Classical Mechanics: From Newtonian to Lagrangian Formulation
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Lecture – 15
Central forces – 08

So we derived an expression for the eccentricity for central orbit especially for inverse square force law.

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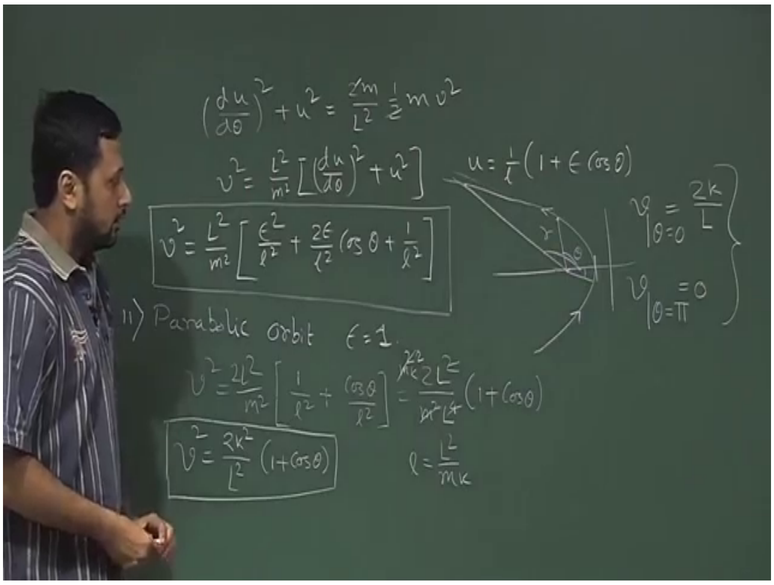

Now, we want to derive an expression for the velocity, now again we start with this equation which is du the d theta square plus u square equal to 2 m by L square e minus v, recall that e minus v is nothing, but the kinetic energy t which we can write as half m v square, v being the speed at the orbit at any point.

So, once we do this substitution we get an relation 2, 2 cancels out, we get a relation that v square is equal to e square is equal to l square by m square d u d theta whole square plus u square now. So, this is a general expression for any velocity for any central orbit.

Now, we want to see what is the specific form for a conic section; that means, if we consider the inverse square force field then we know that u equal to 1 by l, 1 plus epsilon cos theta. Once again we are setting theta 0 equal to 0 so; that means, our reference line is at already at 0. Now once we do that we see we if we substitute this into the equation once again we have to evaluate du d theta and then we have to do, we have to perform this summation of du d theta square plus u square and we will see that it will reduce to. I will just give you the final form because the similar calculation we just very recently did. So, I will just give you the final form of this, it will be epsilon square by l square plus 2 epsilon by l square cos theta plus 1 by l square.

So, this is the general expression for velocity in an inverse square force field; that means, when the path of the path of the particle is a conic section could be a circle, could be a parabola, could be hyperbola, could be an ellipse this is the general expression.

Now, for specific cases, we have to substitute specific values of epsilon, we know that epsilon has different ranges we will see that in little more details very soon.

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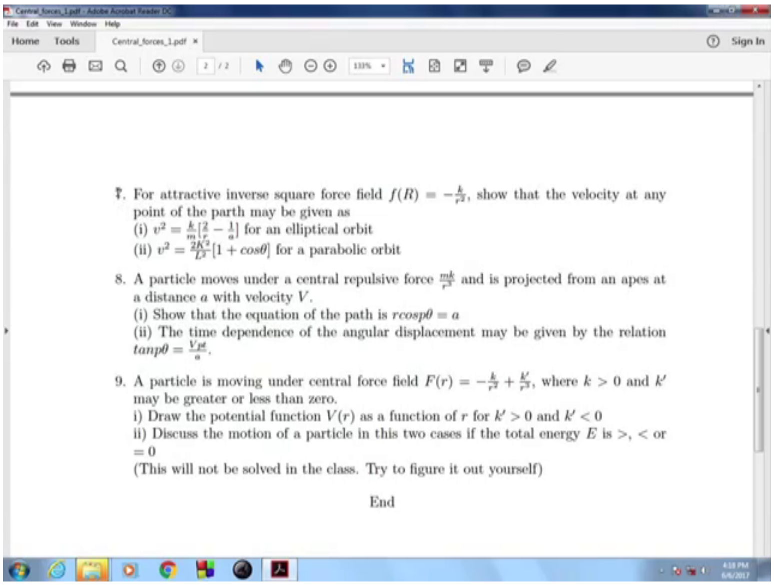
7. For attractive inverse square force field $f(R) = -\frac{k}{r^2}$, show that the velocity at any point of the parth may be given as
(i) $v^2 = \frac{k}{m}[\frac{2}{r} - \frac{1}{a}]$ for an elliptical orbit
(ii) $v^2 = \frac{2K^2}{L^2}[1 + cos\theta]$ for a parabolic orbit

8. A particle moves under a central repulsive force $\frac{mk}{r^3}$ and is projected from an apes at a distance a with velocity V.
(i) Show that the equation of the path is $rcosp\theta = a$
(ii) The time dependence of the angular displacement may be given by the relation $tanp\theta = \frac{Vpt}{a}$.

9. A particle is moving under central force field $F(r) = -\frac{k}{r^2} + \frac{k'}{r^3}$, where $k > 0$ and k' may be greater or less than zero.
i) Draw the potential function $V(r)$ as a function of r for $k' > 0$ and $k' < 0$
ii) Discuss the motion of a particle in this two cases if the total energy E is $>$, $<$ or $= 0$
(This will not be solved in the class. Try to figure it out yourself)

End

Now, let us focus on the problems set, problem number seven is what we are trying to attend now the attractive inverse square force field fR is minus k by r square show that the speed at any point of the path may be given by v square equal to k by m 2 r minus 1 by a for an elliptical orbit and another expression given for a parabolic orbit. So, what we need to do is, we need to we need to introduce we need to take start from this particular expression and for the first part we have to put some, we have to use some properties of the elliptical orbit in order to get the expression which is given in the problem set and for the second part we can use the parabola, I mean separate set of parameters for the or separate set of modification to this particulars relation in order to get it for parabolic orbit.

So, we will start with the second one which because this is slightly more trivial. So, what happens in a parabolic orbit? So, we are doing the part 2 first. So, what happens in a parabolic orbit, we have epsilon equal to 1 just a minute then this is better right epsilon equal to 1. Now, if we put epsilon equal to one into this expression then we see that v square is equal to l square by m square. So, see it will if you put one here this expression will look alike similar and so it will add up to give you 2 by l square there is a 2 here. So, that 2 will come out and you see it reduces to 1 by l square plus cos theta by l square, also l square can be taken out and we can write it as 2 capital L square by m square small l square 1 plus cos theta.

Now, it is time to substitute for small l you remember this expression for small l is also for (Refer Time: 05:54) always forget this expression. So, it will be yeah it will be l square by mk capital L square by m k, if you do the substitution here then we see that it will be L; capital L to the power 4 m square k square right. So, it will be capital L to the power 4 you have m squared you have k square, m square m square goes we have 1 l square left and without surprise you get the expression which is given in the problem set as 2 2 k square by capital L square into 1 plus cos theta. So, exactly the expression which is given in the problem set is it. So, yeah right so it is fine so we are good here.

Now, if you look at it carefully so we can now if you try to I mean using this relation if you try to see; how does the speed vary along the orbit of a parabola. If the equation is l by r equal to 1 plus epsilon cos theta, then I have discussed already then this will be our focus right and your parabola will be something like this, right. So, this is your for any point this is your r and this is your theta or for any point here, this is your r and this is your theta things like that.

So, if you put theta equal to 0 that will correspond to this particular point and if you put theta equal to 180 degree; that means, you have to extend the radial vector to a point which is essentially which is at infinity. As the particle or as the object moves away from the force center sorry, if it moves away from the force center your theta becomes more and more and in the limiting case when r goes to infinity your theta goes to 180 degree.

Now, if you put theta equal to 180 degree you will get a velocity equal to 0 right and if you put theta equal to 0 you get a velocity v or v corresponding to theta equal to 0 which is given by it will be 2 into 2 4 k square by l square v square equal to 4 k square by l square. So, v will be equal to 2 k by l, 2 k by l of course, I am omitting the negative sign because speed cannot be negative velocity can be. So, this is at v equal to v at theta equal to 0 and v at theta equal to pi which is a point at infinity if you put it in here it will be one minus one equal to 0 right.

So, these 2 points which one is here and the other one is at infinity these are called the apsidal points for a for an orbit we will come to that come back to that in a moment.

But for now let us focus on the first part. So, we did the second part we have to now go back and do it for a elliptical orbit, all right same velocity expression, same everything for elliptical or now we have to.

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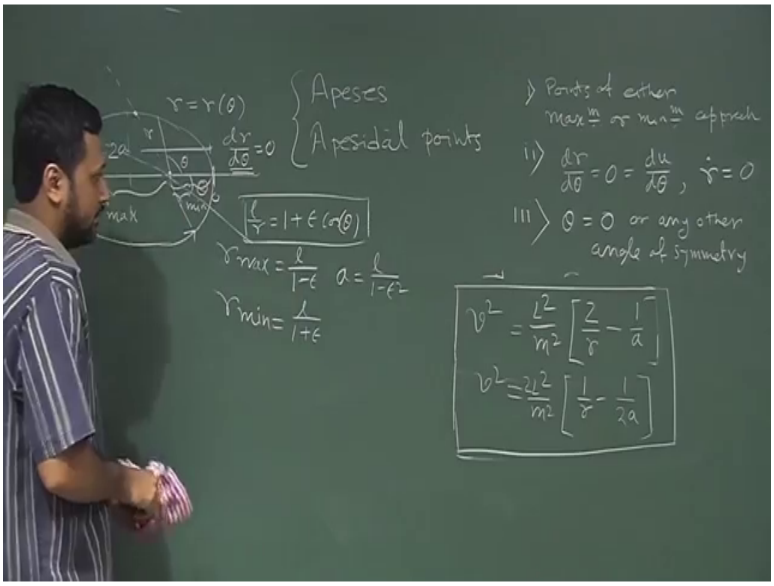

For elliptical orbit what we have to do is we need to rearrange this equation a bit and we can write it as v square is equal to I will just use this space here, want write the same thing do not have to write the same thing again I can just use the space here what we can do is we can put a 2 here right yeah if we take yeah. So, I can put this give me a second sorry not her, we can put a 2 here and it is just a mathematical manipulation and we can write it like this.

Now, this is equal to now if you take these 2 terms separately and take these 2 terms in another group in the first group you get L square by m square, 2 by l square into one plus epsilon cos theta. So, it is 2 by l square into 1 plus epsilon cos theta and from the second group it is plus 1 by l square epsilon square minus square minus 1 right. So, we are doing it for and now remember that we are doing this calculation for elliptical orbit now what happens for an elliptical orbit is we can if you recall for an elliptical orbit a the length of the semi major axis is given by l by 1 minus epsilon square, right.

Now, from the first bracket inside we call that l by r is equal to 1 plus epsilon cos theta. So, 1 plus epsi; we can substitute l by r for 1 plus epsilon cos theta, right. So, if we do that we get or we can just l by r and for the second one we can use this relation to substitute for now we just keep it like this and we have. So, one minus epsilon square is l by a. So, we can just put l by a; now it looks better, now if you simplify this I think, we are almost done see l, 1 l cancels here 1 l cancels here leaving behind 2 by r minus 1 by a. So, v square is equal to this for an elliptical orbit, right.

So, we are once again we got an expression of velocity which in terms of angular momentum m semi length of semi major axis, m being the mass length of semi major axis and the radial displacement at any point on the elliptical orbit. Now this result has a slightly more I mean slightly higher, I mean slightly more significant things hidden inside if you just do one more mathematical step and write this as 2 l square by m square 1 by r minus 1 by 2 a; right.

Now, So, v square is equal this, is the same expression we just wrote it in a slightly different manner also the same from the first expression also it is not very different now put r equal to 2 a in any of the expression either this or that you will immediately see that the velocity goes to 0, but in principle r can never be 2 a why because in an elliptical orbit how does orbit look, this is my elliptical orbit this is my origin let us set. So, this is first focus this is the second focus. So, this is minimum and maximum length of r. So, this length is r min this length is r max, this length from here to here is 2 a. So, maximum of r is r max which if you go by the equation is given by l by the equation once again is the l by r is 1 plus epsilon cos theta.

So, maximum of r is reachable if you put theta equal to. So, please remember for any displacement this is your r this is your theta. So, if you put theta equal to 180 degree you get the maximum value for r. So, r max is l by 1 minus epsilon r, min is l by 1 plus epsilon and if you recall a and r they are related by a sorry a and a and l they are related by a equal to l 1 minus epsilon square. So, r can never be l sorry r can never be a. So, what is the significance of this 2 a here? Significance is at r equal to we can consider this as a situation as if we if, we draw a circuit circle with the center at this particular focus and with the radius of 2 a. So, to a will be here the same to a will be somewhere here. So, it will be a big circle.

So, same way yeah so it will something like I do not know how good this circle is I hope it is good and it will be you have to finish it I mean close the thing, now think of it this way when your mass point is here on this big circle let us say this is at this is stationary here you allow it to drop it if you allow it to drop from this height there is a force center here. So, it will be attracted towards this force center, by the time it reaches, at reaches this position which is the location of this particular, I mean when the which is a valid location on the orbit it will attain a velocity v square velocity v which is given by this particular relation. So, the velocity at any point on this elliptical orbit is the same velocity of mass point will achieve if it is being dropped from any point on this big circle and ends up on the point, on a point on that or that that point of the elliptical orbit. So, this is the physical significance of this particular result, please keep that picture in mind because this is sometime very useful in understanding certain phenomena. So, we have discussed hopes; sorry, I need this; sorry, sorry, sorry, sorry this is r min is equal to 1 minus epsilon r or rather r max is equal to 1 by 1 minus epsilon and r min is equal to l by 1 plus epsilon.

Now, let us forget about this for a moment and we have derived the velocity expression for 2 types of orbit in inverse square force field, one is elliptical orbit which is where the velocity is given by this relation and one is the parabolic orbit which we did previously.

Now, for both the orbits we have seen that there are extreme points the points which is closest to the force center or the closest approach of the mass which is moving or the point mass which is moving on this particular orbit, it can come to a shortest distance from the force center and it can go to another longest possible distance from the force center. Now this 2 points where it the distance between the force center and the mass is either shortest or longest has certain names they are called the apeses or points for example, this r min and r max they are 2 apsidal points on an elliptical path right.

Now, this apsidal point has certain properties that we need to keep in mind apsidal points are points which are one points of either maximum or minimum approach that is one definition, second definition is these are as you see this orbit is given as a you know function of r; I mean r as a function of theta. So, when r goes through a minima or a maxima, we can immediately infer that d r d theta will be equal to 0, in these 2 extreme points and that is true in general for any orbit. Now apsidal point is something which is defined for any central orbit it is not specific to inverse square force filled orbits, but will be using mostly for inverse square force field I mean later on we will be making I mean we doing calculation based on inverse square force field.

So, we are focusing primarily on inverse square force field, but please keep in mind that for any orbit if the, you know what you call the particle goes through an apsidal points this relation will be valid. So, d r theta will be equal to 0 that essentially means d r theta will be equal to 0 same as d u d theta without surprise because they are related u is equal to 1 over r. So, this is another property also not only the spatial derivative, but also the time derivative of r will vanish here because if you think of it see when the particle moves when a particle moves on this particular orbit let us say we are just taking this as an example. So, at each point the velocity changes right, as over time each point the velocity I mean the value of radial vector changes and these are extremum points. So, not only the theta I mean theta derivative of r will vanish, but also the time derivative of r has to vanish because it is also an extremum point in the space time directory, either minimum or x maximum point, right. So, r dot will also vanish.

Now, what happens is the way we write our equation for this for this particular case the equation is l by r equal to 1 plus epsilon cos theta, but typically we have chosen to write our equation in a particular form where the reference line is the line where the apeses line what I what I am trying to tell you that if I have instead of cos theta, theta minus theta 0. That means, our axis system is not this, but this with an angle theta 0 here right, but we are not doing that we are assuming most of the time we are writing equations as you mean theta equal to theta 0 and when we do that we describe this angle theta with respect to a line from which the angle theta is measured. So, anything on that line is theta equal to 0.

Now by simple geometrical consideration we can understand that the apeses or apsidal points has to lie on this particular point, this particular line because maximum and distance of maximum and minimum approach will take place for either theta equal to 0 or theta equal to pi or some symmetric position and. So, that symmetry can be achieved if and only if the apeses lie on this particular point.

So, the third important conclusion is for apsidal point theta is equal to 0 or any other angle of symmetry. So, I mean 0 pi 2 pi I mean 0 and 2 pi is essentially the same, I mean the third law again it is a choice we make by writing this equation in this particular form. This is not a rule, but by using this relation we can define or let us assume that for the choice we make theta is equal to 0 for an apsidal point, right, now oops once again I removed it, sorry.

Now, let us look at this carefully we have few relation or few condition for apsidal points one of the condition is d u d theta will vanish.

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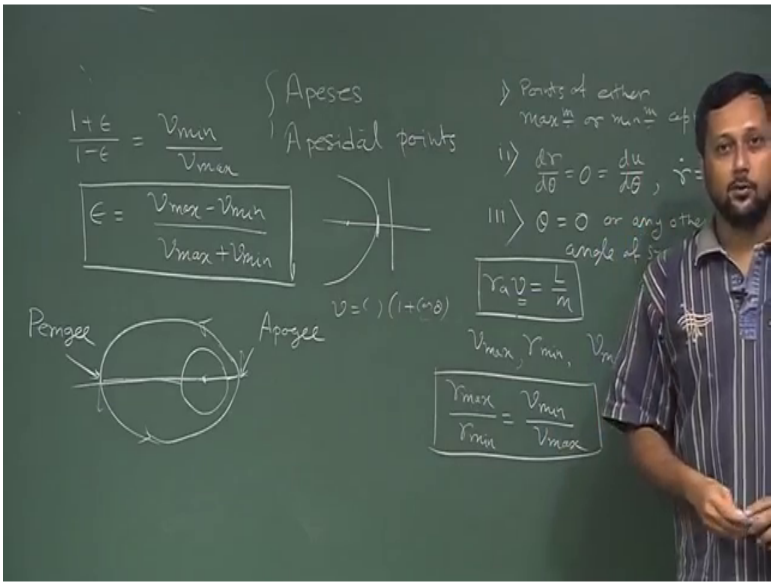

Now if you go back to the original equation of velocity we wrote that was d u d theta square plus v square is equal to and this d u d theta sorry not d square u square whole multiplied by l square by m square equal to v square. Now, for when we hit during the during the path or during the trajectory when the particle hits an apsidal point this term vanishes. So, at at particular point if we write an apsidal distance by r a then we immediately see from this relation that l square by r a square m square is equal to v square once again we can take the root of this particular equation and we can write a relation which is v equal to or r a v equal to l by m this is a very very very crucial relation we have in hand, right.

So, how are we going to use this relation let us understand this, now r a is either a minima or maxima in the orbit so; that means, the v velocity is either a velocity maxima or minima because r and v they are inversely related. So, if we write the velocity at the maximum point as v max corresponding to r min and v min corresponding to r max then we immediately see from this relation that r max by r min will be equal to what it will be, it will be v min by v max right. So, this is relation one relation we have.

Now, here for this particular case if I removed it again this is something I have to work on. So, for an elliptical orbit r max and r min are given by this expression if we put this into the, into this equation we get r max. So, it will be 1 plus epsilon by 1 minus epsilon equal to v min by v max, now I do not need it anymore right.

Now, if you start adding and subtracting the numerator and denominator or basically we have to first do the subtraction. So, we immediately see that epsilon equal to v oh. So, r max is equal to l by 1 minus epsilon right this is max. So, this will stay here remain yeah looks good right. So, if we perform this we see that it will be v max minus v min divided by v max and plus v min and please remember that this is specially for the elliptical orbit it is not a general relation this relation is a general relation, but this one is not.

Now, this relation is extremely useful because in an elliptical orbit if we know so; that means, if we know the speed at the apsidal points by the way this if this is my elliptical orbit, this is my apsidal line, this is the r min and there is there are specific names for it especially for a elliptical orbit. So, the names are this form is called apogee and this one is called Pemgee.

So, we have we have a name for it this is called apogee and this is called Pemgee this is the force center. So, minimum distance is this maximum distance is that. So, if we only know the velocities at these 2 points; for example, let us say we are talking about an artificial satellite, if we know the velocity and this is my earth. So, let us say we have this is my earth almost as sphere which is not true and this is the closest approach to earth and this is the farthest approach to earth of this particular satellite and we know the velocities at this position and velocity at this position assuming that it is going in this direction. So, using this relation we can immediately find out what is the eccentricity of the orbit, right.

So, also we see that as v max equal to if v max is equal to v min then what happens then this term vanishes and we have an uniform epsilon equal to 0 and that essentially means there is a uniform velocity and that exactly what happens in a circular orbit are circular orbit is there is only one velocity possible. If it is under central force we will see that later in more details that there is only one velocity possible and it comes with an eccentricity of 0. So, this is how it is and also the parabola we draw previously and the velocity was some constant times 1 plus cos theta. So, one of the app is the apogee distance is this and the Pemgee distance was at infinity.

Now, for a parabolic orbit in inverse square force field velocity at infinity essentially means that it is I mean the particle has gone to infinity with 0 energy because at infinity the effect of attractive force is also 0 and that essentially means that the escape velocity path possible the lowest, lowest energy possible escape velocity path with lowest energy is a parabolic orbit with the 0 total energy on it. We will discuss all this little bit more details later, but for now, we will stop here and in the next class we will start doing some problems on this.

Thank you.