Classical Mechanics : From Newtonian to Lagrangian Formulation
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Lecture - 16
Central forces - 9

So, we continue our discussion on apsidal points. First of all let me correct spelling mistake.

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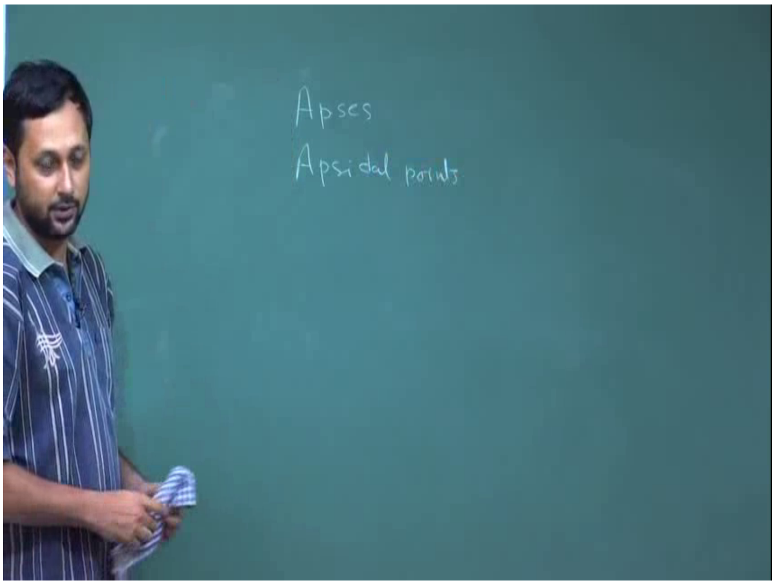

I wrote apeses, actually it will be apses not apeses. So, instead of apsidal point, we will call it apsidal, apsidal points, but the spelling for apogee and perigee was correct ok.

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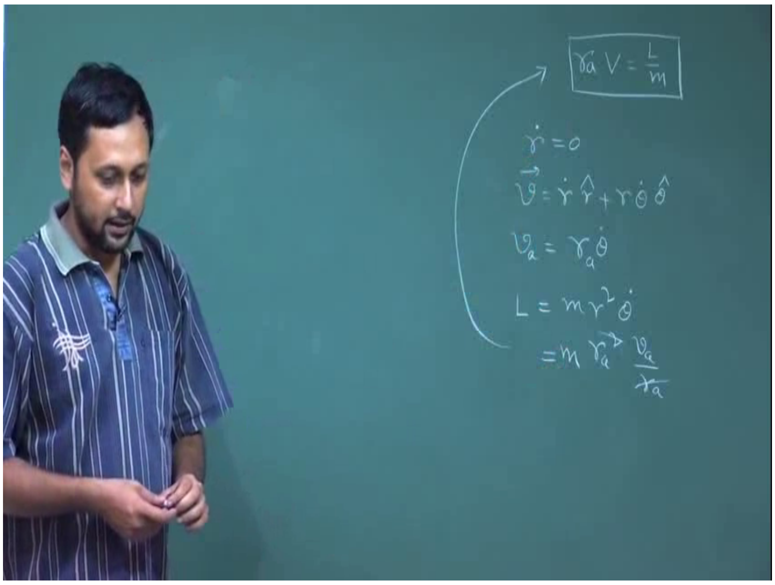

So, we have derived this relation that r a times V is equal to L by m, we wrote it. I am writing it here, because we will be needing it during the calculations we are going to do now. So, I will just want to keep the board empty.

Now, there is an alternative proof to this, what we did was. We started with a differential equation. We set d theta d u d theta equal to 0, and got this relation also. Please remember that for an apsidal point r dot is also equal to 0. So, that is; that means, the velocity V which is r dot r cap plus r theta dot theta cap at an apsidal point V a becomes r a theta dot theta cap only. So, the magnitude also is simply r a theta dot.

Now, the angular momentum which is given by m r square theta dot, so for theta dot we can substitute V a by r a, if we do that at an apsidal point, the expression becomes r a square into V a, divided by r a. So, this will again gives back the same relation just by. I mean you know this one will cancel out and rearrangement will give us the same relation.

So, we have seen that these are the, this is the relation which we can prove it in two or three different ways and. So, the significance of this relation is at a point, extremum point of the orbit, where r becomes the minima or maxima, we have velocity component which is totally perpendicular to the direction. I mean to the radial vector, that joins that particular point to the origin, and that essentially means that an apsidal point theta is equal to 0 or pi or any other symmetric for the symmetric angle right.

Now, with this background, let us move on to do some problems.

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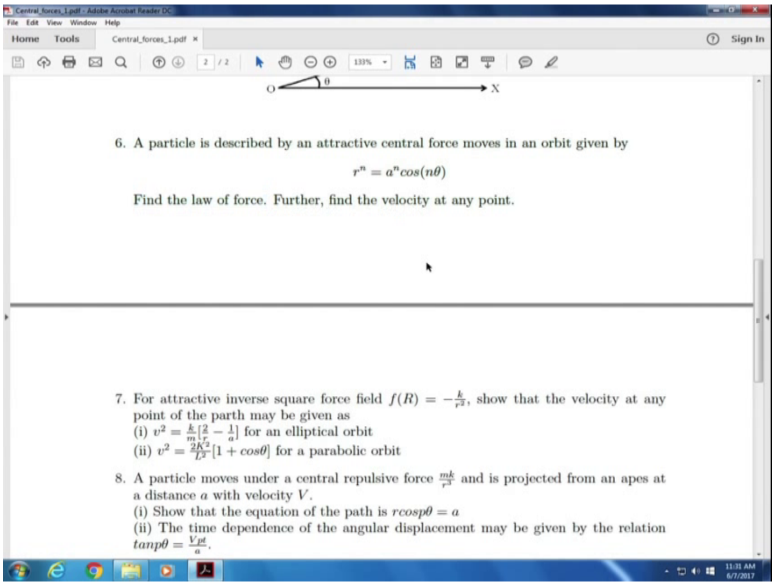

So, we start again by problem number 6, problem number 6 is here; the orbit is given in terms of this equation, which is r to the power n at equal to a to the power n cos n theta. We need to find the law of force, and also we have to find the velocity at any point of the orbit right. So, it is given that r to the power n is equal to a to the power n cos n theta, do not be scared by this form.

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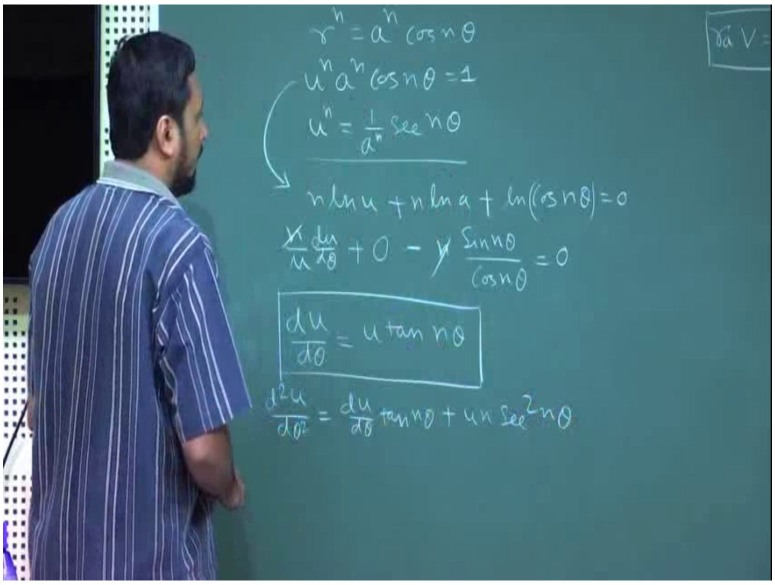

I mean it is not very simple form I agree, but will see that it can be reduced, the calculation can be reduced to a very simple form. So, what we do is, we write u to the power n a to the power n cos n theta equal to 1 right, and; that means, u to the power n is equal to 1 by a to the power n sec n theta right.

So, I mean we will use this later on, let us take this and take logarithm of this equation. So, it will be n log u plus n log a plus log cos. So, we have taken, we are taking log in the base of e. So, it is log. Actually log u plus log a plus log n theta equal to 0. If u, if we take derivative with respect to theta from of this equation, then we get n by u du d theta plus the second term, will become 0, and this term become minus n sin n theta by cos n theta, which is once again equal to 0.

So, this will give you 1 log of cos n theta will give you 1 over cos n theta, derivative of cos n theta will give you minus n sin n theta. So, from this we can write. So, you see this n n will go off. We can write du d theta is equal to u tan n theta. So, this is one equation which is very useful.

Now, we have to compute the second derivative of u. So, we write d 2 u d theta 2, which will be equal to du d theta tan n theta plus u tan n theta will give you a derivative of sec square n theta right. Now if we substitute this whole thing back into the equation. So, we just need this two expressions right du d theta and du d theta 2, and of course, we need to use this form once again we will see soon.

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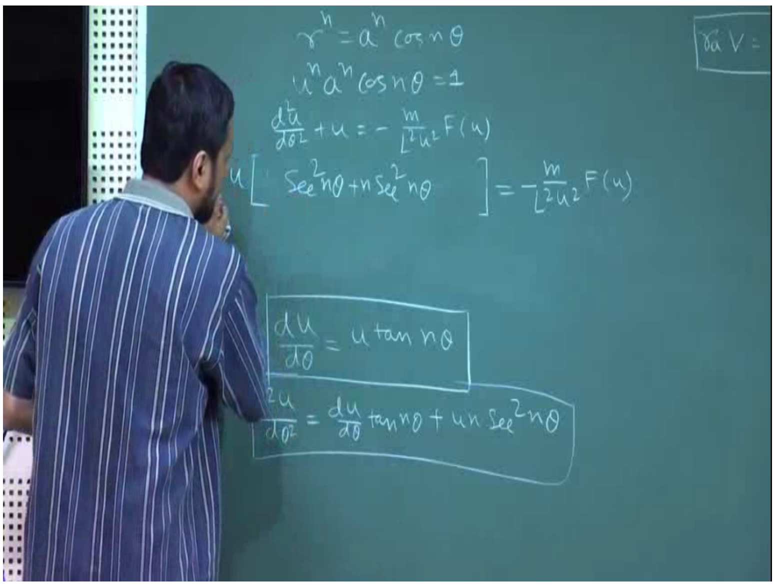

Now, if we substitute this thing into the equation that d 2 u d theta 2 plus u is equal minus minus n by L square u square F of u right. Now if we substitute this once again we have to substitute du d theta in this equation, which will give you u tan square n theta. So, this equation left hand side is u tan square n theta plus un sec square n theta plus u is what.

So, we have to substitute u here right. You have to just keep it like this minus m by L square u square F of u. Now if we take what do we do now? We have to take u common from all this expressions to that one here. Now you see tan square n theta plus one will give you sec square n theta right, because sec square theta minus tan square theta is equal to one.

So, tan square n theta plus 1. We can get rid of this, and we can substitute it with sec square n theta plus n sec square n theta. So, all together we will have n plus 1 sec square n theta times u.

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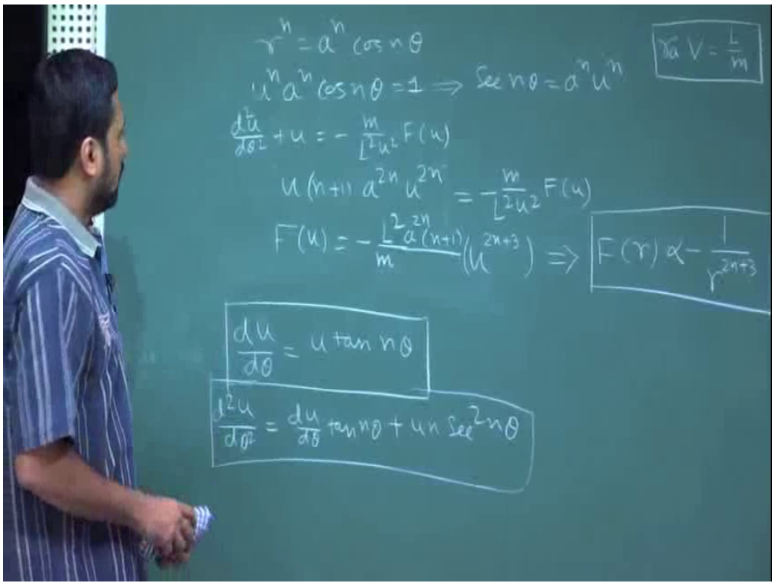

So, this whole thing reduces to u times n plus 1 sec square n theta, now let us. So, I removed it sorry. So, once again if we look into the expression for sec and theta from this relation sec n theta is simply a to the power n u to the power n a to the power n to the power n. So, if we substitute it here, we immediately get a to the power 2 n u to the power 2 n right.

So, sec square n theta sec n theta is this. So, sec square n theta is a to the power 2 n u to the power 2 n, if we. So, we have u square, sorry you have u to the power 2 n u and u square. So, altogether if we change sides of this, we get L square by m times a to the power 2 n n plus 1; this whole thing is a constant. So, we do not worry much about it, u to the power 2 n plus 3 right.

So, that is my final expression right 2 n plus 3 right. So, if this is my final expression I can tell you that F of r is proportional to minus 1 by r to the power 2 n plus 3. So, this is one answer. We have now for the second part, we have to find out the velocity expression. The velocity at any point of the orbit as we have focused, we do not need this anymore, and also the second derivative we do not need anymore, because the first derivative I mean for velocity expression we only need the first derivative.

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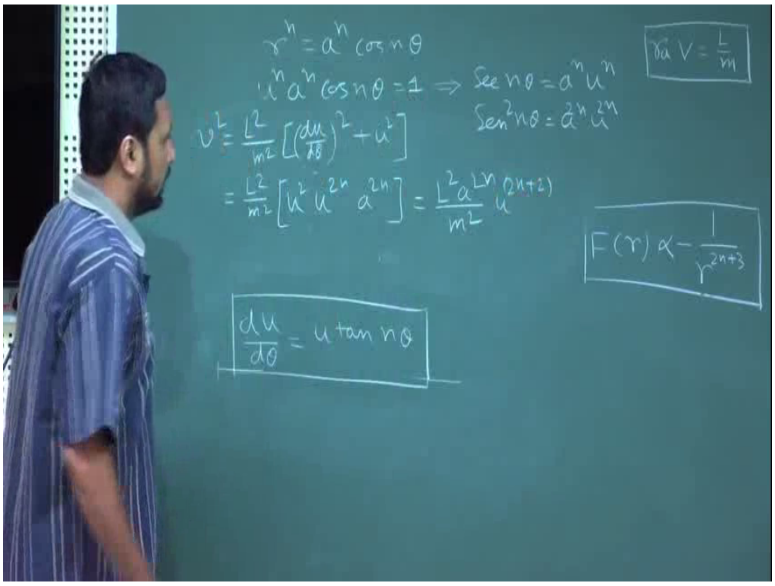

Now, V square is L square by m square du d theta whole square plus u square. Please remember that this is not, this force is not inverse square force. The force we have got just now. Its not an inverse square force. So, we cannot use the expressions we have derived for either parabola or ellipse. So, what we need to do is, we need to find out expression for velocity V from this relation. So, if we substitute here du d theta whole square.

So, we have to have, we will get u square inside bracket tan square theta plus 1 tan square n theta tan square n theta plus 1. And once again just like we did before, we will substitute this with sec square n theta. Now sec square n theta from this we did already, but let us do it once again a to the power n a to the power 2 n 2 n.

So, sec square n theta is u to the power 2 n a to the power 2 n. So, final expression L square a square. Sorry this is a to the power 2 n by n square, not a square it will be u to the power 2 n plus 2 u to the power 2 n plus 2 ok.

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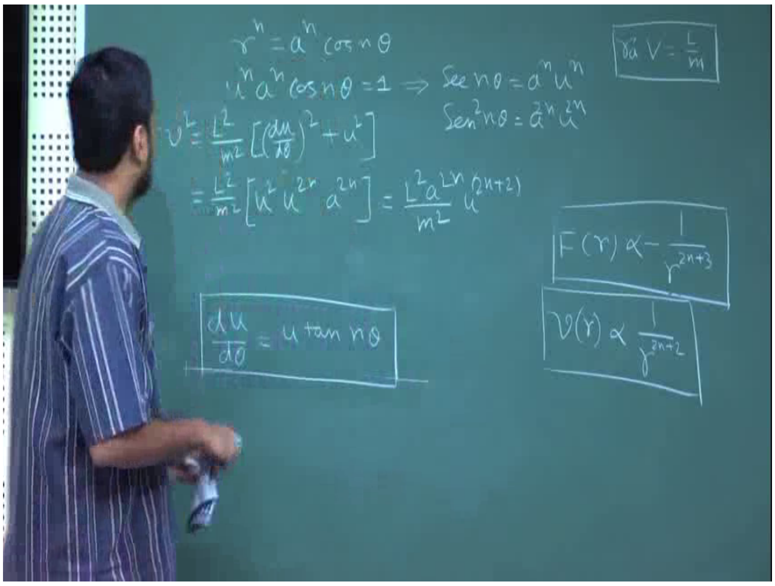

So, velocity as a function of r is proportional to 1 by r to the power 2 n plus 2 right ok.

So, this is the answer. So, if we start with this orbit, we see that the force law is given by this, and the velocity at any point of the orbit is given by this particular expression, which is proportional to 1 by r to the power 2 n plus 2. So, this solves our problem here.

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Now, let us move to the next problem. Next this we need this, specially for the next problem.

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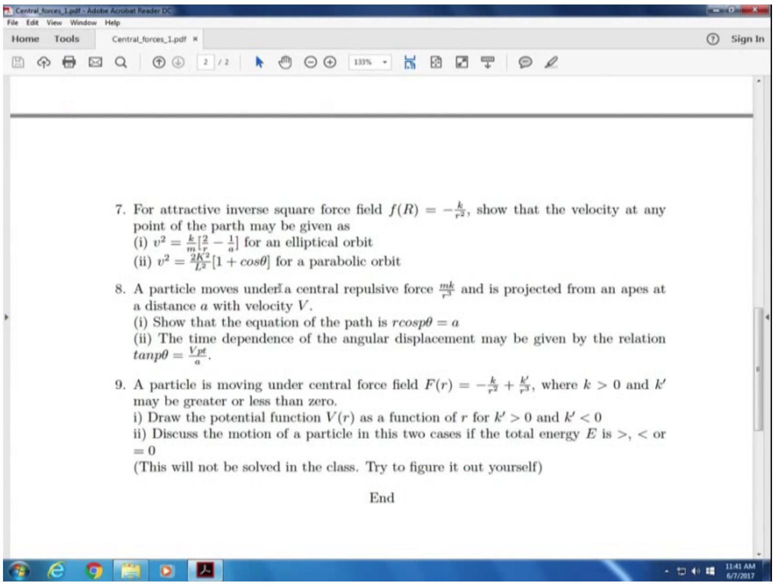
7. For attractive inverse square force field $f(R) = -\frac{k}{r^2}$, show that the velocity at any point of the parth may be given as
(i) $v^2 = \frac{k}{m}[\frac{2}{r} - \frac{1}{a}]$ for an elliptical orbit
(ii) $v^2 = \frac{2K^2}{L^2}[1 + cos\theta]$ for a parabolic orbit

8. A particle moves under a central repulsive force $\frac{mk}{r^3}$ and is projected from an apes at a distance a with velocity V.
(i) Show that the equation of the path is $rcosp\theta = a$
(ii) The time dependence of the angular displacement may be given by the relation $tanp\theta = \frac{Vpt}{a}$.

9. A particle is moving under central force field $F(r) = -\frac{k}{r^2} + \frac{k'}{r^3}$, where $k > 0$ and k' may be greater or less than zero.
i) Draw the potential function $V(r)$ as a function of r for $k' > 0$ and $k' < 0$
ii) Discuss the motion of a particle in this two cases if the total energy E is $>$, $<$ or $= 0$
(This will not be solved in the class. Try to figure it out yourself)

End

What is the description? what is the statement of next problem? Next problem says a particle moves under the central repulsive force proportional to mk by r cubed, and is projected from an apes at a distance a with velocity v. So, ra equal to a, and the velocity is given by V, show that the path is given by r cos p theta equal to a. So, this is slightly wrong here. I have not given an expression for p here, but. So, we will see what is p actually and the time dependence of the angular displacement may be given by the relation tan p theta equal to V pt by a. Once again there is a p here and p here.

So, we first have to. Well I have not given you what is p, but when we start the problem we will see that p is actually given by 1 plus k by V square a square V square. So, we will see that. So, what is the description of the problem we have a force.

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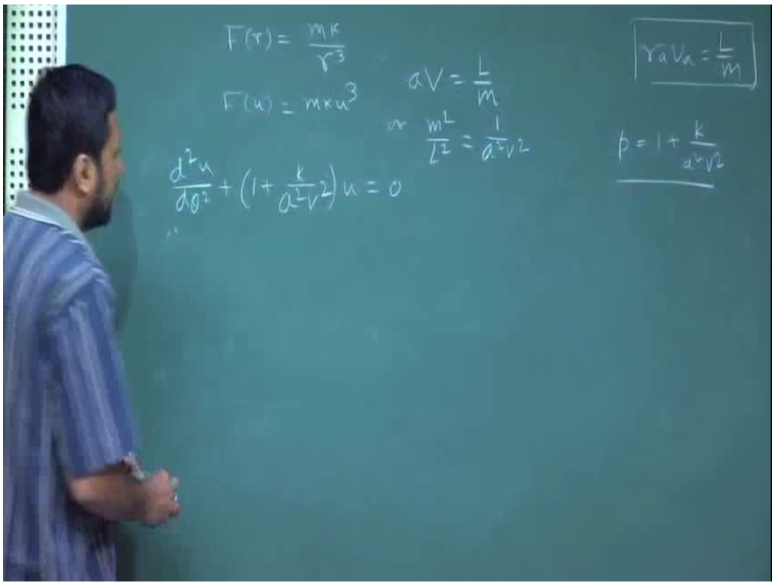

Which is given by F r is equal to s is the repulsive force. So, it will come with a positive sign. So, m k by r cubed m k by r cubed right. This is the force then F of u is mk u cubed right.

Now, if we put it back into the differential equation d 2 u d theta 2 plus u equal to minus l. Once again I keep forgetting this particular expression m by L square u square F of u. If we do that, we see we have m square k, and you have an u cubed here which will out of that u square will cancel out and you have this.

Now, taking it to this side we can write the differential equation as well. I will just modified here m square k by L square m square k by L square times u equal to 0 right. It is also given that the particle was projected from an apes apsidal point, which is at a distance a with the velocity V. Now using this relation we see that a times V is equal to L by m. See if we substitute this for and. So, or m square by L square will be 1 by a square b square.

So, if we substitute this for m square by L square, this relation becomes k by a square b square right, which is exactly the expression of p which is given here. So, we replace this substitute, this with p and we write the equation d 2 ud theta 2 plus p of u equal to 0.

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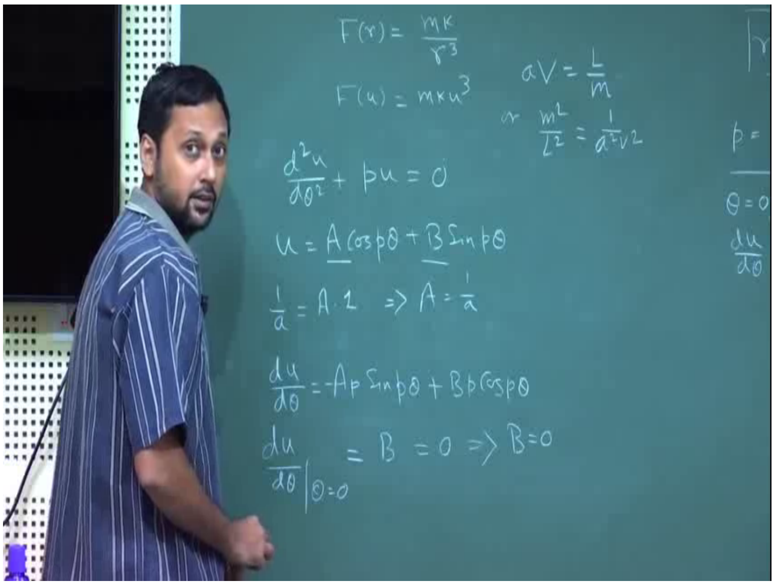

So, this is an equation in a very familiar form, and this equation has a solution u equal to a cos p theta plus b sin p theta A and B are two constants, which has to be determined from the initial condition.

Now, what are the initial condition again, once again this is the initial condition which is given to us A is the velocity. So, A is the velocity sorry A is the distance of initial projection V is the velocity at time V is the initial velocity, and as we know for a apses is we can safely consider that it happened at theta equal to 0.

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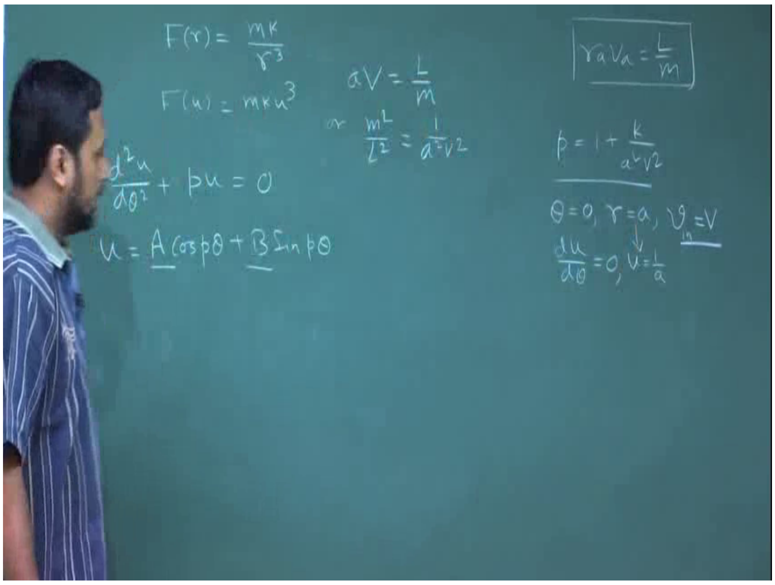

So, the initial condition is theta equal to 0 r equal to a, and V is the initial velocity and at an apses the initial velocity is ok.

So, the initial velocity we will just call it v, initial which is equal to V right, and also we call that in an apses du d theta equal to 0 right. So, we need two initial condition. We essentially have three initial condition, we have a theta equal to 0 r equal to a du du d theta equal to 0 and V v initial V initial is given by V right. For the second part of the problem we will need this one right. Now we just need at theta equal to 0 du d theta du d theta equal to 0, and u equal to 1 over a that is just from this relation right.

Now, if we put u equal to 1 over a and theta equal to 0. We see that the solution of this equation gives a u equal to 1 by a. So, if you put theta equal to 0, this term goes and this will give you 1, which will give you a equal to 1 over a. Now if you compute du d theta which will be Ap sin p theta plus Bp cos p theta du.

So, theta equal to 0. Once again this term will vanish, which will give you B equal to 0. So, first relation gives A equal to 1 over a second relation gives B equal to 0. So, your solution to this equation is the final, final solution to this equation becomes including all these condition, the solution becomes 1 by a cos theta right.

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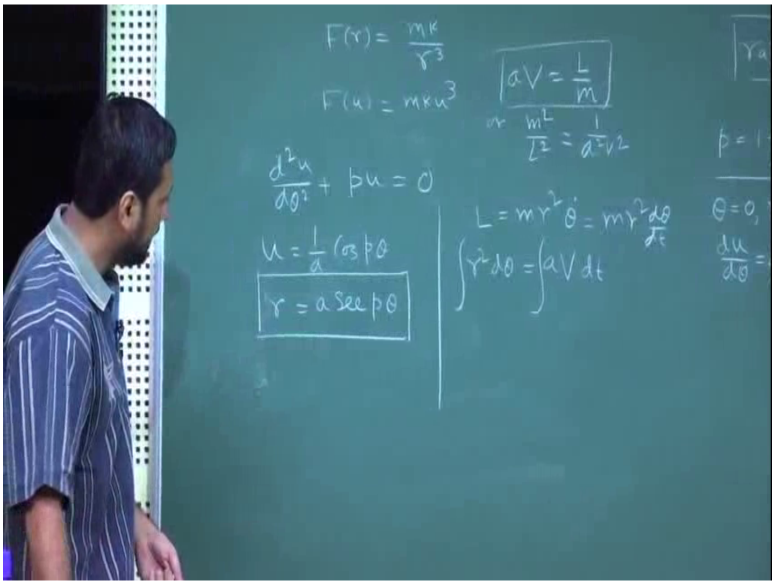

Write it in terms of r and you will see r equal to a sec p theta right, and what is given is the solution. So, r cos p theta equal to a, it is the same thing eventually.

So, whatever I have got, here is what is given in the problem. Now for the velocity part once again we have to compute the velocity by using this relation, that. Oh sorry its the angular velocity, we are looking for just a minute. So, its the angular velocity. So, for angular velocity we do not have to, we just have to use this relation that L is equal to mr square theta dot ok.

So, theta dot essentially its means mr square d theta dt right. So, rearranging we get r square d theta is equal to L by m dt integral L by m has to be replaced by a times v. So, do the substitution by this relation right, and r has to be replaced by this equation. So, r square will be a square sec square p theta p d theta right.

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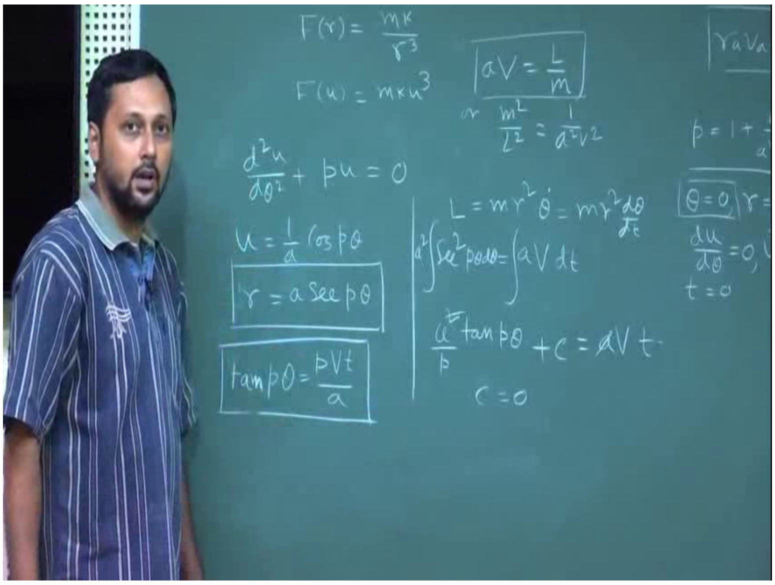

Now, if you integrate this part you will get a square tan p theta, there will be a 1 by p coming plus some constant equal to a times V times t right. No actually we do not need this relation explicitly, because the first relation itself is sufficient that theta equal to 0 and. So, we assume that the projection was has taken place initiate at the initial point.

So, at initial point theta was equal to 0 at t equal to 0. So, if we put that in this relation we see, we put theta equal to 0 p t equal to 0, we get c equal to 0. So, the final relation what we get from here is, tan p theta is equal to. So, there is 1 a that will survive pvt by a, and this is exactly what is given as the solution ok.

So, we solved problem 6 and problem 8 of this. There is a problem 9 which will be. Well I said that it will not be solved in the class, but we can discuss it briefly later on now, but there is one more problem, which I want to do in the class, which is the statement of the problem is as follows. Just give me a second I will find. So, we have one more problem to solve in this class, and the problem is, there is a particular orbit given, and we have to find out the law of force.

Now, the orbit the equation of the orbit is not given, why I am doing this? There is a reason for it, which you will realize once we solve the problem. There is a significance to it. The problem is under the influence of central force. Please remember it is not in the problem set.

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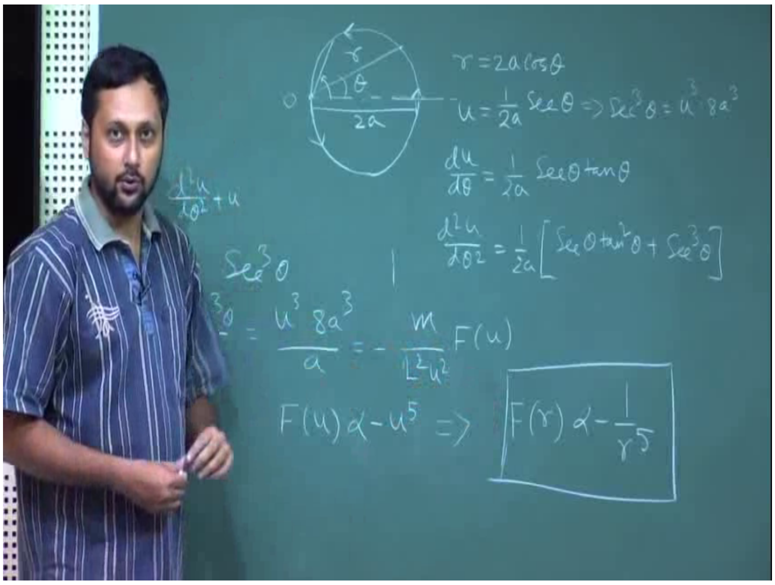

So, I will, you have to listen carefully under the influence of central force, the particle is moving in a circular orbit; such that the circle during its path, it passes through the center of force.

So, circular, typically for circular orbits we assume that the force center of force is at the middle right. So, we set our origin in the middle, but this is not the case here, this is the orbit. Let us say its moving in this direction at some point here, this is the force center. So, your origin is here, and this is your axis. So, this much is given, and we have to find out the law of force.

Now it is a straightforward problem, we do not need the concept of apsidal velocities. We do not need the concepts of you know velocity of the orbit, why we are doing it, because it has some other significance to it. First of all what is the equation of this orbit. Please recall, if the origin is at the center, then the velocity I mean the. Sorry the equation of a circle is r equal to a right.

If the origin is at the center, but in this case origin is shifted here. If a is the radius then this length is to 2 a right. So, we can write the equation of the circle as r equal to 2 a cos theta for any point. This is my r and this is my theta. What happens is a. What happens at r equal to 0 at r equal to. Sorry theta equal to 0. At theta equal to 0 that essentially means this particular point right.

So, this particular point means, sorry theta equal to 0 means this particular point, and that this particular point we have r equal to 2 a, when we put theta equal to 90 degree, which is. Please understand this closer and closer we are coming to this particular, I mean coming to the orbit, coming to the origin theta goes on increasing, and the limiting value is theta equal to 90 degree, at theta equal to 90 degree we have r equal to 0. So, this is the equation we are looking for.

Now, if this is the equation, then u is equal to 1 by 2 a sec theta right. From here if we go on with our calculation, we have to calculate the first derivative du d theta is equal to 1 by 2 a sec theta tan theta d 2 u d theta 2 is equal to 1 by 2 a sec theta tan square theta, and sec cube theta right. I will sec theta tan square theta sec cube theta right.

So, d 2 u d theta 2 plus u is equal to 1 by 2 a sec, just write it in the next line; so this term plus this term. Now from this, if we add these two terms, and if we take sec theta common from the first, this term and this term, then we have sec theta into 1 plus tan square theta right plus sec cube theta right.

Now, 1 plus tan square theta is once again sec cube sec square theta multiplied by this. It will be a sec cube theta adding to this. So, we have a sec cube theta here. There is one more here. So, we will simply have two times sec cube theta, which will be sec cube theta by a. now what is sec cube theta? From this relation we say sec cube theta is equal to u cubed into 8 a cubed right. So, put it in here. So, it will be u cubed into 8 a cubed divided by a.

Now, this whole thing is equal to minus L by m square u square F of u right, or sorry once again I think I messed up, it will be m by L square u square y m by. I always forget this expression anyway. So, there is an u square here, and there is an u cubed here. So, essentially the law of force F of u will be proportional to minus u to the power 5, which tells us that F of r is proportional to minus 1 by r cubed. Sorry r to the power 5 ok.

So, it is a problem which was not included in my the problems set I have created for the class, but still I wanted to do it, because this is a very typical example of force. Please remember this number. Please remember the power it is inverse r to the power fifth power which makes the particle go through the force centre, during its orbit. Now this result has a very. I mean this result has some significance in terms of effective potential which will be discussing towards the end of this course.

Thank you.