Classical Mechanics: From Newtonian to Lagrangian Formulation
Prof. Debmalya Banerjee
Department of Physics
Indian Institute of Technology, Kharagpur

Lecture - 14
Central Forces – 7

We continue our discussion on Central Forces, one very important aspect of central force which we have already discussed briefly is that. So, far we have considered the force center to be stationary.

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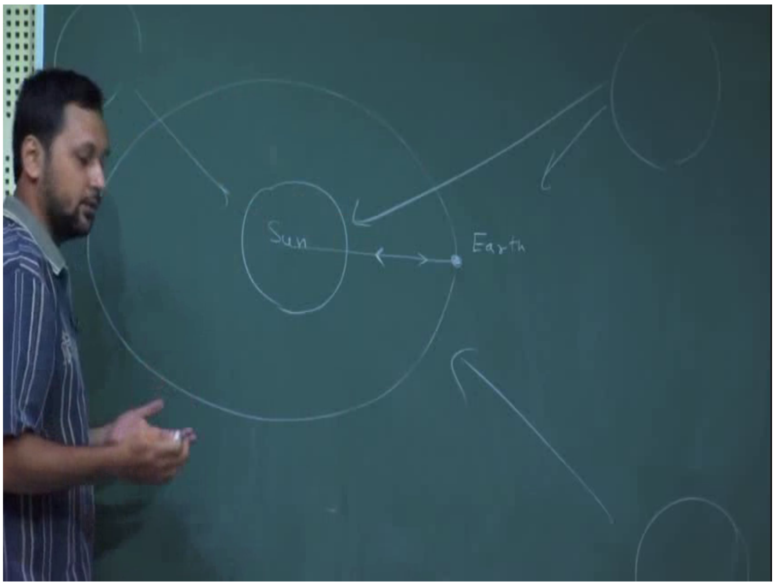

Let us say we have assumed that this is the center of force and at any point in space the force f is proportional to the distance from this force center and it is towards or away from this particular force now typically what happens if we consider the interaction between 2 gravitational interaction between 2 bodies or electrostatic interaction between 2 charged particles then typically what happens, we have a bigger body for example, the sun and we have a smaller body, which is let us say could be any of the planet let us consider it to be earth and the smaller body is evolving, the gravitational force of attraction is taking place acting along the line joining the center of masses and it is an attractive interaction in nature.

Now, for this interaction earth is evolving around in an elliptical path around sun. Now in this entire process earth is experiencing the gravitational pull of the sun, but at the same time there is a gravitational pull of earth that is being inserted on the sun and, so for we have neglected that, because we have assumed that for center is stationary; that means, sun is not moving under the attraction of earth, but earth is moving under the attraction of sun.

Now, we have to systematically check whether this is a valid assumption or not because typically what happens although very small this force of attraction which sun experienced due to the presence of earth or any other planet combined combining all the planet this interaction is also not very small and there is a good chance that this whole sun position is also shifting just because of this interactions. And also as we all know might we all might know that sun is just a simple star in this galaxy. So galaxy is also not very special there are many other galaxies in this universe and inside the galaxy there are millions and millions of stars sun is just one of them and it is placed not at the center of the earth, the center of the galaxy, which was a believe at some point of time, but it is also placed slightly off center. Due to there are many other heavier masses and also sun mass is a very moderate mass compared to other stars.

There are many other masses around this solar system, which are you know attracting this not only sun, but sun and earth alike towards them. The whole solar system although the sun to earth the force for example, I mean this a distance is not probably changing with timing, not the orbit is not probably changing with time, which is not true again it is changing slightly, but most importantly the entire solar system is being pulled in different directions. So, the solar system as a whole is moving the galaxy as the whole is moving. There is a huge many types of gravitational interaction taking place. If we try to model this and see if the effect of earth is indeed significant of sun or not what we need to do is.

We need to just consider this 2 body system let us assume that there are 2 point masses m 1 and m 2.

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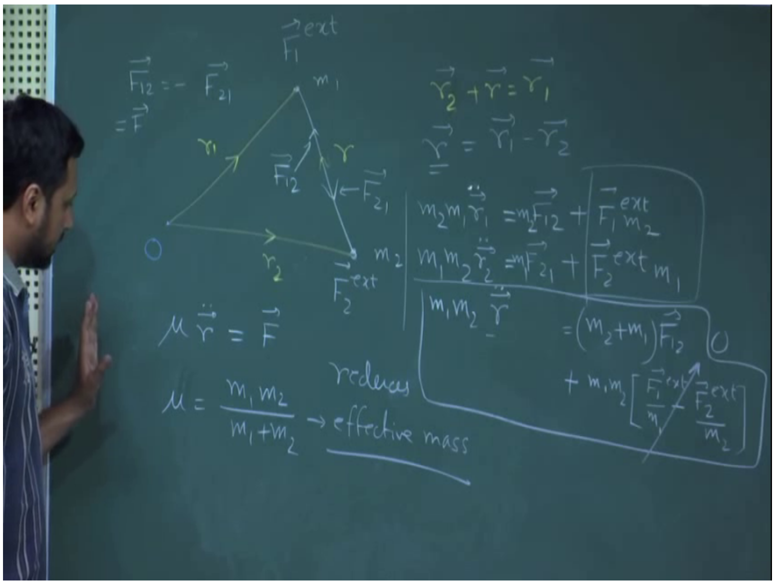

And let us assume that m 1 and m 2, they are the position vector is measured with respect to some origin O and position vector of this 1 is r 1 position vector of this 1 is r 2 and let us assume this to be some small r. We can immediately see r 2 plus r is equal to r 1 which we can rearrange to write r. These are vectors sorry excuse me if I miss the mix the color a bit, but I think it is. So, r is equal to r 1 minus r two.

We have 2 point masses and let us assume that overall external force, which is which is being experienced by this 2 mass system at the position of m 1 this magnitude of the force is F 1 external and here the force is F 2 external right. If we try to write the force equations and of course, there are internal attractive forces or internal forces we can just call it force on m 1 due to m 2. We call it m 1 2 which is acting along this direction force on m 2 on due to m 1, is F 2 1 which is acting along this direction. F 1 2 is let us forget about this arrow F 1 2 is acting in this particular direction F 2 1 is acting in that particular direction, but by from Newton’s third law of dynamics, we can immediately tell that F 1 2 is equal to minus 2 a minus F 2 1. These are vector equations of course, the magnitude of these 2 forces are essentially equal right.

Now if we try to write the equations of motion for this to mass points m 1 and m 2, what we get is m 1 r 1 double dot, I hope this is visible to you equal to F 1 2 plus F 1 external please understand this F 1 external and F 2 external, these are 2 vector quantities similarly at site m 2, we can write m 2 r 2 double dot is equal to f 2 1 plus f 2 external.

Now, if this happens, what now what we can do is we can multiply this with m 2, we have m 2 here and m 2 here we can multiply this with m 1, we have m 1 here and m 1 here, now if we add or rather we subtract this 2 we have m 1 m 2 r one minus r 1 double dot minus r 2 double dot, which is equal to, now you see if we subtract it will be m 2 F 1 2 minus m 1 F 2 one by using this relation we can write the entire thing in terms of F 1 2 and it will be m 2 plus m 1 F 1 2.

Now, for the last term once again what we can do is we can take out or rather I will just write it just below this, so that easy for you to see. Once again we can just take out m 1 m 2 common from the last 2 terms; that means, this 2 terms. So, the first term will be F 1, external by m 1 minus and the second term will be F 2, external by m 2 is it visible it is visible.

We have the equation altogether can be written as, now what we are going to do is in the left hand side we are just going to use r to be equal to r one minus r 2. So, left hand side we have and then. We can just rearrange it r double dot which will be equal to m 1 plus I will just I will not write it again what I will do is I will simply replace left hand side by m 1 m 2 r double dot. So, this is our equation of interest right.

Now, look at this third term here which is F 1 by m 1 divided by minus F 2 by m 2, now F one by m 1 is what F 1 is the net external force that is being acted on mass m 1 and F 2 is the net external force which is being acted on m 2. Now consider the situation let us say we have this sun and earth system, which is somewhere in the galaxy all the big stars, and all the rest of the galaxies I mean exerting some force on the system.

Now due to this force, assuming that the net effect of all this force combined on this 2 body system is that both the masses are accelerated equally then F 1 external by m 1 is the acceleration please understand this F 1 external by m 1 is the acceleration of mass m 1, due to this external force. Similarly, F 2 external by m 2 is the acceleration due to mass or acceleration of mass m 2 due to this external force if the overall effect of all these forces is not affecting the internal interaction, but it is shifting the 2 body system I mean or it is producing equal acceleration in the 2 body system then this term essentially cancels out.

What we are doing is we are taking the entire effect of rest of the universe; as this a including these into the external forces and we are saying that this extra the acceleration all of these 2 masses are equal due to effect of this external forces. If that is the case then this term essentially goes to 0, because the accelerations of this 2 masses are equal due to the combined effect of our combined external force, which leaves us with this 2 terms only and we can rearrange the equation to write r double dot is equal to or rather, we can write this as mu r double dot to be equal to F 1 2 or we can just write this as F assuming that this is the magnitude of internal forces I am just equating this equal to F.

So, this is the equation with mu equal to m 1 m 2 by m 1 plus m 2 and this is called the effective mass. We see assuming that all the external forces combined is producing an equal acceleration in this 2 point masses, the equation of motion is strictly a function of both the masses with this effective mass or reduced mass or you can also call it the reduced mass, if you like which is given as m 1 m 2 divided by m 1 plus m 2.

Strictly speaking this is how we should write equations. When we are writing the gravitational interact equation of motion due to gravitational interaction between 2 masses this is the equation we should to a we need to a employ we need to convert everything in terms of you know reduced mass and I can tell you that the reduced mass will be essentially the center of mass of this system will be and will be placed at a radius vector r which is given by I will just write it here maybe.

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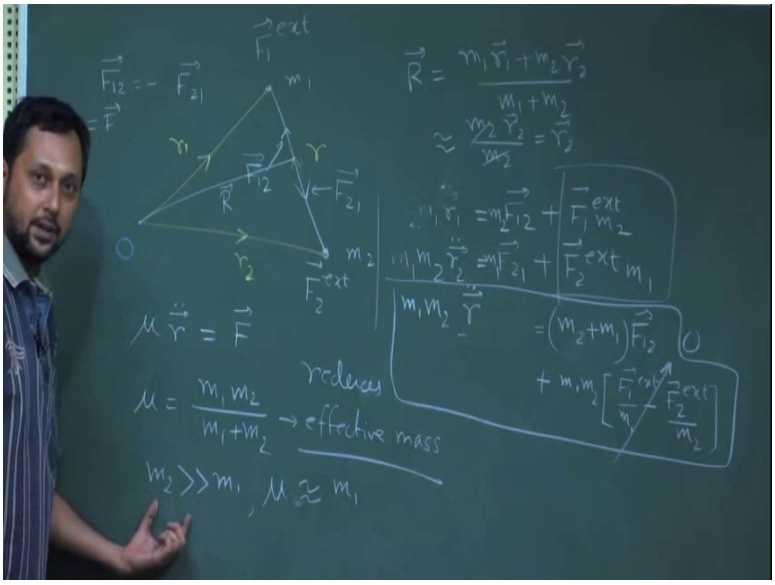

So, r will be the center of mass location of the center of mass where this reduced mass will be located which will be given by m 1 r 1 plus m 2 r 2 divided by m 1 plus m 2. We will discuss this in details later on; that is not really essential at this moment.

But now what is important, now let us consider the situation let us consider earth and sun system. We know that solar mass is many times greater than the mass of this planet. Now in this particular case we can consider m 2 or let us say m 2 is way why greater than m 1 if that happens look at this expression of the reduced mass, what happens is if we put m to lot greater than m 1 then the denominator is essentially m 2 because, m 1 has hardly any effect on it and we get mu even this particular case mu is almost equal to m 1.

Now, if this happens right and then if we look back into this expression then what happens in this under this particular assumption is a that m 2 is lot greater than m 1 this 1 essentially becomes m 2 r 2 by m 2, which is equal to r 2. When see the first term will be neglected, what we see is when one mass is very dominant over the others then the equation of motion we can essentially replace the reduced mass with the smaller mass and the center of mass essentially shifts on the bigger one r 2.

That is what we see here and this is why especially when we are discussing gravitation, we can consider the bigger mass most of the time when we are discussing solar system the bigger mass is the sun. Because solar mass is lot greater than any of the planet masses you can find the numbers from internet exact numbers. That is why it is always safe to assume that effective potential. And with the equations we are writing essentially we will reduce to the mass of the planet times r double dot is equal to F being the gravitational attraction, but please remember this particular form.

And when we are dealing with 2 systems or 2 point masses or point charges for example, which has almost equal mass and they are attracting each other. And if we write want to write the equation of motion of that particular system we have to employ the concept of reduces mass or effective mass to get it right. We will look into examples later on where we are we will see with or without considering as a reduces mass what will happen to the results that will do will come later.

Right now, let us move on and continue our discussion on central orbit. Now when we were discussing central orbit in the last class, so for we have proved 3 laws of Kepler and now we are we have pretty I mean we are in a play we have you know developed the platform from where we can jump forward. We can essentially take up little advanced dynamic problem, but before that we need to find we need to you know dig deep into the fundamentals once again and try to figure out some of the fundamental properties of central orbit especially under inverse square force field.

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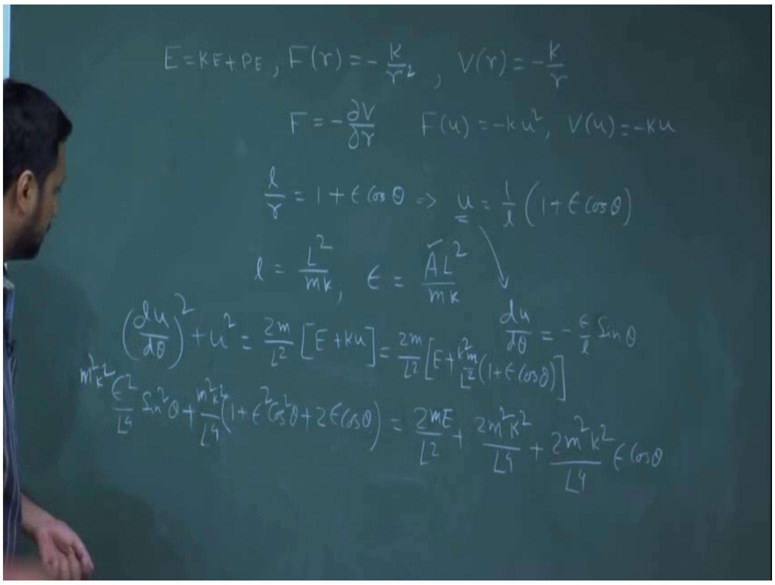

Now, in inverse square force field we have seen work which is given as F r is equal to minus k by r square V of r the potential corresponding potential function this minus k by r.

So, that F is equal to minus del V del r this equation is well this equation is valid and we can very well write the same in terms of u. So, that F of u equal to minus k u square v of u equal to minus k u and we have already seen that the orbit is given by l by r equal to 1 plus epsilon cos theta minus theta 0 or what we can do is we can simply consider theta 0 to be equal to 0 and we can simply write cos theta that is also assuming that our axis is at theta equal to theta 0 and this can also be written as u equal to 1 by l 1 plus epsilon cos theta. Now l is equal to what l is equal to l square by m k where l is the angular momentum, k is the force constant, which is here and epsilon is equal to, for what we have derived is a l square by m k.

Now there we have an undetermined term a now it is time that we get rid of this undetermined term and we get an expression of epsilon, which is given which is in terms of the known variables. Now there is another thing which is known to us in principle which is known to us is the total energy E which is kinetic energy plus potential energy although it is not known a priory, but we know that this is a conserved quantity. What we are going to do now we need to find we are going to find an expression of A in terms of E and m and k and l which are fundamental parameters of a central orbit. Now what we are going to do is in order to do that what we are going to do is, we are going to use the equation du d theta square plus u square is equal to 2 m by l square E minus v right.

Now, this is the equation we see d u d theta will be given by minus epsilon by l sine theta right. If we put it here then we immediately we immediately get the left hand side as epsilon square by l square sine square theta and then we have u square which will be given as 1 by l square 1 plus epsilon square cos square theta plus 2 epsilon cos theta, on the right hand side please remember v can be replaced by minus k u. So, we can for again we are doing everything for inverse square force field. We can write k times u which can be written as 2 m by l square e plus k times u will be this we use this expression for u. It is k by l 1 plus epsilon cos theta.

Now, we open the brackets here. So, we have 2 e by l square plus and l please remember L is L square capital L square by m k. We substitute it here it will be L square and it will be k square m right if we open the bracket the first term will be 2 m E by L square and second term will have, 2 m square k square by L to the power 4 into 1 plus epsilon cos theta.Now here also we have to substitute the L square L square here will be replaced by L to the power 4 m square k square m square k square here also we substitute it by k square by L to the power 4.

Now, look at the second term. This we 1 term here this plus the same term 2 m square k square by L to the power 4 epsilon cos theta, you see this cos theta term m square L square 2 m square L square e to the power 4 by epsilon into epsilon cos theta, it will cancel from both sides. If we cancel that what we are left with is these and of course, we have this epsilon cos theta and epsilon sine epsilon square cos square theta plus epsilon square sine squared theta with the same pre factor m square k square by L to the power four. This will simply give us epsilon square.

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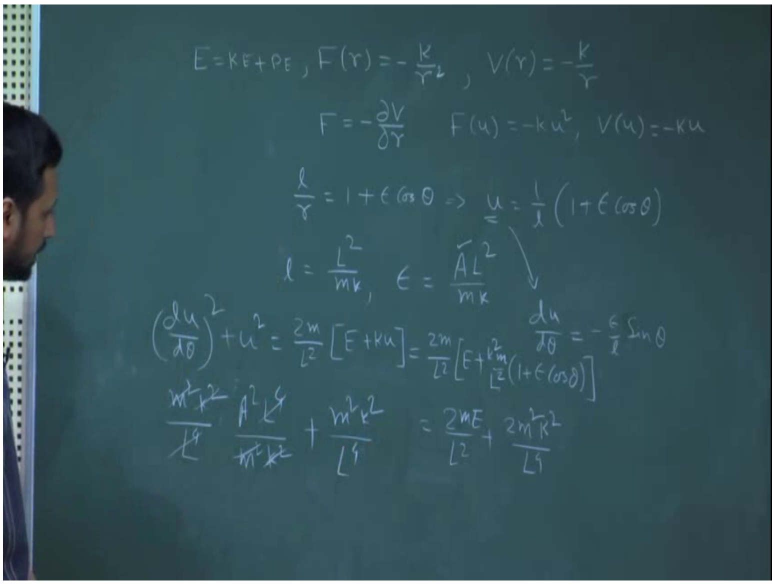

The left hand side will be m square k square by L to the power 4, here we will have epsilon square plus, now if we put the value of epsilon it will be A square L to the power 4 by m square k square. There will be an 1 additional term which will be square k square. There was one there. So, it will be m square k square by L to the power 4 here.

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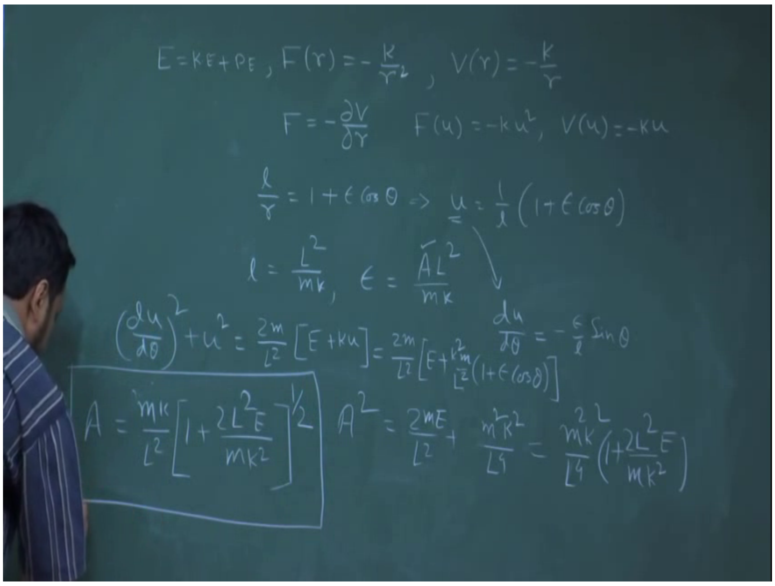

What happens is all this cancels out leaving only A square in the left hand side. So, we can rearrange this and you see this term is m square k square L to the power 4 we have 2 here. We can rearrange this to make this A square were equal to this right.

Now, we can take m k by m square k square by L to the power 4 common. This term will be 1 plus. This will be L and this will be this will be L square by mk square k plus mk square I think E here right there will be a 2 term coming right.

Simplifying we can write A equal to mk by L square into 1 plus 2 l square E by mk square to the power half right. This is the expression final expression right. I think I made a slight mistake somewhere in the calculation. So, this is a bit complicated calculation. We will get an expression for a. And finally, if we substitute in if we substitute this expression for A in this for epsilon then we will get epsilon equal to this. So, you I i request you kindly go through it once more it is also figured out in many of the books. This will be the final expression for epsilon mk I will just check it once again no sorry not this it will be simply this, this will I think it will cancel out with this one; so leaving behind epsilon.

The final expression for epsilon will be epsilon equal to right I think it was right actually the expression for A was right because if we multiply it with l square by mk right the expression for a was right. So, if we simply multiply with l square by mk the pre factor will cancel out and then we have epsilon equal to one plus 2 l square e by mk square to the power half.

Now, this is the very useful expression for the eccentricity. The eccentricity is given in terms of mass the force constant the total angular momentum and total energy and this is a very useful expression for epsilon as we will see later. And we will keep on using this expression for solving many important problems related to the orbit sense central forces that will come slowly and slowly.

Now, we have to stop this. And next class we are taking up the general equivalent one dimensional motion.

Thank you.