Classical Mechanics : From Newtonian to Lagrangian Formulation
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Lecture - 11
Central forces – 4

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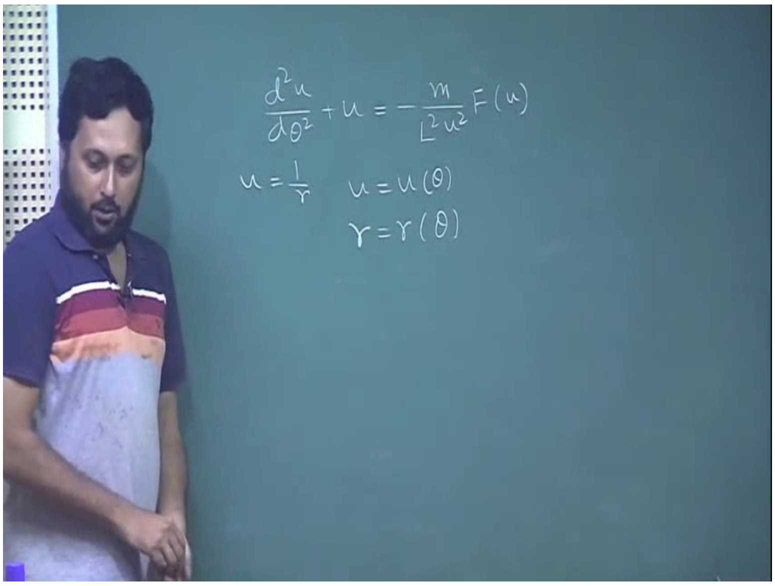

So, this is the relation which we derived in the last class that is for any central orbit. We have d 2 u d theta 2 plus u equal to minus m by L square u square f of u m being the mass of the object, L being the angular momentum, u is a parameter which is nothing, but 1 over r. So, by solving this equation we get u as a function of theta, which is equivalent of saying that we get r as a function of theta because u and dot they are just inversely related to each other.

Now, this we have derived in the last class we solved couple of problems.

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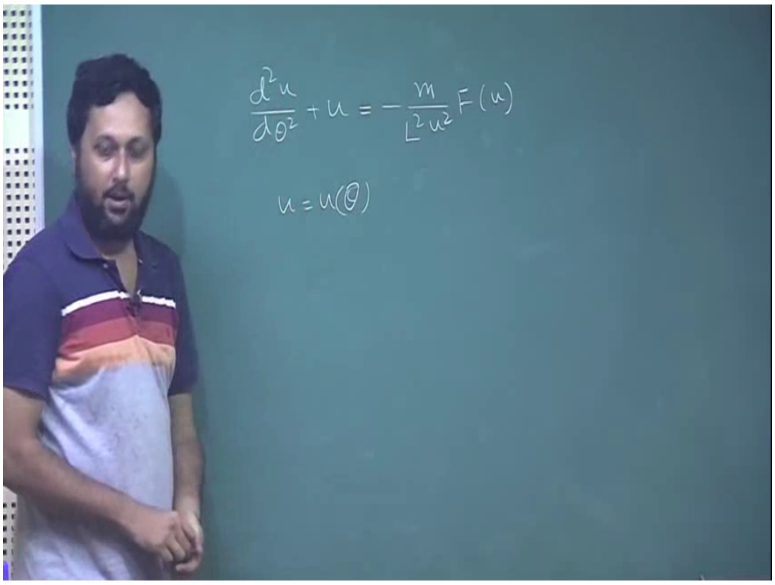

We saw how to if we know the force law, if we know f of u which is the force law then we can get a differential equation and in principle we can solve this differential equation to get the equation of the orbit and we have seen if we have the differential equation in hand, we can also get the force law or rather sorry.

If we have u as a function of theta that is equation of orbit in hand, then we can compute d 2 u d theta 2 put into this equation and we can get the force law. So, it worked both ways.

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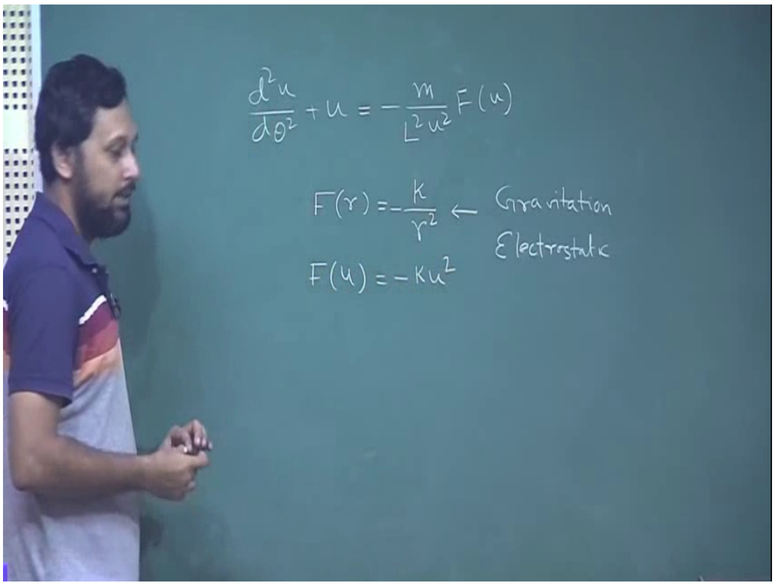

Now, what we are going to do today is we would try a very specific type of force law and see how does this equation behave, now the force we are going to try is the inverse square law of force, negative sign means attraction we can also have a positive sign here, but let us for now let us do it with negative sign. Now this is a force which is very common to us, we have it in gravitation and we have similar forces in coulomb attraction that is electrostatic. So, gravitation is something that governs ourselves solar system or actually that governs the universe.

So, that way it is a very universal law of force and the very existence of universe, we as we know today depends I mean is the it is depends on this particular force law right anyway. So, we would like to see how if we instead I mean if we you use this force law in this particular equation, do we get any special information about the orbit. Now in order to do that we first write Fu; which will be minus Ku square simply.

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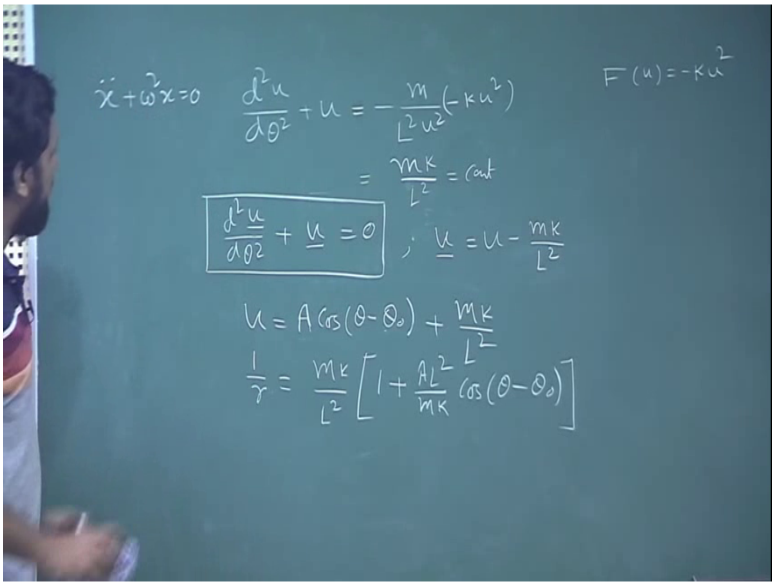

So, what we are going to do is for inverse square force law, we are going to substitute minus ku square into this equation we will just keep it in side. So, our force law is f of u equal to minus ku square right. So, if we do that we immediately see that right hand side becomes u square u square cancels this becomes a plus sign. So, m k by L square which is a constant right.

So, we can rearrange this equation to write d 2 u d theta 2 or u bar v theta 2, let us do it u bar is equal to 0. So, we define a new parameter u bar which is u minus m k by L square this constant. So, it if we took this on the left hand side we defined a new parameter u bar equal to u minus mk square, because it is a constant the second derivative is not going to vary. So, we can simply write it as d 2 u bar d theta 2 plus u equal to 0, now this is a very familiar form remembers simple harmonic oscillation under no resistance.

Under no external force the simple harmonic oscillation has this familiar form, if you do not recall I will just remind you it will be x double dot plus omega square x equal to 0, in this case just it is equivalent with omega square equal to 1 . So, this 1 will have a solution of the form u bar is equal to a cos theta minus theta 0. As it is a second order differential equation, we need to have 2 constants in the solution. So, you can either write a cos theta plus b sin theta or we can simply write u bar is equal to a cos theta minus theta 0, whole idea is we need to have 2 unknown coefficients a and theta 0 right. So, we substitute once again for u bar which is u minus mk by L square right and we once again we take it to the right hand side.

So, the left hand side remains as u and right hand side becomes mk by L square r right, now if we take mk by L square out of this; then the first term there is a 1 and then there is a L square by mk cos theta minus theta 0 and left hand side is u now what we can do is we can write 1 by r for u right. So, this equation as we see again can be slightly rearranged. So, I am removing this I hope you all understand this equation by we all you all remember it by heart by now. So, let us take it up here once again.

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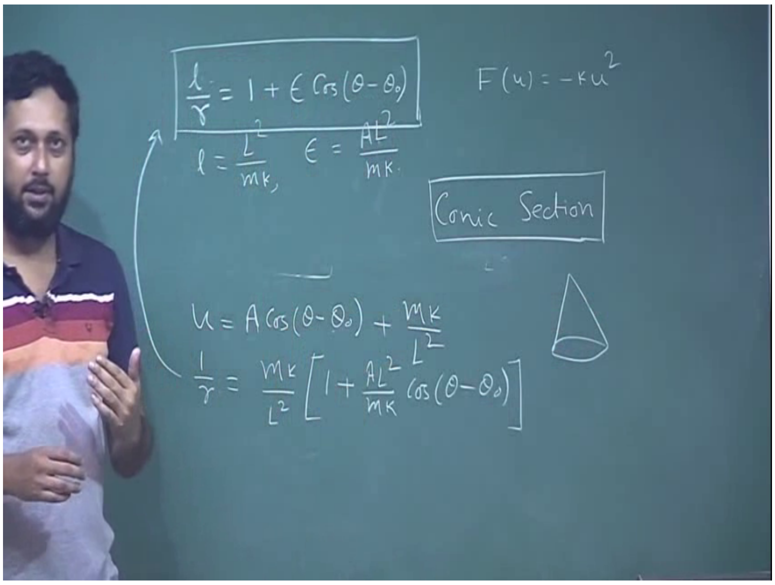

So, we can write this equation as L by r equal to 1 plus epsilon cos theta minus theta 0 right. Where or actually I can use this place. So, let it be here let it be here where L is equal to nothing, but L square by mk and epsilon is equal to a L square or a L square by mk. So, we see that the solution for this particular law of force in general can be reduced to a form which is given as L by r equal to 1 plus epsilon cos theta minus theta 0, where L and epsilon are 2 constants which depends on the angular momentum, the force constant k and some arbitrary constant A. So, this equation is a very special class of equation which describes a conic section. So, what is the conic section and how it is special conic section is essentially, definition of a conic section is you we will we all know cones right. So, these are 3 d structure which is something like it looks like this.

So, this is called cone a solid object. Now if we take a plane let us let us consider this cone is not solid it is hollow inside and it. So, basically it has material only along the surface and the inside there is nothing, now what we do is we take a plane and make that plane intercept into this cone the curve, that will be traced out by such interception is called a conic section.

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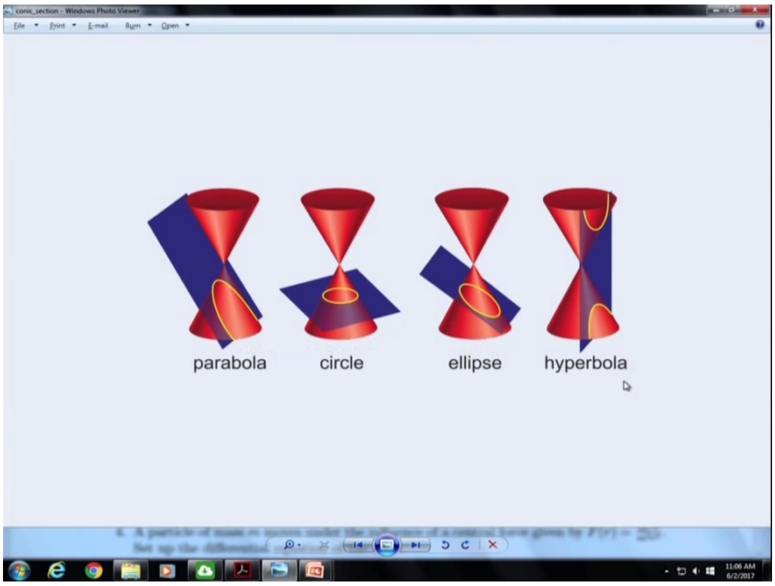

Let us look at some examples I took a picture of it from internet, now if the plane if that plane is obliquely cutting this cutting the cone then we get a shape which is called a parabola. If the plane cuts the cone with remaining horizontal to the ground, then it traced out a curve called a circle, if the cut is like this it is an ellipse and if the cut is like this it is an hyperbola.

Now, why we need 2 cones especially for hyperbola because in hyperbola if you recall from your coordinate geometry, lesson hyperbola essentially has 2 components; so one component is coming due to the intersection with the lower cone and the upper co other component is coming within from interception of the upper cone, which is vertically placed on top of the lower. So, we see that depending on the angle at which this plane cuts this 2 conic system, we have 1 conic 1 cone which is placed on the ground and 1 cone which is put vertically on top of the side and cone like this. So, if the plane cuts it like this then we have a parabola a circle or an ellipse hyperbola depending on the nature of the interception.

Now, what is so special about conic section now we know that the equation which describes here we do not know, but we I am telling you that this set of equation, is the equation of general equation of a conic section. Now how do you define, I mean is there any mathematical way of definition of conic section and happen to be there is a definition which is also a very powerful definition and this powerful definition. So, I will just it is this bit for a while, I hope that is fine and we can always gain it get it back.

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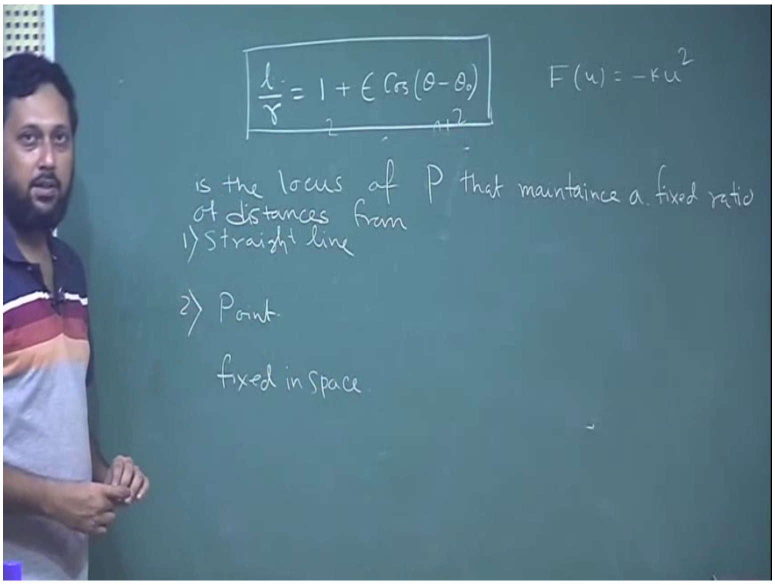

The definition is conic section is the locus of a point p, which maintains a fixed ratio of length with respect to a straight line and second a point fixed in space.

That maintain a fixed ratio I hope it is readable. So, conic section is a point is a locus of a point p that maintains a fixed ratio of distance from a straight line and a point which is fixed in space, now it is a bit complicated definition. So, let us not go into the writing we will just try to define it on board that will be easier I guess. So, what we need in order to define a conic section is a straight line and the point which is fixed in space.

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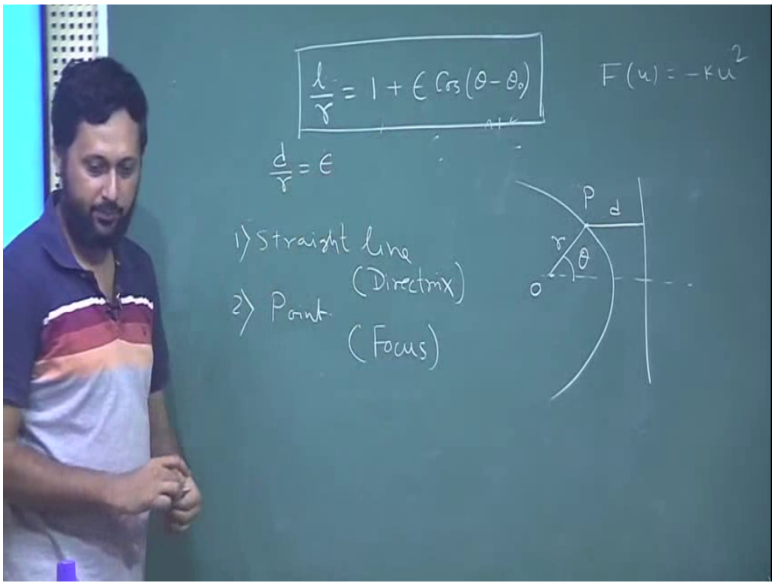

So, let us say this is my straight line and this is my point which are fixed, at this point they are happen to be very defined name for this thing, the straight line is called a directrix and the fixed point is called focus I hope this terms are familiar for you.

From your coordinate geometry now this is the focus we call this o and this is the directrix, we do not have a name for it because it is a straight line. So, this is the locus of the point let us say which we are interested in. I mean I am not intentionally I am not writing it in, what you call very symmetric manner. Maybe I should actually make it symmetric otherwise it is difficult to visualize all right. Now let us take the location of this point p which is given by r and theta. So, we are placing we are placing our origin at the focus.

So, that is how we get to this type of equation, if we place our origin anywhere arbitrarily then probably we will have a different form of equation also. We will see that here we will get theta 0 equal to 0 because as we as we said I put it very symmetrically, if theta 0 if the symmetry is broken then we have a value of theta 0, we will come to that in the moment. Now what we see is so what is the definition is it has a distance. So, any point on this particular curve maintains a distance d from this directrix and a distance r from this focus and this ratio is fixed. So, let us call this ratio d by r is equal to epsilon.

So, this epsilon turned out to be this epsilon, we will see that in the moment let us call it epsilon for now.

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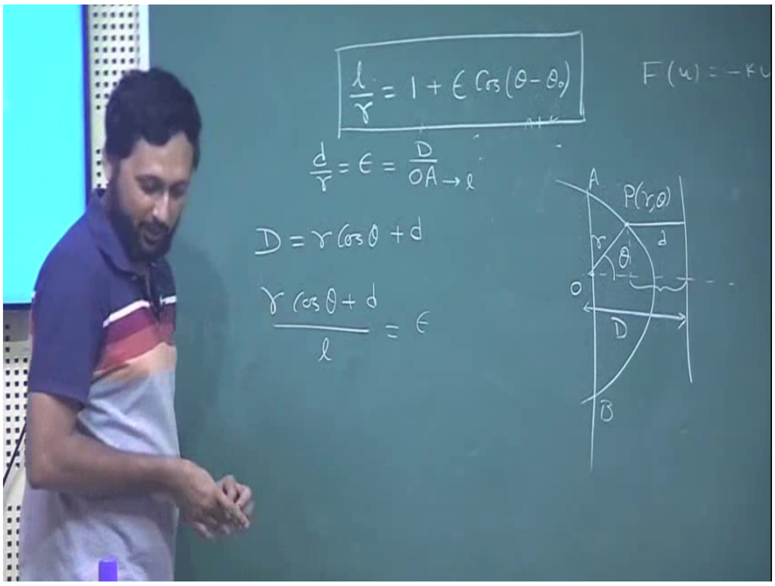

So, how do we get to the equation let us call this distance d. So, we get a definition for epsilon, now we need a definition for l. Let us draw a straight line which goes through focus and remains parallel to the directrix let us call this intercept a and this intercept b right, I mean it is not a very symmetric nice drawing, but you can you get the idea that we are drawing a straight line which is parallel to directrix and goes through the focus and we are finding out the intercept of this line with these 2 points right. Now if this ratio is fixed for all the look, I mean for all points on this particular curve then this will be the ratio for of d by oa as well is it clear.

See this point p which has a coordinate of r theta is moving along, I mean it can move along this curve. So, if we take this point or that point or any point in between or any point on this side, this ratio of d versus r should always be epsilon. Now when this point p is at point a coinciding at point a, then your d is equal to capital d and your r is equal to oa. So, d by r equal to epsilon equal to d by oa. So, also oh that comes later fine now. So, it is r cos theta plus d, now d is equal to what d is equal to this length once again is small d and this length is r cos theta. So, d is equal to r cos theta plus d so and let us call this length oa to be L, this L you as you guessed is the same l. So, r cos theta plus d whole divided by L is equal to epsilon, now we start rearranging things is take. So, sorry.

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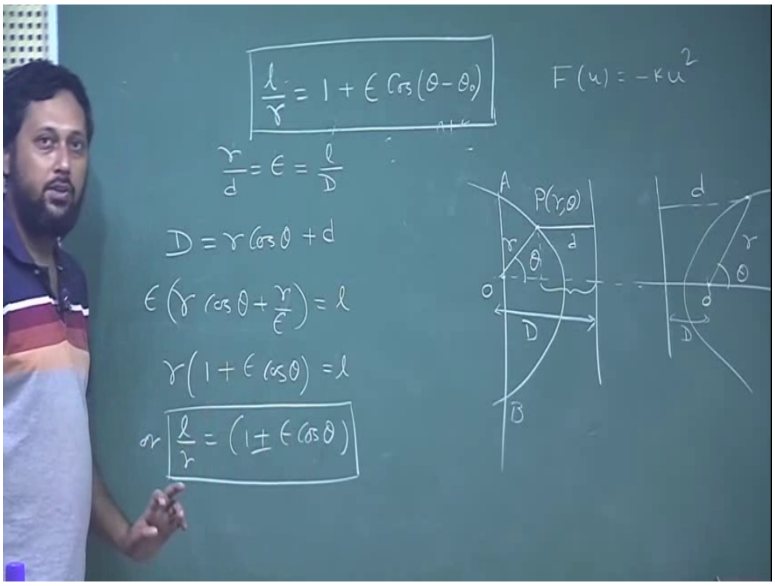

So, the ratio is defined as r by d not d by r. So, what I defined is a inverse of epsilon 1 over epsilon not epsilon. So, it will be r by d which will be once again L by D right. So, essentially what we have is this, d equal to r epsilon d is equal to L right. So, if we take d out of here we immediately see 1 plus r by d cos theta time’s epsilon t is equal to L right. oh sorry it is not needed. We just have to replace d with r by epsilon and that will solve the problem r cos theta equal to d. So, d is with r by epsilon right. So, if we take epsilon in so it will be r 1 plus epsilon cos theta to be equal to l, which will give L by r equal to 1 plus epsilon cos theta right.

So, because we took it symmetrically we got theta 0 equal to 0 and we got L by r is equal to 1 plus epsilon cos theta. If our picture is slightly tilted with a reference angle theta 0, then instead of theta we have to put theta minus theta 0 right. So, I can also show you that if I take this point somewhere here there is a alternative definition or basically if I change the picture slightly from this side, if we you know look into a mirror image something like this where this is your focus o and this is your directrix and this is your locus of the point and you take a point here which is given as r. So, this equation essentially reduces to L by r equal to 1 minus epsilon cos theta.

So, what I can conclude from this I am not showing that to you, but I can tell you that if you try this picture, if you then this will be your d. So, once again you have to define r by d equal to epsilon and this distance is your d. So, these are equally valid picture both of these are equally valid picture. So, with this we got L by r equal to 1 plus epsilon cos theta if you try this out yourself. So, in that case this will be your theta, as you can immediately see that this theta and etc is there is a phase difference of 90 degree between them or rather 1 80 degree between them. So, you will immediately see that there is a minus sign coming in. So, L by r equal to 1 minus epsilon cos theta is also a valid equation for a conic section and we will also be using this equation sometime all depends on how you choose your chose your focus.

We will see later on that depending on so for example, we all know that ellipse is a symmetric shape where there is 1 focus here and 1 focus here right.

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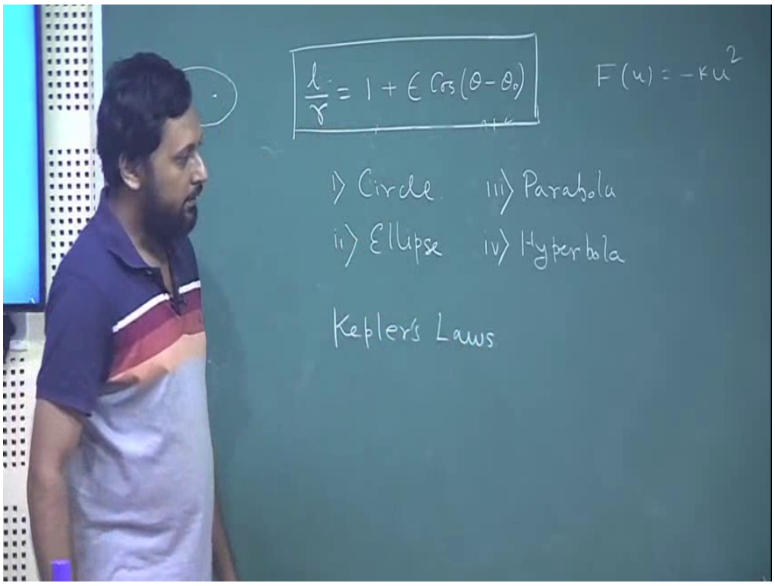

So, we will see that if we take our original this position, then we will get an equation of L by r equal to 1 plus epsilon cos theta whereas, if I can show you that if I take the shift origin from here to here, we get an equation which has the form that L by r equal to 1 minus epsilon cos theta. So, both of these are a valid form of conic section equation. Now that we know that inverse square law of force gives rise to a conic section, what does it imply; what is the implication physical implication of that? physical implication is not very straightforward I would say, at this at this point what we can suggest.

That any object moving under a gravitational field or an electrostatic attraction, attractive or repulsive forces. We will have 1 of this following orbit it can be a circle, it can be an ellipse, it can be a parabola or it can be a hyperbola 1 of this 4 right. Now it so happens that we can actually depending on the energy of the entire systems energy of the particle, which is moving under the action of central or inverse square attractive or repulsive force. We can comment whether this particular object is a circle or a parabola or an ellipse or a hyperbola, but that comes in a slightly later stage, right now what we are going to do is we are going to move.

We are going to move out of this equation I mean we keep this equation in mind, but now we are going to switch topic will be discussing Kepler's laws of planetary motion.

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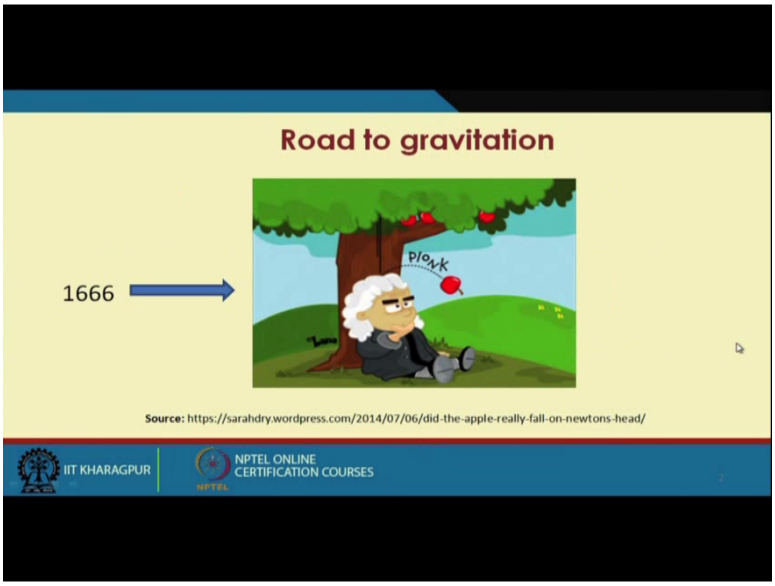

Now, this is not a very straightforward discussion, but I can show you some. I can take you through step by step manner, now we all know this very famous picture of sir Isaac Newton sitting under an apple tree and an apple falls on his head and we all we have this very notion, that immediately after this a miracle happened and he got an idea of gravitation, How does how the laws of gravitation work and all. But it is not that straightforward, see like in any branch of science or we all know what happens in sports for example, there is a new record only when there was a previous record, Virat Kohli comes into picture only of because there was a Sachin Tendulkar and Sachin Tendulkar comes into picture, because there was a Sunil Gavaskar prior to him records were made to be broken like that similar.

In a very similar manner in science, what happens new theory is developed only when there is a history behind it. So, it is not like 1 single day sir Isaac Newton was sitting under the apple tree without any previous notion of what could be a gravity, what could be the motion of planets and suddenly an apple fell on his head and he got the spark of inspiration it is not like that.

So, what we are going to do in the next section is we will systematically, I will try to give you a systematic estimate, I mean systematic what you call chronological description of how the force law of very famous force law of Newton I mean Newton's law of gravitation was derived and what is the history behind that, I can tell you that is a it has a history of many hundreds of years behind it is not a 1 day work.

So, we will in the next beginning of next class, we will I will go take you through a very small presentation on very on uncertain, very famous names you are familiar with probably and I will show you how their work was motivate; I mean their work motivated the discovery of gravitational field ok.

Thank you.