Classical Mechanics: From Newtonian to Lagrangian Formulation
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Lecture - 19
Central forces – 12

So we are back. And this time let us talk about different types of orbit and different types of initial condition that will lead to different types of orbit.

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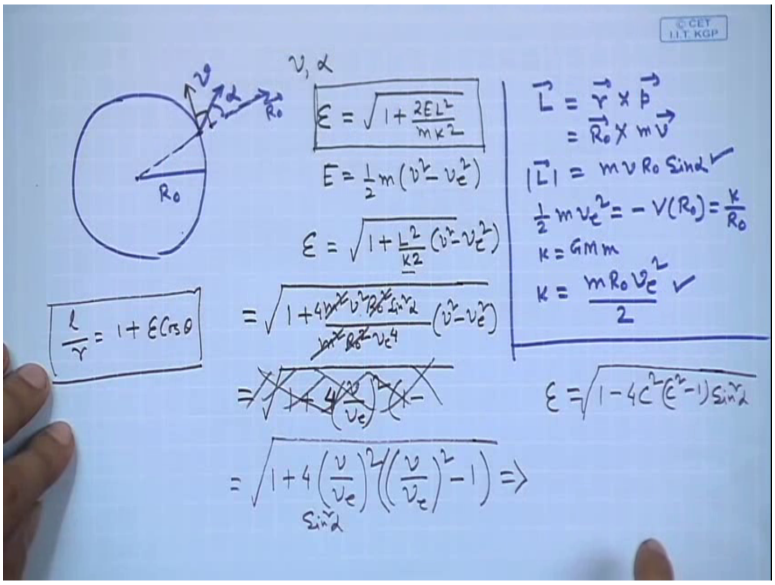

Now first consider the situation; we have let us say; this is our earth surface and we want to fire a rocket. So, this is my radius of earth which is R 0 and some point on earth we are firing a rocket at a velocity V with an angle alpha with a horizontal direction.

Now, in this chapter in this discussion; what we are going to do is we are to; we will try to find different conditions of on this V and alpha. So, we will see check out 4 different conditions of V and alpha which will lead to different types of orbit namely the circular orbit the parabolic orbit elliptical orbit and hyperbolic orbit. Now all this will be discussed in terms of this eccentricity expression which is given by one plus 2 e L square by m K square, right.

Now, e as we have discussed just now can be replaced by half m v square minus v e square right. So, eccentricity can effectively be written as root over one plus L square by

K square V square minus v e square right. So, now, let us consider this particular case we have to find. And so with some we somehow need to replace this K and L which are present in this expression with the parameters V and alpha and escape velocity ve now it turned out it is not very difficult as we can write expressions for L the angular momentum in terms of this 2 parameters. So, I am just calculating L here L is by definition is r cross p here r is given by R 0 let us say this is my R 0 vector and my p is m times V.

So, m and v vector; so, R 0 cross V is m v. So, the magnitude; so, this is my vector L and my magnitude L here will be sin alpha and we are only interested in this magnitude here, similarly K can be rep replaced by the expression half m v e square is equal to minus V of R 0 which is given by K by R 0 K being the force constant right now actually K we know that K is equal to gm, but gmm, but we are not go with writing it in the explicit form now once we do that and now once we. So, we using this expression we can write K is equal to m R 0 v e square divided by 2. So, we have one value for L one value for K both in terms of the known parameters are 0 m m being the mass of this particle and V v e and V. Now we substitute this back into the equation here .

We can write the expression for a centricity as 1 plus. So, it will be m square V square R 0 square sin square alpha divided by m square. So, there will be 2 will go up as four. So, it will be m square R 0 square v e to the power 4 and we have V square minus ve square right, now these 2 cancels; these 2 cancels leaving behind and then what we can do is we can there is a V square here and V to the power 4 here. And so we can take V square from it, inside this bracket and we can write this whole thing as 1 plus 4 V by v e whole square sorry; into 1 minus or sorry.

We can write this as one plus 4 times V by v e whole square into V by v e whole square minus 1 right. Now this can be further simplified to give epsilon is equal to root over 1 minus 4 C square into C square minus 1; sorry totally missed there is a sin squared alpha term here. So, it will come here in the final expression.

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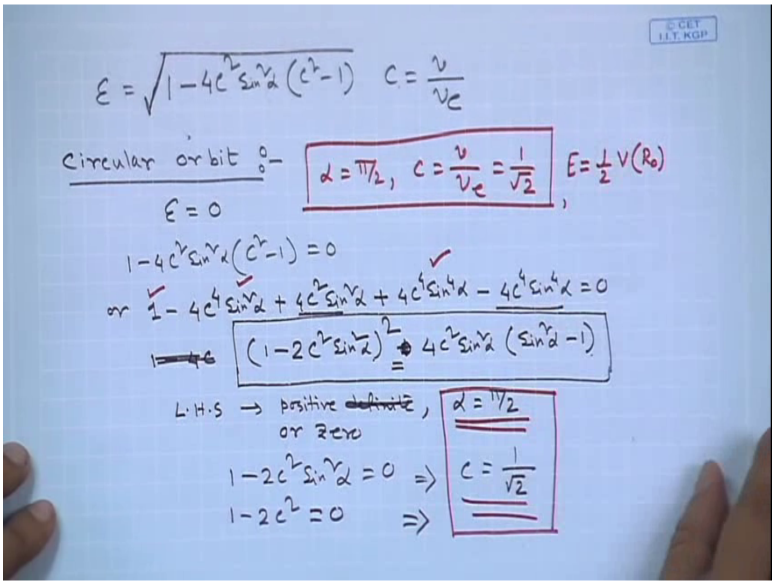

So, we get an expression for epsilon which is root over one minus 4 C square sin square alpha C square minus 1 and where C is equal to V by v e.

So, we got everything in terms of the known parameter see V by v e; v e is something that we typically we know or we can calculate easily V is the velocity of initial velocity of the mass projected mass which is also known to us. So, that way C is known alpha is the angle and that is all we need. So, you see essentially we started off with an expression of epsilon which was of this particular form and that we in one of the previous classes, we proved. I mean, we got this expression by using 2 differential equations we got for central orbit for 1 inverse square law of force or rather we used the equation of conic section L by r equal to 1 plus epsilon cos theta and 2 sets of differential equation and essentially got this expression in terms of energy angular momentum and force constant.

Now, this entire thing can be reduced to this simple looking expression which is in terms of the velocity of projection not even the mass of the object comes into the picture. So, it is all in terms of velocity of projection the escape velocity from earth surface and the angle of projection it is an amazing transformation I mean it is a very simple equation, but it is that this entire transformation is amazing, because how simplified things can be if we just you know; if we are smart enough and this is the starting point of the you know this is the master equation I would say if we want to determine the shape of the orbit for a given value of alpha and C.

Now, let us first look into the condition for circular orbit because we have already discussed circular orbit many times and every time I have I have discussed circular orbit I told you that circular orbit is a special case of elliptical orbit and there are certain conditions that has to be satisfied in order to get a circular orbit now we are going to show you I am going to show you what are these conditions which has to be satisfied in order to get a circular orbit for an artificial satellite. So, we here we are considering the cases which is typically the cases where a mass is projected from earth's surface towards outer space it could be a satellite it could be a missile it could be a rocket and we are trying to see what will be the nature of the orbit, but this particular analysis is equally valid if we want to look into the; if we want to calculate the you know trajectory for different heavenly bodies for example, a comet shooting star.

But of course, equations these are simplified equations we are not taking into account the air resistance not taking into account any other non-linear effect that might be present in the atmosphere we are not taking into account any of the attractions that it can be exhorted by any other heavenly bodies. So, it is a simplified model, but it is a very effective model. So, the equation as I said again; I am repeating myself once again the same set of equation can be used to determine the trajectory of a shooting star or a comet as well only thing is we need to have a very good idea of the escape velocity the or the set of initial conditions which describes the initial velocity the escape velocity at that particular point and the launch angle at which it was launched right.

Now, let us come back to this circular orbit circular orbits is defined with epsilon equal to 0 now if epsilon is equal to 0, then we have we see that from it this expression 1 minus 4 C square sin square alpha C square minus 1 equal to 0 or which gives one minus 4 C to the power 4 sin square alpha plus 4 C square sin square alpha; we can put equal to 0 here, but we are not doing that what we are doing; what we are going to do is we can we are going to add 4 C to the power 4 sin to the power 4 alpha minus again we subtract the same quantity. So, it is just a mathematical manipulation to bring it to a particular shape.

Now, let us look at this if we take this term this term and this term if we take this together these three terms will give us one minus 4 or rather. So, sorry, sorry, sorry, it will give us one minus 2 C square sin square alpha whole square or 2 say C square sin square alpha minus 1 whole square; does not matter really and the other 2 terms in the expression if we paired them up and take 4 C square sin square alpha common or actually; we can take them to the other side. So, if we do that we get this sin square alpha. So, it will give us sin square alpha minus 1, right. Now, focus on this expression. So, on one side, we have 1 minus 2 C square sin square alpha whole square and the other side, we have 4; 3 square sin square alpha into sin square alpha minus one now the left hand side is positive definite because its square of some term.

So, LHS is a positive definite; IHS is generally negative right because sin square alpha is never greater than one. So, sin square alpha minus one is equal to negative all the time except when we have alpha equal to pi by 2, then my right hand side becomes 0. So, it can have I should not say positive definite it ca it can be positive definite or 0. So, it can be either positive or 0 and right hand side can only have a 0 value at alpha equal to pi by 2 for any other alpha which is less than pi by 2 or greater than pi by 2, we can have this side as negative the only time these 2 sides can match is for this particular value of alpha. So, this is the condition which has to be met in order to satisfy this equation to be satisfied now if this is equal to 0, then we have from the from the left hand side we have.

1 minus 2 C square sin square alpha is equal to 0 which gives C equal to 1 by or and we have to put sin. So, we have to put the value of sin square alpha. So, it will be 1 minus 2 C square. So, if we put alpha equal to pi by 2 sin squared alpha is equal to 1 equal to. So, 1 minus 2 C square equal to 0 which gives C equal to 1 over root 2. So, this is another condition. So, we see that the condition for circular orbit by looking at this calculation is alpha equal to pi by 2 C equal to C which is equal to V by v e is equal to 1 by root 2, right. So, if and only if this condition is met we can have a circular orbit any other no other initial condition will give us a circular orbit, right.

So, now we know and also we can calculate the total energy of the; of a circular orbit and we can show immediately that total energy e of a circular orbit is given by half V of r 0. So, I am just leaving this to prove up to you for us as an exercise to prove is very easy if we start from the consideration; once again I mean initial values once again this is very easy right now.

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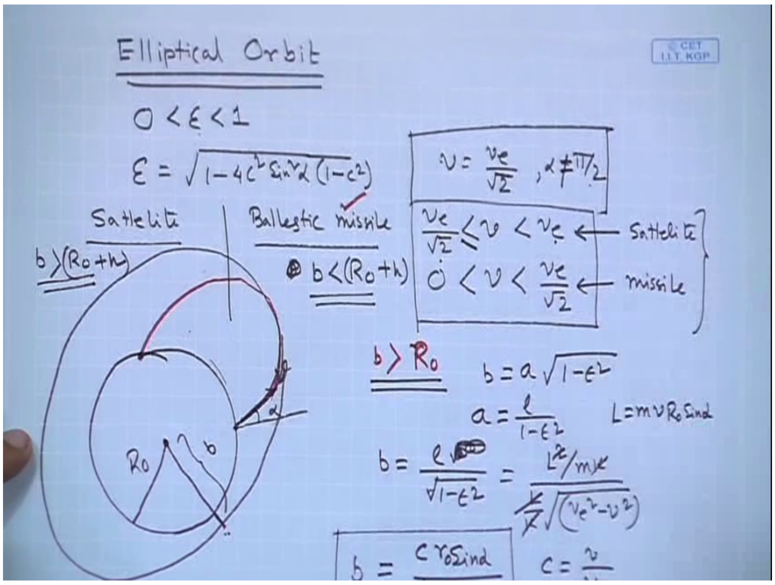

Now, let us go into the more general case of an elliptical orbit right. Now elliptical orbit epsilon we know has to be between 0 and 1 right. Now once again, we look into the expression for epsilon which is given by one minus 4 C square sin square alpha into 1.

So, you can either have a plus and write C square minus 1 or you can write 1 minus C square the way which whichever way you prefer, but for this particular case its easy I mean its better visualized if we write minus and 1 minus C square. So, that is why I am doing it otherwise previously you know, we can also write C square minus 1 and we can put a plus sign here, right. Now if this is the case we can and we have already seen that for V equal to V v equal to v e by root 2 and alpha equal to pi by 2 we can get a circular orbit now the elliptical orbit can be achieved either for a velocity V which stays between v e by root 2 to v e or we can also get it get an elliptical orbit for velocity V which is greater than 0 and between v e by root 2 why I am classifying these 2 ranges specifically we will see towards the end we will see why it is important to have this.

Also we can have an elliptical orbit if we have V equal to v e by root 2, but alpha e is not equal to pi by 2 if alpha is equal to pi by 2, we get a circular orbit if this is the case the velocity remains same, but we do not have a vertical launch. So, alpha equal to pi by 2 essentially means if you recall this picture alpha has to be equal to sorry; I think right alpha equal to sorry; alpha equal to pi by 2 means this. So, it has to be launched horizontally, right.

So, this is the condition for a circular orbit now if this condition is not met for any other alpha which even if we if our velocity condition is met for a circular orbit, but our angle condition is not met; we can also have an elliptical orbit. So, essentially these are the three domains in which we can have elliptical orbit right. Now inside elliptical orbit we can have 2 broad classifications one is a satellite and one is a ballistic missile how to define how to how to distinguish between this 2 let us say this is my earth surface.

It is given the radius is given by R 0 and I am launching an object from this particular point with some velocity V and angle alpha alpha; right and let us say this velocity V is less than v e. So, we will get an elliptical orbit. Now if the condition is such that you know that this particle can continue on this elliptical path. So, that. So, it will be an it will be a what you call it is a ellipse with because it is the gravitational pull coming from earth surface. So, earth surface has to be in one of the focus of this particular ellipse, right. Now if earth has to be in one of the focus the shortest distance is this distance which is given by the length of semi minor axis b.

If a shortest of r is greater than b; only then it can continue on this orbit and we have a satellite, but on the other hand if we have a situation where after launch this; this condition is not made and it hits, I mean your length of the semi minor axis is not greater than the shortest distance R 0 which is the shortest possible approach to this to the earth surface then what happens is it will come back and hit the earth surface at some point, right. So, in this case we have a ballistic missile.

So, we have one case where r is or b is greater than R 0 or sorry capital R 0, that is the case of a satellite or I will just write it there may be yeah. So, this is the case when b is greater than R 0 and this is the case when b is less than R 0. So, here why are 0 because R 0 is their shortest possible approach to the force centre which is in principle allowed of course, we have to give it a relaxation a satellite cannot pass you know pass very close to the earth surface there has to be a certain gap here.

Which for the calculation sake we are ignoring and we are assuming if b is greater than R 0 then it is a satellite and b is less than R 0 you say. So, in principle there has to be a minimum gap which is R 0 plus h and the R 0 minus h. Now let us look into the calculations. So, for first we have to set the condition b greater than R 0 now what is b if you recall b is equal to a root over one minus epsilon square and a was given by L into one minus epsilon square. So, in terms of L b is equal to L root over or sorry L divided by root over 1 minus epsilon square right. Now if we substitute the values for L and epsilon.

We see it is L square by m K divided by L by K root over v e square minus V square. So, we just have to go back to the definition of epsilon and substitute this; here if we do that and if we do this reduction at the end into a. So, this will cancelled out this K will cancelled out. So, if we do all this perform all this calculation. So, essentially we will be leave we will be left with C R 0 sin alpha to C R 0 sin alpha root over one plus or one minus C square. So, this is your b, now if b is this.

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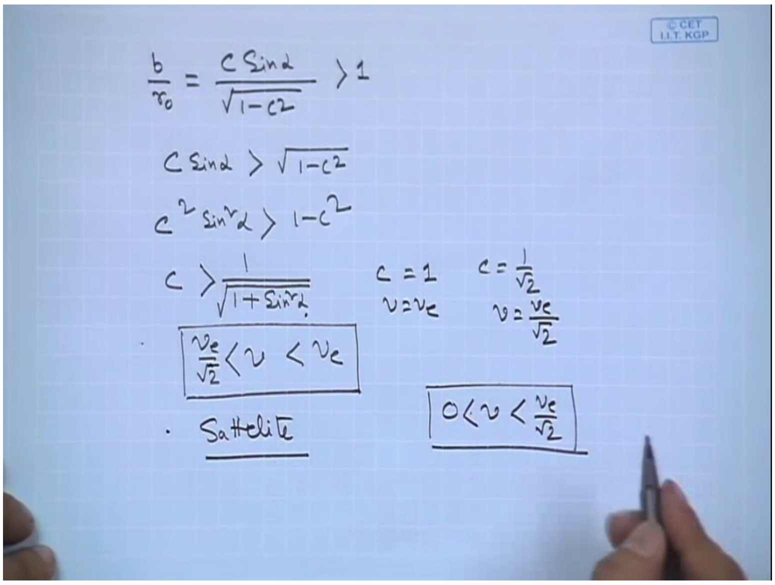

Then b by R 0 is equal to C sin alpha into 1 minus C square. So, here I did not show the calculation.

All you need to do in it to substitute the again you are initially launching at a velocity V and r with an angle alpha. So, you have to substitute L equal to m v R 0 sin alpha into this expression and you have to write C equal to V by v e you immediately get rc get this expression that b is equal to C R 0 sin alpha into one minus C square, right. So, for satellite as I have discussed already, this ratio has to be greater than 1. So, we have C sin alpha greater than root over 1 minus C square and which essentially gives C square sin square alpha greater than 1 minus C square simplifying, we get an expression C greater than one by one plus sin square alpha whole root, right.

So, range of sin square alpha is now between 0 and 1. So, the highest we can have in C is the highest of C can be equal to 1 and the lowest of C can be C equal to 1 by root 2. So, we see velocity has to be enclosed between v e by root 2 to v e in order to have a satellite C equal to 1 if it essentially means V is equal to v e C equal to 1 by root 2 essentially means V is equal to V equal to v e by root 2. So, that defines. So, range in sin square alpha because alpha can have range between 0, I mean sin square alpha can have range between 0 and 1 it defines the range as V 2 V by root 2. So, if this condition is made we have the condition if this particular condition is made then we have a satellite and if this ratio if we go back here and said this ratio less than one we will find out a ray.

Similarly, we will find out the range of V which is given by this and for this particular case as I have discussed already as the length of semi minor axis is less than R 0, the object which will be projected from earth will come back and hit that surface at some point and this particular case as we have discussed is called the ballistic design.

So, we see that we have 2 cases as we have why we have specified this because in this particular case we have we can have a satellite also we put a equal sign here because of we considered this as a special case of this one. So, if we have a velocity condition which is between escape velocity and escape velocity by root 2, then we can have a satellite. So, with the object we will end up going into the outer space and into an orbit around sun around earth and if this is the case then it will come back and we have a missile.

So, with this we end today's class; next class, we will take up some problems on this and also there are different other types of orbit where situations like parabolic orbit and hyperbolic orbit which will be discussed briefly. And then of course we will do some problem, and then we will move on to the next topic from central forces.

Thank you.