Classical Mechanics: From Newtonian to Lagrangian Formulation

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So, we continue our discussion on Kepler’s laws. We had discussed about 3 laws of 3 Kepler’s laws. The first one is the, all of this laws are valid for our planetary system. So, essentially K of or rather F of r is equal to minus K by r square.

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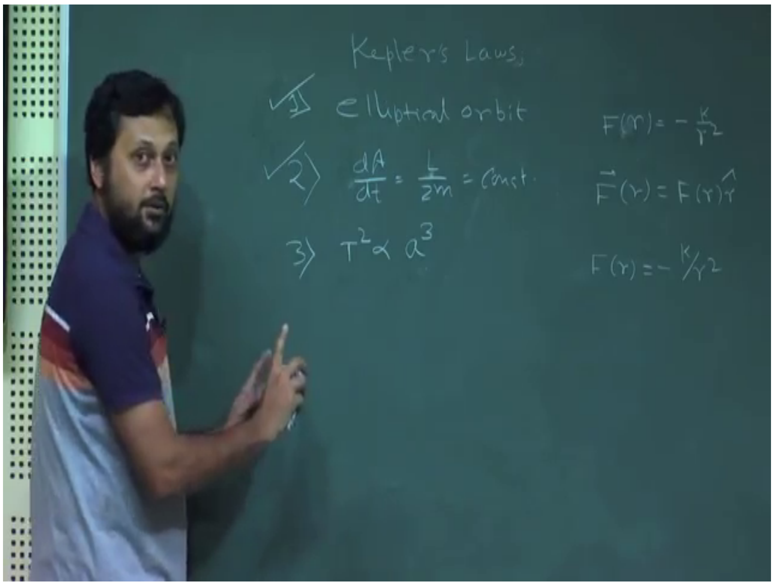

So, this is the force law for which this 3 sets 3 laws are valid. So, first laws is every planet will have an elliptical orbit with sun as one, of it is 4 surposy this we have proved there is no formal proof needed, but because of this particular law of force it is an obvious conclusion. Second law it turned out it is a general law for any central force which has a form F r equal to F r r cap. This is this also we have proved.

Now it is time for the 3rd law the 3rd law once again we will see that it is valid only for Fr equal to minus K by r square and it is not any. So, the that is to say it is not a universal law, but it is strictly valid for inverse square force field. So; that means, gravitation or electrostatic force field. Now this law essentially says the T, T is the time period of evaluation in elliptical orbit and a is the length of the semi major axis.

So, in order to prove this law formally, we first have to look into some of the parameters of an elliptical orbit in a more formal manner. Let us do that, will just remove this because we need the space.

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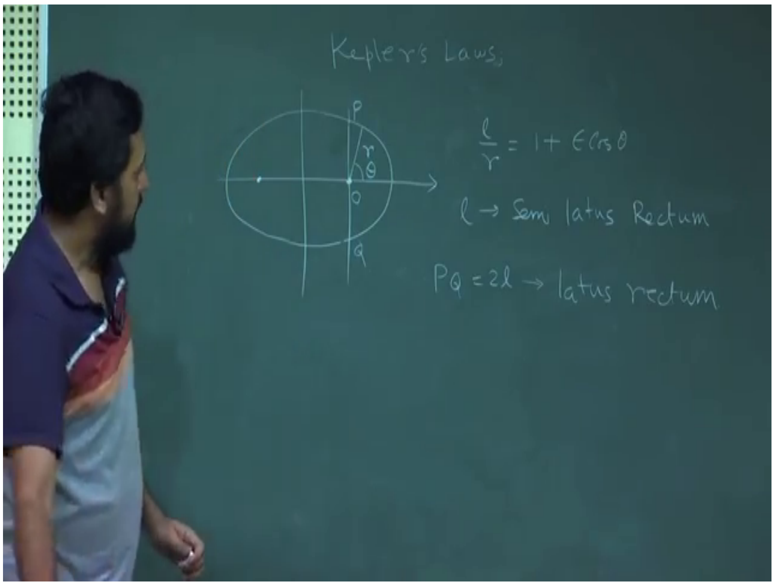

This is an ellipse this is one of the focus. We take our origin here. This is the major axis this is the minor axis of the ellipse. So, this is our center of coordinate system O. Location of any point is given in terms of sorry location of any point is given in terms of r and theta. So, theta is the angle that is made with this particular axis with between this at particular axis and the radial vector.

So, the equation is l by r equal to 1 plus epsilon cos theta. And it could also be 1 minus epsilon cos theta because we can just shift the origin from this focus. This focus we have discussed it already what is l? L is called semi latus, rectum. Semi latus rectum and it is the perpendicular that could be drawn through the origin through the focus and it will cut this 2 points at let us say position P and Q. So, length PQ is equal to twice l, it is called the latus rectum. This entire length the PQ is called latus rectum half of it is O P equal to l equal to semi latus rectum.

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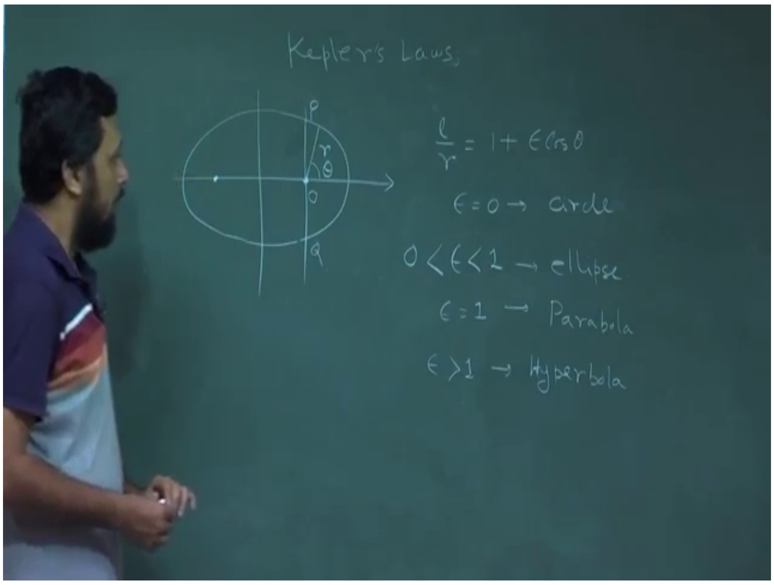

So, this is we know what is l what is r and what is theta; epsilon of course, is the eccentricity and we will we can show that for epsilon equal to 0, we get circle. For epsilon less than 1 and greater than 0; that means, if epsilon lies between 0 and 1 we get an ellipse. If epsilon is greater than 1 or sorry epsilon is equal to 1, we get a parabola. And epsilon greater than 1 we get hyperbola right. This we have not discussed earlier, but this is the scenario. So, for an ellipse the equation is l equal to r sorry. For a circle the equation is l equal to r and this is a familiar equation of r equal to constant for a circle. For ellipse it will be some number epsilon should be some number between 1 and 0, with this general form for parabola epsilon equal to 1. So, the equation is l by r equal to 1 plus cos theta. Similarly, epsilon greater than 1 this equation gives the hyperbola.

So, we are right now for in order to prove Kepler's third law. We are more interested in ellipse because we are talking about closed orbit and we will see later that this epsilon equal to 0 means a certain value of energy which only which will give you a circular orbit, that will come later, right now we are talking about elliptical orbit this general equation with this condition on epsilon. And we know that this length is l this length is 2 l. Also now we have to define 2 parameters a and b.

Let us try to do that this is the major axis 2 a.

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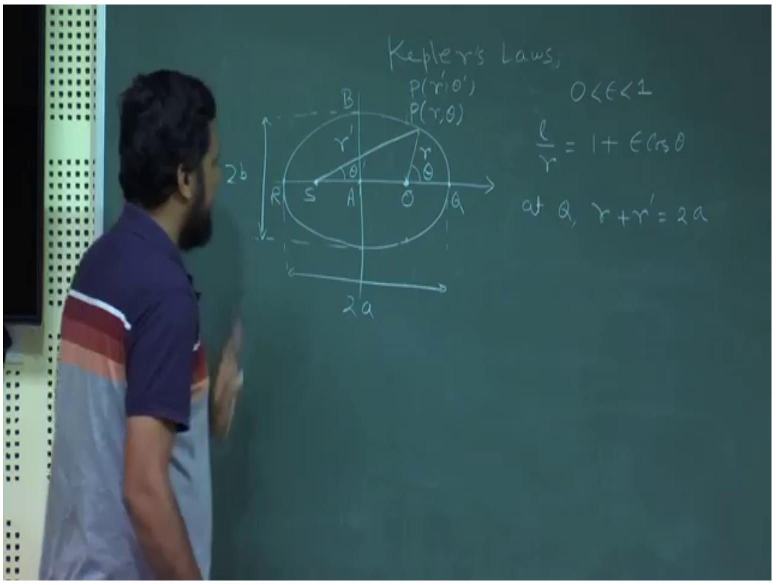

So, half of this is a, that is the minor axis 2 b and half of this is b. Now if we if the position of this point P is r theta, when we have the origin at this focus and if we shift the origin at this particular focus. We have a coordinate which is given by P r prime theta. So, this will be your r prime and this will be your theta prime. So, what I am trying to tell you that if we have, once we have the origin here which will give you a position vector r theta and then we shift the move the origin to here. And this will correspond to a for the same point it will correspond to a coordinate r prime theta prime. Then it can very easily be shown that. So, we will just move it here this is a condition.

Now, if please consider this point P to b here that said this is the point we give it a name maybe, I do not know let us call this Q, let us call this AB this point remains O. So, these are the points we need essentially. So, when the P vector when the sorry P point coincides with Q point. So, what happens then what is r, r is this length and what is r prime r prime is this length. Now this length and this length is equal because the focus is symmetrically placed. So, if this 2 points coincide we can say that at Q r plus r prime is equal to twice a. Think of it r prime, when this point moves here r prime will be we will also need a name for this PQRS. So, r prime will be S Q and r will be OQ, which will be equal to we also need a name for this point is R yeah S R. So, this will be equal to this. So, essentially r plus r prime equal to 2 a.

Now, it can be shown that for an elliptical orbit anywhere for any point up, I mean any point P any arbitrary point P this is a valid relation this can be formally shown, but that is not terribly important we you just you can take my word for it. So, r plus r prime is equal to 2a at any point. Now if we now move to another point. Just bear with me a bit more because let us a little complicated geometry involved here. So, we have to be very precise about it.

Now, let us consider this point.

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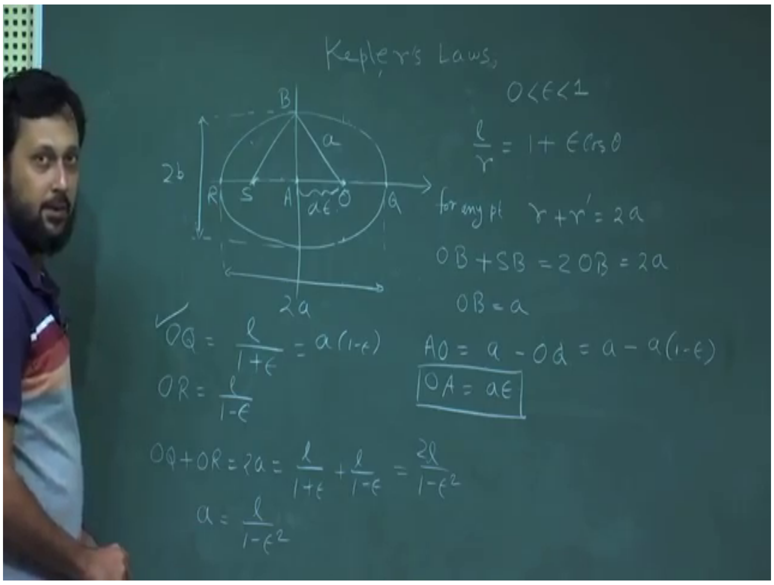

So, as I said this relation is valid not only for this point Q, but it is a general relation for any point. Any point of this elliptical circle a elliptical path this is the general relation. So, for this point B this is also generally true now for this point r is equal to or rather this is r and this is r prime. And we can see from the symmetry of the problem this r is equal to r prime right.

So, OB plus SB which is once again OB and SB, this 2 are equal length we can write it as 2 OB equal to 2a. So, that essentially tells us that OB is equal to a. So, this length will just we have just proved that this length is a. Now why I am doing this why I am doing this a you have to bear with me slide for a little longer, then you will know immediately now this length. Now this length segment OA, AO. So, AO is equal to AQ minus OQ AQ minus OQ right. AQ is nothing, but this length which is once again equal to a. So, we replace it with a and what is OQ? OQ is in this relation if we put theta equal to 0 we get then our corresponding r is our OQ.

So, from this relation we can calculate OQ is equal to l by 1 plus putting theta equal to 0 only sub glides it 1 plus epsilon right. So, this is 1 plus epsilon. And now if we repeat the same procedure for OR, then please understand that if we put theta equal to 180 here. If we get theta put theta equal to 180 with respect to this origin r will move to this point. So, or is equal to l by 1 minus epsilon. Now OQ plus or is equal to qr which is equal to 2A which is a summation of this 2. So, OQ plus or is equal to 2A equal to l by 1 plus epsilon plus l by 1 minus epsilon, which on simplification gives you l by 1 minus epsilon square or there will be a factor of 2 here. So, 2A is equal to 12 by 1 minus epsilon square. So, immediately we see that A is equal to l by 1 minus epsilon square.

Now we have a expression for OQ. And OQ is l times 1 plus epsilon, with now if we substitute the value of l in terms of A here. We immediately get l is equal to A times 1 minus epsilon square divided by 1 plus epsilon, which will be nothing more than 1 minus epsilon into 1 plus epsilon and it will be divided by that. So, it will be 1 minus epsilon right. So, we see that OQ is equal to A times 1 minus epsilon, I hope it is clear.

Now, we put go back to this relation and we see a into minus A 1 minus epsilon which will simply equal to A epsilon. So, OA is or AO is equal to A times epsilon right. So, this length is A this length is A epsilon. So, finding this length AB is easy right. That will be simply I am I do not need this part anymore. I just used the calculations we need.

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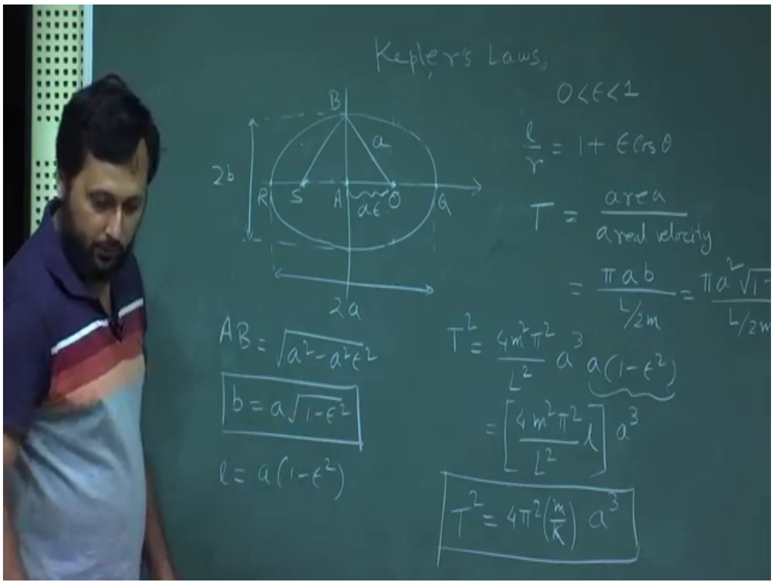

So, AB equal to root over S square minus A square epsilon square, which will be a root over 1 minus epsilon square. So, we and this AB is what? AB is this length which is the length of semi major axis. So, which will be b. So, b is equal to 1 by a into 1 minus epsilon square right.

Now, why do we need all this? We need this because so we do not use these anymore. We will just keep this same because, we have to define the time period T. Now there are many ways of doing defining time period the definition will be adapting is area divided by areal velocity. This is also a valid definition. Because all we need is a time it takes to get back to the same point over and over again in this in this path of oscillation. So, we can use many other definition area by areal velocity also valid definition.

Now, what is the area of an ellipse, it happened to be given by pi ab, that you know from your geometry lesson and aerial velocity we already have derived an expression for areal velocity. And the expression is L by twice m L by twice m, which is constant for any central orbit right. Now you see if I replace this for B, we immediately get pi a square into root over 1 minus epsilon square divided by L by twice m. So, T square is equal to and we have to rearranged slightly. So, T square equal to 4 m square pi square 4 m square pi square by L square a to the power 4 into 1 minus epsilon square. So, we what we can do is, we can use the relation that L is equal to that we have derived already l is equal to A times 1 minus epsilon square right. And so we can slightly modify this. As a cubed into A times root over or sorry not no root over anymore L minus epsilon square.

Now, this we can simply substitute with L right, which will leave us 4 m square pi square by L square L. Right now this quantity is a constant. And we can also put the value of L and simplify it slightly. And the final expression what we can do get is essentially T square is equal to 4 pi square m by K times a cubed.

So, this is the term in the bracket will can be simplified to this. I am leaving it on you to do it yourself nothing difficult just have to put the value of L. So, we immediately see a T square we have proved the Kepler's third law because, this relation tells you the T square is some constant times a cubed which essentially means T square is proportional to a cubed. So, we formally proved Kepler's third law. This entire mathematical formulation what we did here the this sorry this yeah. So, the mathematica the geometrical formulation was needed. So, that we can gain and expression we can write b as a times 1 minus F root over 1 minus epsilon squared.

So, hence we have proved 3 Kepler's laws, starting from very fundamental force law. And now it is time to review different aspect for central orbit. So, I am removing this. So, our discussion on Kepler's laws are over now. And not over exactly we will come back to that. Because, we can actually apply the third law, we can actually apply the third law. And we can find it is really helpful in order to find the time period of certain orbits.

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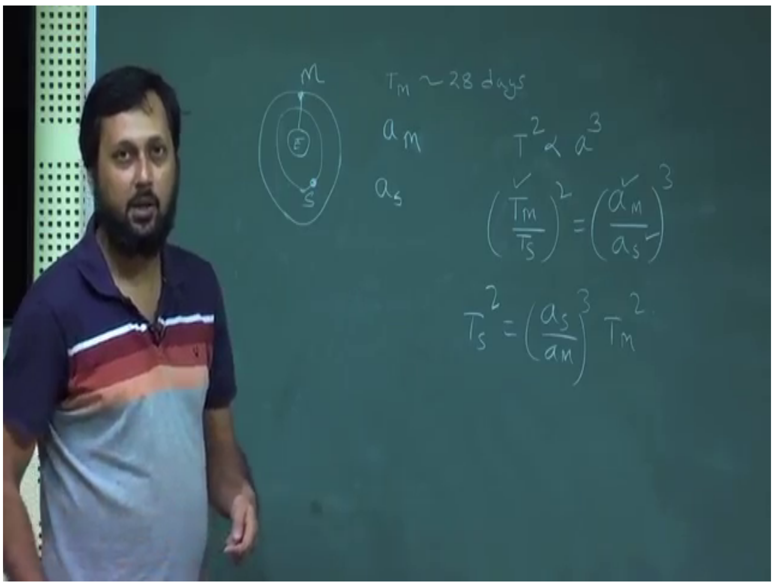

Take an example; let us say we have earth here. Moon is evolving around earth in an elliptical orbit. Once again it has to be an elliptical orbit because this is also gravitational force law.

Now, we have a artificial satellite. Which is revolving around earth. So, let us call this m and let us call this s. We know that the time period of moon is roughly 28 days. And earth to moon distance from this earths surface to moon. We know the discrete we know the height we know the distance. So, in principle we know the semi major axis for the moons orbit. We also know, let us assume that we also know the orbit of this artificial satellite. So, we know a s. So, if this is the case we can always write and please remember that T square is proportional to a cubed. And this is independent of whether it is a satellite or an artificial satellite it remains. The same the this relation this Kepler's law remains I mean it here it holds for both.

So, in that case, we can write a relation of the form that Tm by Ts whole square is equal to am by as whole cubed. So, if we know this 3 parameters, for example, we know this one this one and this one then Ts square can simply be written as by am whole cubed into Tm square.

So, these are the types of application, if we know 3 parameters in an orbit the fourth one can be calculated. Or sorry rather if we know the semi major axis of one orbit with time period and semi major axis of another orbit, then the time period of the second orbit can be calculated. Or also the if we know the time period we can also calculate the length of the orbit sorry the semi major axis of the orbit. Now this we are not doing right now, because there are some more considerations, we need to realize we need to understand before we can do such applications.

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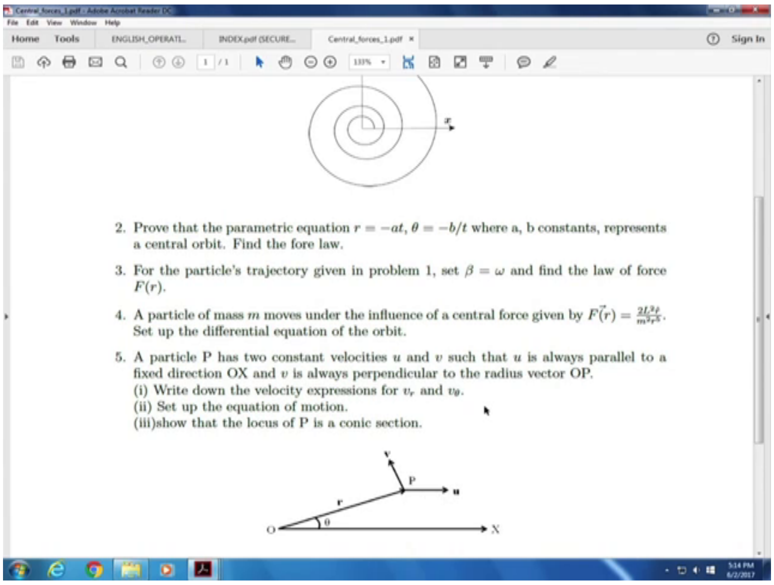

So, we are just stopping it here. Before moving out of this particular topic, let me try to solve this problem for you. The problem is problem number 5 up to problem number 4 we have already done. Problem number 5 says a particle P has 2 constant velocities u and v such that u is always parallel to a fix direction OX, and v is always perpendicular to the radius vector OP.

So, a particle P is moving under the influence of some force we are not sure what is the exact nature of the force, but it has 2 velocity components, which are very defined the velocity u is always parallel to this way x axis. So, that we can take this OX axis as a reference axis, and we can define this theta angle from there. Now if we come to the board and draw this over here.

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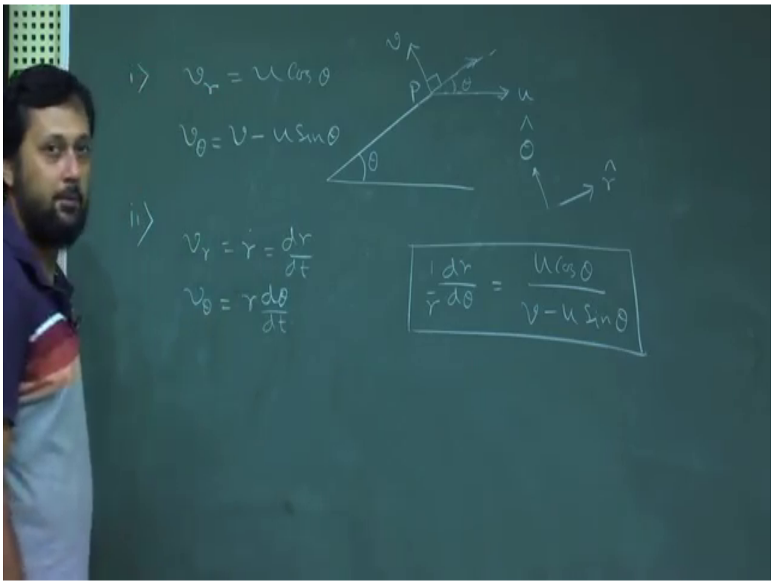

So, this is my theta angle. So, once again this is theta this is v this is sorry. So, this is u here and this is v. So, the question says first, we have to write down the velocity expression in terms of v r and v theta.

Now, what is vr, vr is a radial velocity for this particular point P. It points in this direction. So, vr will be of course, v the so, please do not get confused u and v are 2 numbers. So, probably I will call it something else. I will call it, let us call it vr. I think you will understand. So, vr is the radial velocity. So, this v velocity will have no component along this velocity I mean along the radial direction because this angle is 90 degree. So, if a vector cannot have a component at 90 degree separation. So, that is why v is ruled out. So, vr will be simply u cos theta, v theta; obviously, it is v and r and theta. So, if this is the direction of r which one will be the direction of theta. Sorry if this is the direction of r cap then what will be the direction of theta cap this or this? The standard convention is we go in the anti clockwise direction.

So, if this is r cap, this is our theta cap not this one, this is minus theta cap right. So, if this is my direction of r cap my theta cap direction is this. So, this one have a will have a velocity component which is opposite to this direction. So, v theta can be written as v minus u sin theta right. So, we could write down the velocity components. Second question is, so, this is the part one part 2 of the problem is we have to set up the equation of motion. Please remember v r is equal to r dot is equal to dr dt. And v theta is equal to r d theta dt.

So, if we simply divide one by the other, then we have v r by v theta is equal to d r by 1 over r dr d theta which will be from this we get u cos theta v minus u sin theta right. So, this is my differential equation of motion, we just get rid of this one. This is my differential equation of motion. So, part 2 is done. Now the third part and the last part we have to show that the equation of the chord and the trajectory of this particle is represented when it is a conic section. So, we have to essentially solve this equation integrate this equation and bring it to the familiar form of L by r equal to 1 minus or 1 plus epsilon cos theta.

Let us see, if we can do that should not be a problem it is a simple.

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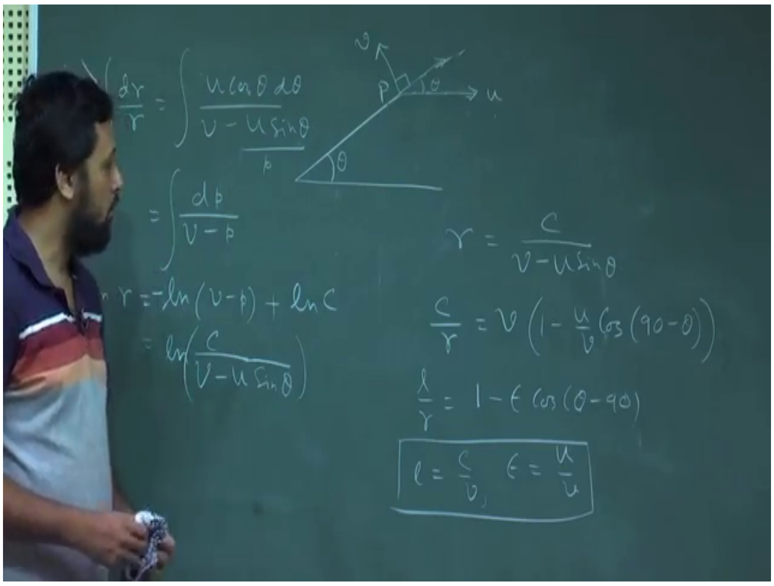

So, see that all the theta dependence is on the right hand side. So, we can simply write d r by r is equal to integration u cos theta equal to v minus u sin theta. So, if you take u sin theta equal to some p. So, you right hand side becomes d sin theta here. So, sin. So, it will be d py v minus p right. So, left hand side. So, if we perform the integration, it is lon r is equal to lon v minus p. There will be minus sign coming plus we have to add a constant here we just take as lon c. So, c by v minus v substitute the p value of p again. So, it is u sin theta. So, lon r is equal to lon c by v minus u sin theta.

So, if we simplify, we immediately get r is equal to c minus v minus u sin theta. Or c by r is equal to u v by u minus 1 minus cos 90 minus theta. Or l by r is equal to oh sorry. It has to be v sorry not u. So, if we take v out it will be 1 minus u by v right. So, we see l by r is equal to 1 minus epsilon cos 90 minus theta with l is equal to c by v epsilon is equal to u by.

So, we could bring it down to the familiar form. Please recall that plus or minus it is does not matter both represents a conic section we have a reference. So, it is of the form of theta minus theta 0 because it is cos theta, we can write this also as theta minus 90 because cos of minus theta is equal to cos theta. So, we get the general familiar form of l by l l by r equal to 1 minus epsilon cos theta minus theta 0 theta 0 90 l is equal to c by v epsilon equal to u by v good.

So, that is it for now. And 2 on the next class onwards what we are going to do is we are going to look into the different cases which will give rise to you know in terms of energy, which will give rise to different types of trajectory. Also we have to formally see why a 2 body problem is sometimes not necessarily has to be considered. In order to derive the motion under central force, considering one particle fix I, sorry the force center fixed will also work that we will see in the next class.

Thank you.