and x equal to 10 y equal to minus 10 at time t equal to 0.

So, now this forms the complete set of equations and we have all the necessary; we have all the necessary information in order to solve this equation. Now in order to truly solve this equation, we have to apply separation of variables and I am pretty sure that you are familiar with the technique which that is why I am not discussing it in the short span of this class. So, with this what we are planning to do is we are moving to the next topic which will be motion in resistive media.

Now, if you recall in the last; end of the last class; I wrote certain forms for the force field. So, we wrote one for the constant. So, force could be a constant quantity force could be proportional to time, I mean as a time varying function it could be a function of position and also it can be a proportional to the velocity now when it is proportional to the velocity and if it is not favoring the motion then it is called a resistive force.

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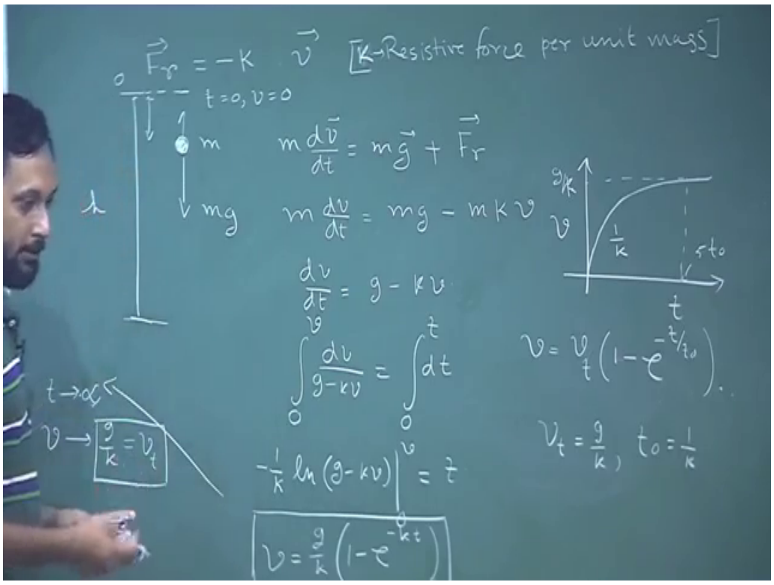

So, we have a force F which is called a resistive force, if it contributes with the functional form of kv; v being the velocity of the particle sorry minus kv; k being some constant and v is the velocity of instantaneous velocity of the particle and please remember that there is a minus sign this essentially means that the force Fr works opposite to the direction of velocity. So, if it works if the if the minus sign is not there; that means, this force is favoring the velocity which is which might be possible, but in physical real physical world we always observe forces especially; especially there is a special type of force called the viscous forces it could be a you know viscous drag it could be a pressure drag.

So, depend whatever it is, but it always works in the opposite direction of the velocity now you might be already familiar with the term terminal velocity if not then let me explain what it is; have you ever realized why rainfall rain drops fall at a constant velocity it does because of the presence of such forces here the k is a constant which is which comes; I mean the constant k depends from the air resistance essentially when a raindrop as a droplet of rain falls you know through the medium.

So, it essentially it; so, it is there is a drag force sorry there is a gravitational force which pulls it downloads which is proportional to mg m being the mass of the droplet and g being the gravitational constant or acceleration due to gravity sorry; not gravitational constant acceleration due to, but at the same time there is a force which is proportional to the velocity and working in the opposite direction. So, essentially there is a force which reacts upwards and at some point these 2 forces balance cancel out exactly and at that point the velocity remains reaches a constant value and that value is constant over time further as the rainfall drops further the velocity does not change anymore; just for the same reason when a paratrooper drops from a height the he never breaks the bone that is simply because by the time; he reaches the ground he has reached a velocity depending on the size of and shape of the parachute there is an air drag which works up board and he reaches a velocity here or she reaches a velocity which is constant and comfortable enough for landing.

So, let us look at; look into it a little more mathematically and then we will understand let us assume that a particle of mass m is falling under the influence of gravity. So, there is a force mg working on it and then there is a resistive force which is proportional to its velocity working upwards. Now for sake of simplicity, we will replace this kv by k m; that means, we are considering resistive force per unit mass. So, k is the resistive mass. So, that is why when we are we are considering a particle of mass m we have to multiply it with an additional factor m.

Now, if you tried to write the equation of motion of this particle, it will be m dv dt is equal to mg plus F r. So, that is in the vector notation; now when we look at into I mean look into component form of this equation it becomes m dv dt is equal to mg minus m k v please understand that here the origin is chosen at the point where this mass is started.

So, let us say we it started at a height h from the ground we are not calculating h here, but just to give you give you an idea. So, if you choose your origin at this point then h will be positive in this direction and as the mass falls h will be decreasing so; that means, you will have a negative sign before your gravity, I mean mass term or this force here, but here in this particular problem in this particular equation what I have done is we have chosen the origin of here. So, this is our origin.

So, as the mass falls it gains h. So, it gains in this direction. So, your velocity in this direction is the positive direction and that is why has the gravitational pull favors the velocity there is no negative sign here now this is very important because in some of the problems, we are taking the origin here for some of the problem we are taking the origin there do not kindly do not blindly follow this equation.

Please understand where is your origin of whether it is down here or a up there and accordingly we had kindly adjust your equation that is very important otherwise you will get very wrong results now as we have the equation is very simple because we have already taken k to be the resistive per unit mass. So, this is k and we have a extra mass term here. So, in this next step itself, we can just write this as g minus kv and now in order to gain insight of the motion we have to integrate this equation.

Integration is also very easy because when we start this is our starting point, let us say at time t equal to 0 v is also equal to 0 because it starts our origin is here. So, if we integrate we simply have dv by let me slightly rearrange this and right now g minus k v and this is the integration we are getting.

So, your integration limit at time t equal to 0 your velocity equal to 0 and at arbitrary time t your velocity is v. Now if we perform this integration it will be lon g minus kv by 1 by k 0 to v is equal to t i case; it is fine minus kv yeah it looks good I tend to make mistakes inside the integration, but. So, if we simplify this further essentially we get a relation I am just writing the final relation here or I will just write it here maybe.

So, v will be equal to g by k one minus e to the power minus kt. So, this is my final expression right, yeah it is right, this will be the final expression. Now if we study this expression down here what do we see if we set a large time let us say if t or I will just write it here may be if in this expression here if t goes to infinity then v goes to. So, what happens at into if we put a very large number for t then this term drops very fast and essentially after sometime it becomes 0. So, only term left is g by k. So, your v essentially as t goes to infinity v goes to g by k and this is called your terminal velocity vt. So, depending on the nature of viscous drag whether; so essentially we reach a velocity which is proportional to g and k and k being the viscous drag per unit mass.

Now, this exponential this is terminal velocity and terminal velocity is reached in by this exponential function. So, if we plot t versus v; this plot will look like something like this at please remember that at t equal to 0 v equal to 0 and a t equal to infinity this will level off to g by k. So, your plot will be something like this and the time constant for this exponential growth is one over k. So, I can rewrite this expression as v is equal to v t 1 minus e to the power minus t by t 0 where vt is simply g by k and t 0 is 1 over k and we know that like all exponential growth it typically takes 5 t 1 or 5 t 0 to reach the terminal the final velocity.

So, if someone ask you; what is the time for this terminal velocity to take place; after how long after starting from this point after how many seconds or how to up here after how many seconds; it will essentially reach this terminal velocity; you can always calculate if you know k, then we can always simply calculate 5 by k and say; this is the terminal velocity. Now please remember; we have used minus k vt here. Now instead of doing; instead of taking resistance force per unit mass; if we simply take this expression as kv, then your equation will be modified.

And also your final expression will be modified; I will explain that in a moment, but for now, I can show you a small trick that even without doing anything simply by using this equation we can get the terminal velocity stability.

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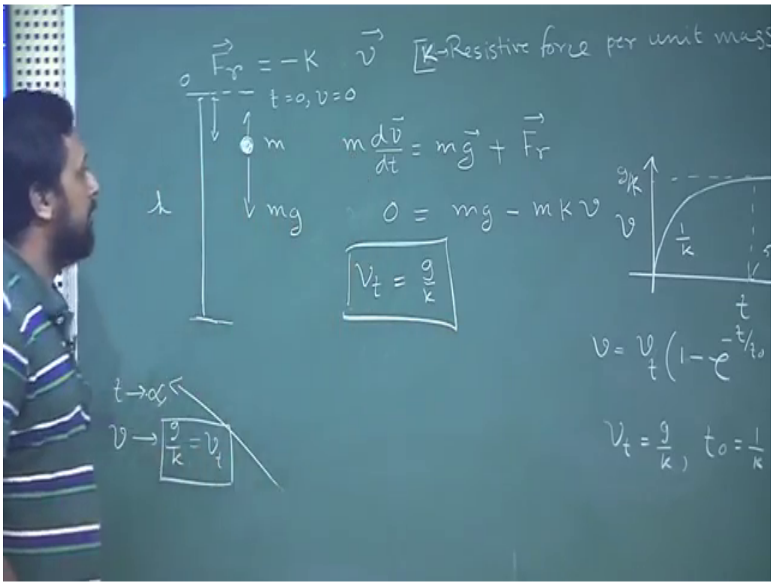

Now, what happens at terminal velocity physically speaking a terminal velocity upward and downward forces balance out and the particle reaches a constant velocity constant velocity means this term is equal to 0 and if this is equal to 0 we can get vt equal to simply g by k directly from here just by setting left hand side is equal to 0.