Classical Mechanics : From Newtonian to Lagrangian Formulation
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Lecture - 18
Central forces – 11

So, we continue our discussion on central forces and today's topic is effective potential.

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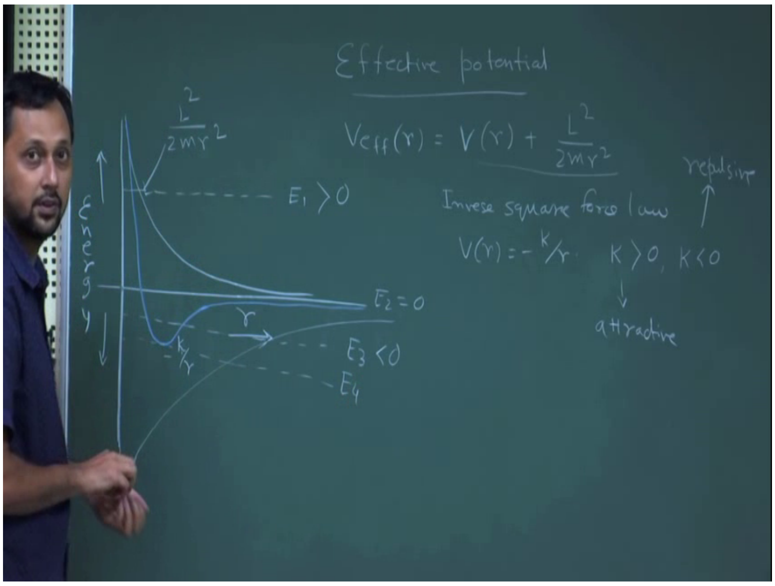

Now yesterday's class, we have seen that there could be many different types of motion in an effective potential. And so that was a very general discussion to begin with we have seen that there could be a scattering phenomena there could be a bound orbit, but strictly speaking bound orbit necessarily does not does not necessarily mean it is a closed orbit, but today we will take up the case of a specific potential. So, let us start with this v effective equal to V r plus L square by 2 mr square, right. So, now, we will take specific examples of V r and see how does the nature of this effective potential changes with specific different properties of V r.

This is the fixed part which is true for any central orbit and it can be shown that this is equivalent to the centripetal acceleration which I am leaving on to you as an exercise you can do that by yourself now let us first take the very common and very well discussed inverse square force law the force is proportional to inverse square. So, the potential V r is given by minus K by r. So, I am trying to show you how this effective potential looks like 4 minus K by r and please remember K is greater than 0. So, right now let us not talk about whether it is a inverse square attractive or repulsive force just let us call this the force and we will take 2 examples and see how the potential changes for K greater than 0.

So; that means, an attractive force and K less than 0 which means a repulsive force because there is a minus sign all together here already. So, K positive means this whole term becomes negative K negative means this whole term becomes positive. So, this means attractive and this means repulsive, right. So, let us try to draw this v effective. And if you recall yesterday when we discussed it in the last class we took know no specific energy level, but here; we are specifying the energy level here we have energy this is the 0 energy level.

And this is the 0 energy point and this direction its r, alright. Now first examine this term for any motion with a finite angular momentum except as we have already discussed that except for simple harmonic oscillation all the motions are with the finite angular momentum. So, for any motion with a finite angular momentum L square is positive definite same for 2 m.

So, the coefficient of this term is positive definite. So, it the magnitude of L might differ magnitude of m might differ, but quantitatively this term will always be looking like this. So, this is my L square by 2 m r squared term. Now let us first take the case where K is greater than 0 which is an attractive potential attractive potential means it will be at. So, the coefficient is overall coefficient is negative. So, at r equal to 0, it goes to minus infinity r equal to infinity it goes to 0, right and it is slower in compared to this one because this one falls off with a with 1 over r square and this 1 falls with 1 over r.

So, if I draw it in the same scale once again the magnitude the magnitude of K is might make the shape slightly different, but the overall contribution of this one will look like this please note that this slope is less than this 1, because this is by it. This one goes by 1 over r, whereas this one goes by 1 over r square. So, this is my potential minus K by r.

So, if we look at the effective potential at very low r values the effective potential will be dominated by this term because this has an 1 over r squared dependence which grows faster at lower r values when we are going to higher r values this one dies faster because this is once again this is because of the 1 over r squared dependence compared to one over r dependence. So, at higher r value this term will sustained, whereas at lower or dominate and lower r value; this term will dominate.

So, if we now try to draw the effective potential the effective potential as I said for lower r value it will be following this term for higher r value it will be following this term and the overall effective potential will look like this I am drawing it in blue for. So, that you can see it clearly something like this. So, you see; this one gives a minima at certain position certain value of r which we can find out.

Because we already have this expression we know we can put minus v equal to minus K r here take a derivative and try to find out what is the position of this minimum value that we can do that is not terribly important at present. Now if we try to discuss the motion of particles coming up coming with different energy; let us assume that one particle comes with an energy E 1 another particle comes with an energy E 2 equal to 0 E 1 is greater than 0 another particle comes with an energy E 3 which is less than 0 another particle has energy E 4 which is equal to the minimum value of this potential sorry this is the drawings are not very good what I will do is I will just switch to the pen and paper mode.

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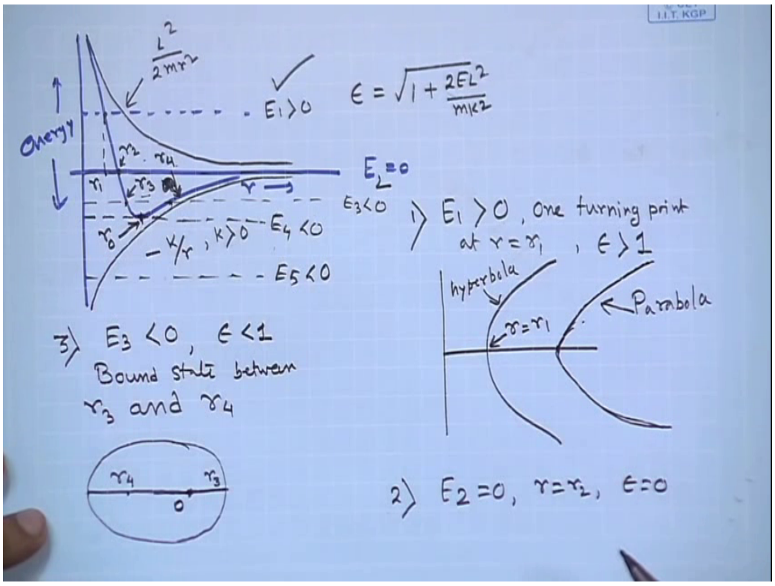

So, let us say this is the axis once again this is my 0 level. So, this is E equal to 0 we have energy here we have r here; now as I have discussed already.

We have this one which is L square by 2 m r square and we have this which is minus K by r K greater than 0 and this is the case then the effective potential as I have discussed already will look like this right; now that is a better to . So, we discuss the situation where a particle comes with an energy E 1 which is greater than 0 E 2 which is equal to 0 E 3 which is less than 0 E 4 which is equal to the minimum energy and then any energy E 5 which is again less than 0 and lower than the minimum energy of this.

Now, what happens is now what we are going to do is we are drawing the intercepts for different energy values. So, let us call this one as r one let us call. So, for this E 2 energy we have an intercept point here. So, let us call it r 2 similarly here we have 2 points let us call it r 3 here and r 4 here. So, it is not clear I will just write here r 4 this one let us call it r 0. So, so we have for E 1 greater than 0, we have one intercept which is at r one right now the total energy of the system total energy of the particle is greater than 0 and as per our discussion yesterday this particular case produces an unbound orbit.

Now, unbound orbit energy net energy greater than 0 that essentially means your epsilon which is given by root over one plus one plus 2 E L square by m K square. Now if E energy total energy is greater than 0 then this epsilon must be greater than 0. So, let us take it case by case; case one when particle comes with an energy E 1 greater than 0. So, we have one intercept 1 turning point I would call it because this is a turning point as we have discussed in the last class. So, turning point at r equal to r 1. So, the orbit looks like this; this r is equal to r one at this particular point.

So, this is this case and a centricity E greater than 1 as we can see from this particular expression if E is greater than 0, then this whole term is epsilon has to be greater than 1. Now for case 2 we see that E 2 which is equal to 0 immediately tells us that there is one turning point at r equal to r 2 and epsilon equal to 0. Now r 2 is slightly higher r r 2 has a slightly higher value than r one and this one corresponds to an parabolic orbit. So, this one is a hyperbola and this one is a parabola right. Now let us look into the third case when we have an energy E 3 which is less than 0 and it intercepts this effective potential at these 2 points. So, there are 2 intercepts this and this and as we have discussed yesterday it will be a bound orbit bound orbit between the limits r 3 and r 4.

Now, a bound orbit and of course, we see that epsilon is less than one. So, this one produces a bound state between r 3 and r 4 and any and only bound state which is possible in an inverse square force field is an ellipse. So, this one is an ellipse with r ranging from the smallest to the highest. So, this is if this is my focus o or the origin at o one of this focus, then the smallest r 1 is. So, our smallest of the r is this. So, this will be my r 3 in this case and the largest of the r is this length which will be given by r 4. So, this is this will be the. So, the if the energy is E 3 which is less than 0 and has 2 intercepts with that effective potential then this will be described by an ellipse.

Now, what happens in case of a circle and or in the case where E 4 is less than 0 and also it is the minimum point of this effective potential?

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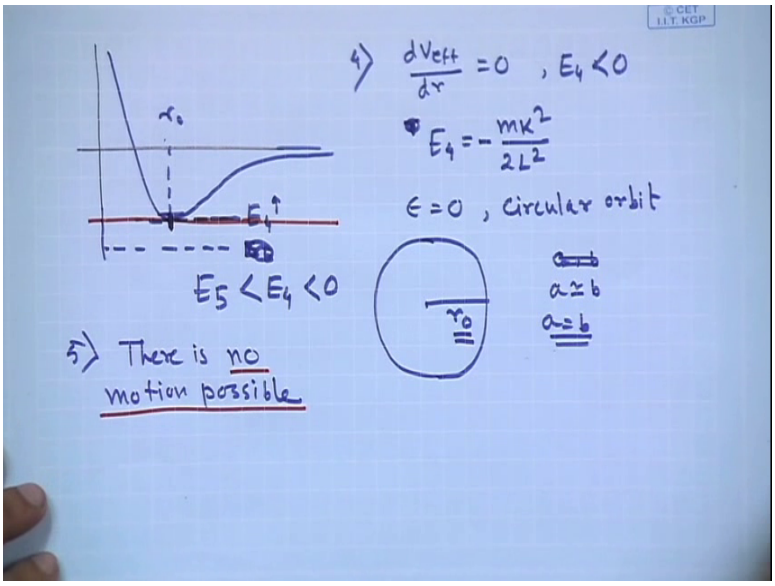

Now, what happens is we have this effective potential which is something like this and the lowest point is given by this now what we can do is we can take d v effective d r and set that equal to 0 and find an expression for this r 0 which we are which uh which is a minimum point in this particular potential and it can be shown I am leaving it to you as an exercise that this will give you a value of r 0 which corresponds to an energy. So, I not writing the value of r 0.

But the energy E 4 can be written as minus m K square by 2 L square. So, that is the fourth case when E 4 E or energy is equal to E 4 which is less than 0. Now if you put that value in your eccentricity expression which is once again which is given by this you will immediately see that this particular expression corresponds to an eccentricity equal to 0. So, it is a why circle is. So, it will give you a circular orbit circular orbit with only one value of r which is given by this r 0 right. Now why it is called as special case of ellipse because if you leaf; I mean if you lift the energy slightly higher from this level slightly higher then immediately you see it has 2 intercepts and 2 intercepts means.

It will have it; it will be a an ellipse, but for that particular ellipse the value of the semi major axis and semi minor axis will be very much equal to each other or we should write for that particular ellipse a will almost be equal to b. So, in an limiting case when energy E 4 total energy heats the minimum of this potential well then we have a equal to b strictly and we have 2 bound, but 2 bounds for r merging at one particular bound which is given as r 0 all right; very soon we are going to discuss what are the general how can one get an elliptical orbit; for example, for a artificial satellite right

Now, there is this energy which is E 5 or sorry E 5 which is less than E 4 of course, it is less than 0, if this is the energy of a particle in this particular force field then the result is there is no motion. So, if the energy is less than this energy 4 this is my lowest level of energy with which a particle can move in this particular force field below this energy there is no motion possible good.

So, I think we have discussed all the necessary details of this effective potential. Now what we are going to do is for the rest of the duration of this class, we are going to discuss about certain important parameters related to artificial satellite.

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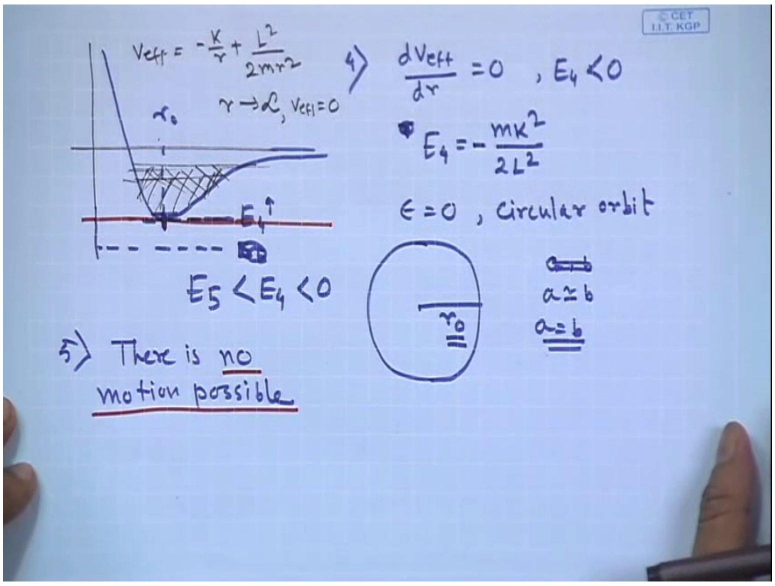

So, the next topic which is the last topic of this discussion is artificial satellite and ballistic epsilon.

Why it is important it is important because you know India is an emerging power in or not an emerging anymore it is a superpower in space and space research nowadays and we are doing excellent work in terms of you know sending spacecraft and of course, our difference technologies are also getting better and better day by day. So, we are making very good ballistic missiles off net. So, this is why I think it is a physics student of this country it is important that we understand how is this artificial satellite is sent to space. And how is ballistic missile is met. So, I just wanted to give you a brief overview of these things again this will be mostly a toy model we will be neglecting air resistance will be neglecting any other non-linear effect.

But at least a fundamental understanding of how an artificial satellite is launched and what determines the shape of its orbit and what is a ballistic missile and what determines its range so on and so forth. So, for this particular discussion first start with this defining a quantity which is also very family quantity to all of you; I have left it for the last because it will be related to this last part of discussion and this is given by escape velocity our escape velocity is defined only for attractive potential.

So, let us say we have an attractive potential we in and we have to go out of this potential with and for. So, if we have to go out of this potential we need to we need to have some energy minimum energy and that minimum energy what is needed can be converted into some kind of a kinetic I mean it can be associated with the kinetic energy. Kinetic energy means it will be associated with the velocity and that minimum velocity that is required to escape from an attractive force field is called the escape velocity.

So, let us take an example let us say this particular force field we have just discussed if we are into this bound state somewhere into this bound state in this region; let us say our particle of interest is has some certain energy which belongs to the shaded region now in order to escape what is; what it needs to do is a escape means it has to go to a distance where the potential will drop to 0. Now for this particular force field if you look at the effective potential v effective is equal to minus K by r plus L square by 2 m r square.

this is 0 if and only if we set r equal to infinity only at r equal to infinity v effective is equal to 0. So, for an inverse square force field specially for an inverse square force field which extends all the way to infinity.

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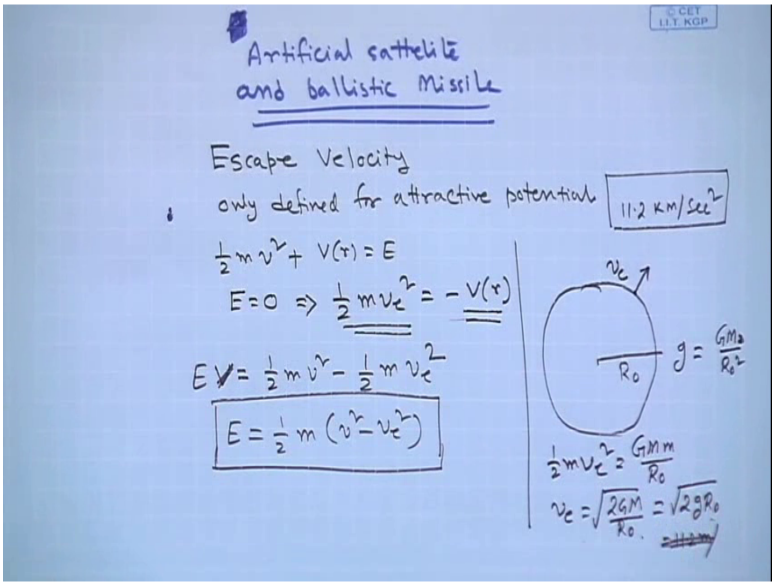

Escape velocity is the minimum velocity requirement which will take an object to infinity now in order to go to infinity or if it E; if it is the minimum energy which will take it to infinity; that means, that infinity it will have 0 velocity right. So, we can write an equation which is given by half which is from the conservation of a momentum half mv square plus v of r is equal to e. So, that is the general equation for any motion in this in the in the attractive force field now for escape velocity.

If we have to if the particle has to escape this force field it has to have of energy which is at least 0 at a distance of infinity and as energy is conserved in this entire motion E has to be 0 even at the beginning itself. So, we immediately see that half m v e square is equal to minus v of r and if we write v e using this relation then the total energy expression for any velocity v can be given as half m v square minus half m v e square which is equivalent of writing half m v square minus v e square sorry not v e. Now if we try to get an expression for the escape velocity, we can use this relation from earth surface for example, if we say this is my radius of my earth which is given as r 0 and if we want to escape from this particular point with the velocity v e, then according to this relation half m v e square is equal to minus V r which is given by gm m by r 0 right.

So, that gives you an expression for v e is equal to root 2 gm by r 0 which is equal to equivalent of writing 2 g r 0, right because this is true because g is equal to gm by r 0 square right now the value of this; as we know on earth surface if we put g equal to 10 meters per second and 6400 kilometer for earth's radius; it will be something about 11.2 meters sorry; it will be something around about 11.2 kilo meters per second square, right. So, this is the discussion of escape velocity.

And from this in the next interval I mean after interval we will go on of to write a different expression for eccentricity. And from there we will go to see how this what is that particular condition at which a circular orbit can be achieved for a artificial satellite what are the condition for elliptical orbit, and what is the condition for a ballistic missile.

Thank you.