Classical Mechanics: From Newtonian to Lagrangian Formulation

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Lecture - 17
Central forces – 10

So, today we are on in this particular class we are going to discuss about the equivalent, one dimensional motion or one dimensional problem for a central force field.

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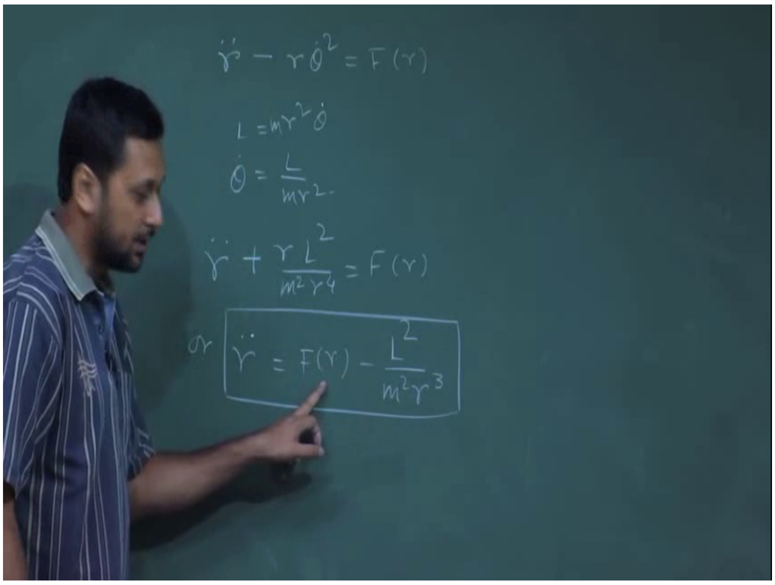

The description of the problem is as follows, if we recall the basic Newtonian differential equation of the motion was R, double dot minus r theta dot square is equal to F of r right.

Now, if you remember, we also had L equal to r square m r square, theta dot which means theta dot is equal to L by m r square. Now, if you substitute for this particular term, then we get r double dot plus r into L square by N square r to the power 4 is equal to F r or r double dot is equal to F r minus L square by m square r cubed.

Now, this right hand side of this term, right hand side of this equation, we have two terms, one is the actual force, which is due to the central orbit and 1 is A 1, is a term which is L square by m square r cubed, which also is proportional to the distance, but it is not a real force, it does not have the components of real attractive, attractive or repulsive forces in it instead, what we have here is a term which consists of angular momentum and mass.

Now, please remember that except for simple harmonic motion, simple harmonic oscillation in one dimension every time an object moves under a central force field, there is a angular momentum, which is constant for the special case of one dimensional simple harmonic oscillation, this angular momentum L is equal to 0. So, except for that particular case, whenever a particle is moving under the action of a central force field, there is this term and this term is also, if you, if you understand L is positive m is definitely positive.

So, L square by m square by r and divided by r cubed. It is a term, which will contribute to the right hand side, of this equation irrespective, whether the motion is you know, if it is A, you know slow movement or is a fast movement does not really matter, this term because angular momentum is a constant, either does not matter; if it is A, if the particle is passing through an apsidal point. If the particle is passing through some sort of minima or maxima does not matter everywhere. This term will be omnipresent.

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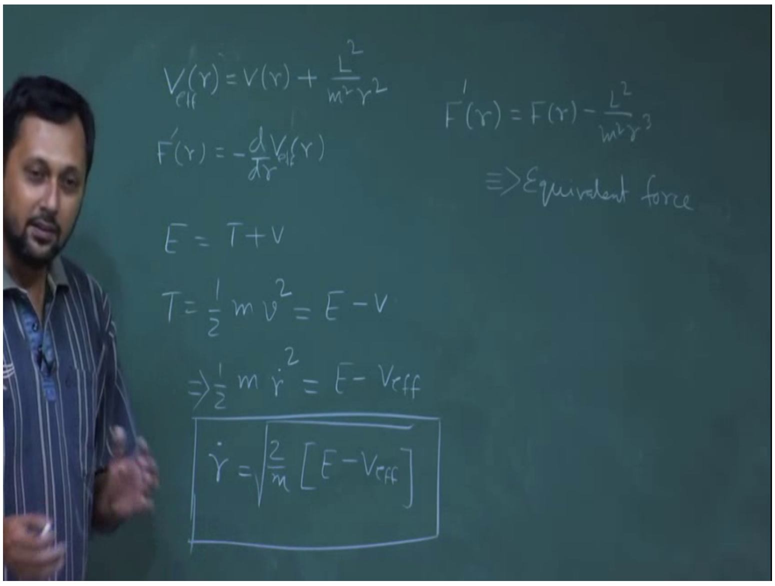

So, we can write an equivalent force term which we denote by F dashed r, please note that, this is not the derivative it is just a representation of the entire right hand side. We can write this as F r minus L squared by m square r cubed. So, that is the, this is nothing, but the equivalent force right. Now, if there is an equivalent force present there will also be an equivalent potential present .

So, if we derive, if we write an equivalent potential V of V dashed of r, which will be V of r plus L square by m square r square. Now, how is this related F of r or rather F dashed of r will be minus d dV, because it is a single variable function dV of r check that the first term will give you the standard force minus dV, dr will be the first term which is the standard force term due to the presence of central field.

Second derivative, the second term under if you perform this derivative will give you L square by m square r there is a minus term coming up and it will be r cubed which is exactly equivalent to this term. So, now, for a moment just go back to this one particular equation e is equal to T plus V. So, T which is half m V square is equivalent to e minus V right.

This V square will have one radial and one transverse part associated with it and we have already seen that the transverse, if it we only take the radial part that is r dot square that will be equal to this particular force term. So, we can substitute for this one and we can write this as half m v r square which will be equal to e minus v effective.

So, we are just, what I am doing is in this kinetic energy term. There will be a contribution from the radial part and one contribution from the transverse part. We have already seen that, if we can rearrange the transverse part, because that has a theta dot term associated with it, to give this particular type of force. So, what we are doing is we are just writing this v r which is nothing, but r dot square equal to this.

So, from this we see r dot is equal to 2 by m e minus v effective, v effective being this or actually, we can write this as v effective this will be easy to visualize. So, r dot is equal to root over of 2 by m into e minus v effective whole root. I said root over once that is why.

Now, if you look at this effect I mean if you look at this particular form of equation, does not it look like a equation of motion for a velocity component for a one dimensional motion, as if the motion is, if you recall this v effective is also a theta independent term v effective the first term is the standard potential, which is already theta independent by definition, the second term is this L square by m square r square.

So, we have included all the theta dependence into this particular term and we converted the theta dependence into an r dependence term. So, as if the motion under central force field can essentially effectively be described by this 1 d description of the problem, where the only velocity is only variable is r, which is changing as a function of t giving rise to velocities and acceleration and the motion is taking place in an effective potential which is given by this particular expression.

Now, right now, we are not specific about the form of vr, I am saying, I am making a statement for a very general force field, which has a very which can have any form of potential, which is given by this particular term. So, let us be general and try to see. So, right now, we are not sure about the functional form of v effective, we know that one term will be L square by m square r square.

Now, this term if you examine carefully at r equal to 0; this term is divergent and r equal to infinity, this term gives you 0. So, whatever may be the may be the functional form of vr, we do not know a priority. There are possibilities that the effective potential will look something like this.

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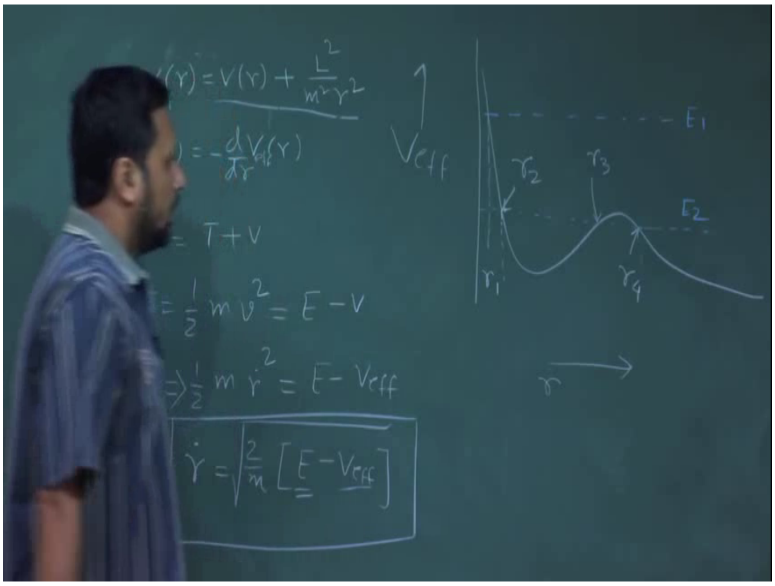

So, this is v effective and I am drawing r here, please understand I am not specifying 0 energy level here right. Now, I am just trying to be very general. I am not specifying 0 energy level.

So, I am not drawing the vertical x axis. I am just marking this as y, y axis as v effective and I know that we know that r is being plotted here right. It as I said, it can have this particular form. It can also have very different types of form; we will come to that in a moment. Now, let us assume that the particle, the total energy e of the particle, which is also a constant of motion.

So, once if we choose the initial condition, I mean a set of initial conditions, which will give a one specific value for energy. We can always find another set of initial condition that will give a different value of energy e. So, e is something, that in principle can be controlled from outside and that depends a lot on how the thing starts, because it is a conservative force field. Once you set energy, the initial energy, the energy will remain constant throughout the motion.

So, let us assume a situation, where we have set energy at this particular level. Let us call it e 1. Now, correspond, I mean this energy, I am just taking the intercept of energy, this energy value with this effective potential, the point we call it r 1. Now, let us assume another energy value, let us say, we lower the energy value to a level which is given by e 2.

Now, this interception, let us say the r value is r 2 for this one. Let us say, it is r 3, let us 1, this 1, let us say it is r 4. Now, for these 2 cases, for energy e 1 and e 2; let us try, let me try to give you a brief account of how the motion in a general, I mean in this force field , how could be the motion a general description.

Examine the case, e equal to e 1 energy, sufficiently high; what I mean by sufficiently high is, see there are some ups and downs in this potential, but the particle which comes with an energy e 1. Does not see all this, the energy is too high. So, energy stays here only time. So, essentially what I am trying to tell you is, if energy is at this particular level then, this particle does not feel that, there are ups and downs in this potential. So, it directly comes to the close to the force center r. This straight line, this vertical line, essentially marks r equal to 0.

So, its r equal to 0 means it is a you know is the origin of the force with this energy tries to come close to the origin of the force and essentially it you know it I mean it intercepts this effective potential at the point r equal to r and because the potential beyond below r equal to r 1. The potential is higher, the vertical cannot come closer than r equal to r 1.

Now, what happens afterwards, afterwards it will. So, the motion of this particle with energy equal to e 1, will be between bound between r 1 and infinity. It can go anywhere without any bound, because assuming that the potential stays in this particular level, it will not have any further interaction with the particle will have any no not have any further interaction with this particular potential, except for this point. So, and beyond this point, it cannot, the particle cannot come closer to the force center than r 1.

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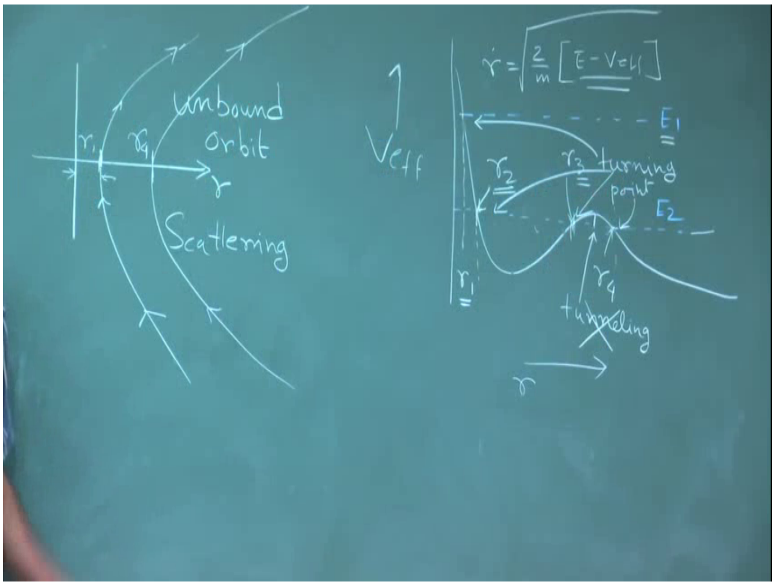

So, if we try to draw a very quantitatively the trajectory of the particle, this is my r axis and the trajectory will look something like this. So, the particle comes from infinity. The shortest distance of approach becomes, this distance which is given by r 1 and then once again it moves away from the force center to infinity.

Once again I am not specifying the shape of this curve. It could be I mean, we do not know the law of force at present. So, we cannot comment on this particular type of orbit, but all I can tell you that this is an unbound orbit. So, the particle essentially comes from infinite distance, the shortest distance of approach is r 1 and it again goes turns back and goes to infinity.

So, it is an example of unbound orbit and these particular phenomena is called scattering. Scattering in central force field is a very well studied and very well observed phenomena for this particular course. We are not going into the very deep of scattering phenomena, because it is slightly advanced topic, we have to consider many other parameters for that.

But I am let me tell you that the first few experiments to which was you know, path finding experiment to determine the structure of atomic structure and all. So, these are all scattering experiments, rutherford experiment was the scattering experiment stern gerlach I mean not stern gerlach exactly, but rutherford experiment was a classical example of scattering experiment.

So, this is one situation, when we can have what we call scattering, what happens for energy equal to e 2, what happens is, please understand that there are 2 very distinct regions, one region is if the particle comes from infinity at a starts coming from infinity, then with energy e 2, then it can come only as close as r 4, from the fourth set not closer than that, because if it comes from infinity at r equal to r 4. It will have an interaction with the potential and beyond r 4 potential goes upwards.

So, it cannot cross this barrier, because it has an energy e 2, which is not sufficient to cross this height, you know jump, this height the energy will be I mean, it will, it cannot come if it comes from this side, it cannot come closer than r 4. So, if the particle comes from r infinite distance, then it will once again be a scattering, but this time instead of r 1, we will have r 4 here, but it will nonetheless. It will be a unbound orbit and we will we will see another scattering.

What happens if somehow the particle ends up in this domain between r 2 and r 3. Now, how does it go here we are not sure about it, maybe it starts the initial condition is such that, it starts somewhere in this between, this r 2 and r 3 2 bounds that is the possible explanation for this, but whatever it may be once, it is inside this well potential, well with energy e 2; that means, it will have two interceptions r 2 and r 3, it cannot go beyond that.

Please understand that, we are discussing classical picture. Here, if it is a quantum mechanical picture, quantum mechanical in quantum mechanical picture, tunneling is allowed, I think you are all familiar with this term. You have, we have all studied particle in a box particle in a finite potential wall and all. So, there, we have seen for if the potential is finite, then we can have tunneling, especially in this type of cases, but because it is, a classical picture tunneling is not allowed. So, there is no tunneling. So, either the particle stays beyond r 4 or it stays between r 2 and r 3.

Now, if it stays between r 2 and r 3. These 2 are the turning points and if you look back to this equation, which was written as r dot equal to 2 by m e minus v, effective hole root at this two points, actually not only at these two points, any point, where e and this potential has an interaction this particular term becomes 0. So, that is why r dot becomes 0 and r dot becomes 0 means as I discussed during the discussion of apsidal point, these are the turning points.

So, that is why, we have turning point means where the velocity radial component velocity vanishes and the particle goes from changes direction from this to that. So, this is the turning point, similarly r 3 and r 4, all these are turning points. So, how many turning points we have in this picture? We have 1 turning point, which is given as r 1, another turning point which is given as r 4 and we have 2 other turning points one is at r 2 and one is at r 3 right.

So, we have now, let us focus only on this particular case, when the particle is bound between the limits r 2 and r 3, that is to say, there it is bound to move within these two turning points. What happens for to, it is motion. We have seen, already seen that the unbound orbit, which is, which results in a scattering with the turning points either r 1 or r 4, depending on the energy. What happens for bound state is we have the particle keeps moving between two circles with radius r 1 and r 2.

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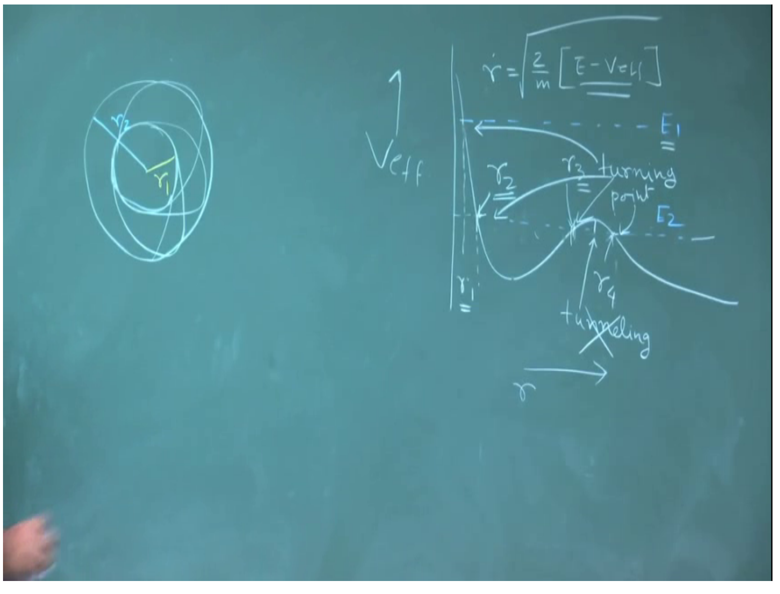

So, let us say this is my. So, this radius is r 2 and this radius is r 1.

So, the particle will move between this two in this manner. So, I do not think it is a very clear picture. I will try to draw it once again. Please excuse me for this. So, we have r 1 and we have r 2.

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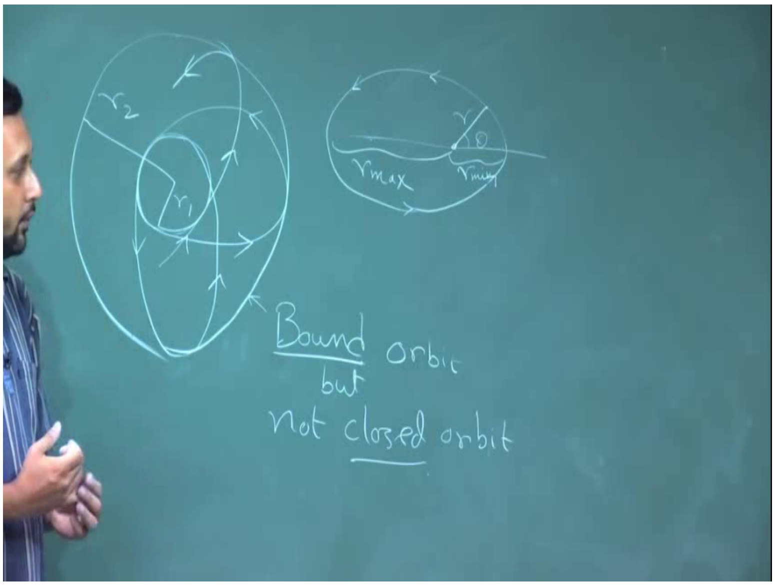

I will just draw r 2 a slightly bigger, that will help us drawing it clearly; so r 1 and r 2.

So, if the particle goes like this, then it comes back like this, goes like this, comes back like this. So, if it goes like this, comes back, goes back like this. So, essentially the motion is confined between two radius two circles of radius r 1 and r 2, but please recall that r 1 and r 2, although it is a bound state bound orbit. It is not a closed orbit. So, this is a bound orbit or, but not closed orbit.

So, this is an example of a bound orbit, but this is not a closed orbit. What is meaning, what is the meaning of bound orbit? Bound orbit means, where the maxima and minima, minimum distance from the center of force is fixed. It cannot go beyond the maximum distance or it cannot come closer to the minimum distance.

Now, what is the meaning of a closed orbit, a closed orbit is the orbit that traces itself over and over again with time. So, that what I mean is now, if we drop this general description and try to describe the, a bound orbit under the action of inverse square law of force, then we know from our previous discussion, that the only possible bound orbit is an ellipse of course, circle which is a circle, is also possible which is also is a special case of ellipse.

Now, in an elliptical orbit what happens is force center is one of the four side one of the focus either this or that and this is my r this is my theta. So, the particle keeps moving on this elliptical orbit, over and over again. So, this is my r min and this is my r max. So, this two are the bounds for this particular orbit r min and r max and at the same time an elliptical orbit, under the action of a central inverse square central force field is also a closed orbit, because it remains the same over time. So, it remains, I means it traces itself back over and over again with time.

So, that is why in planetary motion, each planet will come back to it us previous position, exactly after a certain period of time of course, it is again, it is only the first approximation, it is a closed orbit, it is not exactly close, because we probably know that you probably know that from modern day astronomical observation, it is seeing that each planet the orbit is deviated every year or every single evaluation, the orbit is deviated very slightly, that is due to all other non-linear effects. All other forces that is acting on the orbit, but we are not bringing that into question from purely classical calculation point of view. This elliptical orbit is a closed orbit, because it is not only bound between these two limits, but it will trace itself over and over again.

So, we have seen that in general, in a central force field, we can have a scattering motion or we can have a bound motion, which is, which will provide a bound orbit, but not a closed orbit. Now, tomorrow or in the next class, what we are going to do is, we are going to focus only on inverse square law force and see; how does the equivalent potential picture looks like. And we will try to describe the motion in central force field, in as per the equivalent one, dimensional motion picture.

Thank you.